Spaces with fuzzy partitions and fuzzy sets

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Abstract
A theory of fuzzy objects is derived in the category \( \text{SpaceFP} \) of spaces with fuzzy partitions, which generalize classical fuzzy sets and extensional maps in sets with similarity relations. It is proved that fuzzy objects in \( \text{SpaceFP} \) can be characterized by some morphisms in the category of sets with similarity relations. A powerset object functor \( \mathcal{F} \) in the category \( \text{SpaceFP} \) is introduced and it is proved that \( \mathcal{F} \) defines a CSLAT-powerset theory in the sense of Rodabaugh.

Keywords: Spaces with fuzzy partitions; Category of spaces with fuzzy partitions; Fuzzy objects in spaces with fuzzy partitions; Powerset objects; Powerset theory.

1 Introduction

Since the introduction of fuzzy sets in the paper [18] of L.A.Zadeh, very rapid and extensive development of methods, tools and techniques using this concept appeared. In most of these methods and tools a very significant role is played by the concept of fuzzy relations defined on a set \( X \), i.e., \( R : X \times X \rightarrow [0, 1] \), or, more generally, \( R : X \times X \rightarrow Q \), where \( Q \) is an appropriate ordered structure. Fuzzy relation contributed to a very intensive development of new areas of fuzzy mathematics. These new areas include particularly fuzzy topology, or more generally, a fuzzy closure operators, further fuzzy rough sets and, recently, fuzzy transform as a method successfully used in signal and image processing [1], compression [10], numerical solutions of ordinary and partial differential equations [4, 16], data analysis [11] and many other applications. Fuzzy transform (or \( F \)-transform, shortly), is relatively new, it was firstly proposed byPerfilieva [12]. The main idea of \( F \)-transform is to factorize the precise value of independent variable by a closeness relation. The theory of \( F \)-transform was further elaborated from real valued to lattice-valued functions (e.g., [13], [12]).
Developments in the theory of fuzzy sets lead gradually to change the basic understanding of the universes for these applications. Traditionally, natural universes for these methods were standard sets, or sets with similarity relations instead of the classic relations of equalit. A new universe for many applications becomes a set of fuzzy partitions, as basic building blocks for F-transform and its modifications.

Universes for applications, including mathematical applications are closely connected with powerset structures or powerset operators, based on a given universe. Recall that given a set X, there exists the set \( \mathcal{P}(X) = \{ S : S \subseteq X \} \), called the powerset of X and such that every map \( f : X \to Y \) can be extended to the forward powerset operator \( f^\rightarrow : \mathcal{P}(X) \to \mathcal{P}(Y) \) and backward powerset operator \( f^\leftarrow : \mathcal{P}(Y) \to \mathcal{P}(X) \), such that

\[
\begin{align*}
    f^\rightarrow(S) & = f(S), \\
    f^\leftarrow(T) & = f^{-1}(T) = \{ x \in X : f(x) \in T \}.
\end{align*}
\]

A fuzzy set in a set \( A \) with values in the interval \( I = [0, 1] \) is defined as a map \( A \to I \) and it is then natural that an investigation of powerset objects \( I^X \) of fuzzy sets was of interest. The first approach was done again by L.A.Zadeh [18], who defined \( I^X \) as a new powerset object instead of \( \mathcal{P}(X) \) and introduced new powerset operators \( f^{-}_X : I^X \to I^Y \) and \( f^+ : I^X \to I^X \), such that for \( s \in I^X, t \in I^Y, y \in Y \),

\[
f^{-}_X(s)(y) = \bigvee_{x, f(x) = y} s(x), \quad f^+_X(t) = t \circ f.
\]

Since the original Zadeh’s paper was published, the notion of “fuzzy set” has been changed significantly and it is now more general. The first important modification concerns the value set: instead of real number interval \( I = [0, 1] \), more general lattice structures \( Q \) are considered. Among these lattice structures, complete residuated lattices play important role, (see e.g. [9]), in some terminology unital and commutative quantale, (see [15]), i.e. a structure \( Q = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1) \) such that \( (L, \wedge, \vee) \) is a complete lattice, \( (L, \otimes, 1) \) is a commutative monoid with operation \( \otimes \) isotone in both arguments and \( \rightarrow \) is a binary operation which is adjoint with respect to \( \otimes \).

Further, classical fuzzy sets (or even fuzzy sets with values in residuated lattice \( Q \)) were originally defined on sets. But any set \( A \) can be considered to be a couple \( (A, =) \), where \( = \) is a standard equality relation defined on \( A \). It is then natural instead of the crisp equality relation \( = \), to consider some more “fuzzy” equality relation defined on \( A \), which is called a similarity relation. Hence, instead of a classical set \( A \) as a basic set and a fuzzy set \( s : A \to Q \), we can use a set with similarity relation \( (A, \delta) \) (called a \( Q \)-set) and a map \( s : (A, \delta) \to Q \). Such a map then represents some new “fuzzy object” in \((A, \delta)\). Instead of maps \( A \to Q \), or \((A, \delta) \to Q \), morphisms in some categories can be used. An example of such category is the category \( Q \)-sets as objects and naturally defined morphisms. A morphism \( f : (A, \delta) \to (B, \gamma) \) in the category \( \text{Set}(Q) \) is a map \( f : A \to B \) such that \( \gamma(f(x), f(y)) \geq \delta(x, y) \) for all \( x, y \in A \). It is then natural to speak about a fuzzy object \((A, \delta) \to (Q, \delta)\) in the category \( \text{Set}(Q) \), instead of a “fuzzy set”, where \( \delta \) is the biresiduation operation in \( Q \) defined by \( \alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha) \). These fuzzy objects generalize classical fuzzy sets \( A \to Q \) and in facts, a fuzzy objects \((A, \delta) \to (Q, \delta)\) is nothing else than extensional map in \( Q \)-set (see, e.g. [6, 5, 7]).

Another generalization of universes used in fuzzy theory was done by I. Perfilieva [12, 13], who introduce the notion of F-transform. Basic structure for F-transform is a space with a fuzzy partition. A space with a fuzzy partition is a couple \((X, \alpha)\), where \( X \) is a set and \( \alpha \) is a system of \( Q \)-valued fuzzy sets in \( X \), such that cores of these fuzzy sets are a partition of \( X \). If \((X, \alpha)\) is a space with a fuzzy partition \( \alpha = \{ A_\lambda : \lambda \in \Lambda \} \), then fuzzy transforms (upper and lower) are special maps \( F^+ : \mathcal{P}(X) \to Q^X \), which fuzzify the precise values of independent variable by a closeness relation, and precise values of dependent variables as averages to an approximate values (see, e.g., [13]). Hence, it seems that spaces with fuzzy partitions can be used as a basic category for investigation not only of F-transform, but also of Kuratowski closure and interior operators and approximation spaces and some other notions.

In the paper we deal with fuzzy objects in the universe, which is the category \( \text{SpaceFP} \) of spaces with fuzzy partition. Fuzzy objects in such universe are defined as morphisms \((X, \alpha) \to (Q, \mathcal{D})\), where \( \mathcal{D} \) is an appropriate partition in \( Q \).

In Section 3, we show that such fuzzy objects are natural generalizations of classical fuzzy sets and fuzzy objects in the category \( \text{Set}(Q) \). In the category \( \text{Set}(Q) \), any map \( f : A \to Q \) can be extended to a fuzzy object \( \tilde{f} : (A, \delta) \to (Q, \delta) \). An analogical property for fuzzy objects in the category \( \text{SpaceFP} \) is proved in the same Section. In fact, we prove that fuzzy objects in the category \( \text{SpaceFP} \) correspond, in some sense, fuzzy objects in the category \( \text{Set}(Q) \).

In Section 4 we introduce powerset objects. \( \mathcal{F}(A, \mathcal{D}) = \{ (Q, \mathcal{D})|_{X, \mathcal{D}} \} \subseteq \mathcal{F}(A, \mathcal{D}) \) in the category \( \text{SpaceFP} \) and we show, that these powerset objects define CSLAT-powerset theory in the sense of Rodabaugh [14]. This CSLAT-powerset theory then comprises CSLAT-powerset theories of classical fuzzy sets and fuzzy objects in the category \( \text{Set}(Q) \).
2 Spaces with fuzzy partitions

In the paper, by \( Q \) we denote a complete residuated lattice (see e.g. [9]), in some terminology \textit{unital and commutative quantale}, (see [15]), i.e. a structure \( Q = (L, \wedge, \vee, \otimes, \to, 0_Q, 1_Q) \) such that \( (L, \wedge, \vee) \) is a complete lattice, \( (L, \otimes, 1_Q) \) is a commutative monoid with operation \( \otimes \) isotone in both arguments and \( \to \) is a binary operation which is adjoint with respect to \( \otimes \), i.e.

\[
\alpha \otimes \beta \leq \gamma \text{ iff } \alpha \leq \beta \to \gamma.
\]

We begin with the definition of \( Q \)-valued fuzzy partition, as it was introduced in [13]. Recall that a \textit{core} of a \( (Q\text{-valued}) \) fuzzy set \( f : X \to Q \) is defined by \( \text{core}(f) = \{ x \in X : f(x) = 1_Q \} \). A normal \( (Q\text{-valued}) \) fuzzy set \( f \) in a set \( X \) is such that there exists \( x \in X \), such that \( f(x) = 1_Q \). A fuzzy relation \( R \) in \( X \) is then a fuzzy set \( R : X \times X \to Q \).

An \( F \)-transform in a form introduced by Perfilieva [13] is based on the so called fuzzy partitions on the crisp set.

**Definition 2.1** ([13]). Let \( X \) be a set. A system \( \mathcal{A} = \{ A_\lambda : \lambda \in \Lambda \} \) of normal \( Q \)-valued fuzzy sets in \( X \) is a fuzzy partition of \( X \). If \( \{ \text{core}(A_\lambda) : \lambda \in \Lambda \} \) is a partition of \( X \). A pair \( (X, \mathcal{A}) \) is called a space with a fuzzy partition.

In the paper [8] we introduced the following definition of the category \( \text{SpaceFP} \) of spaces with fuzzy partitions.

**Definition 2.2** ([8]). The category \( \text{SpaceFP} \) is defined by

1. **Fuzzy partitions** \( (X, \mathcal{A}) \), as objects,
2. **Morphisms** \((g, \sigma) : (X, \{A_\lambda : \lambda \in \Lambda \}) \to (Y, \{B_\omega : \omega \in \Omega \})\), such that
   - \( a \) \( g : X \to Y \) is a map,
   - \( b \) \( \sigma : \Lambda \to \Omega \) is a map,
   - \( c \) \( \forall \lambda \in \Lambda, \ g^\sigma(A_\lambda) \leq B_{\sigma(\lambda)} \).
3. The composition of morphisms in \( \text{SpaceFP} \) is defined by \((h, \tau) \circ (g, \sigma) = (h \circ g, \tau \circ \sigma)\).

Instead of \( g^\sigma \) we will use only \( g^{\sigma^\vee} \).

Recall that a set with similarity relation (or \( Q \)-set) is a couple \((A, \delta)\), where \( \delta : A \times A \to Q \) is a map such that

- \( a \) \( (\forall x \in A) \ \delta(x, x) = 1_Q \),
- \( b \) \( (\forall x, y \in A) \ \delta(x, y) = \delta(y, x) \),
- \( c \) \( (\forall x, y, z \in A) \ \delta(x, y) \otimes \delta(y, z) \leq \delta(x, z) \) (generalized transitivity).

We will use the category \( \text{Set}(Q) \) with \( Q \)-sets as objects and with morphisms \( f : (A, \delta) \to (B, \gamma) \) defined as a map \( f : A \to B \), such that \( \gamma(f(x), f(y)) \geq \delta(x, y) \), for all \( x, y \in A \). The category \( \text{Set}(Q) \) has its origin in Wyler’s category introduced in [17], and in a more general way it was developed by U. Höhle in [3]. Within morphisms in the category \( \text{Set}(Q) \), we will be specially interested in morphisms \( (A, \delta) \to (Q, +) \), which are called \textit{fuzzy objects} in a corresponding category, or \textit{extensional sets} ([15]). Let \( F(A, \delta) = (\{Q, +\}^{\delta(A, \delta)}, \leq) \) be the ordered set of all such fuzzy objects in \( \text{Set}(Q) \), ordered point-wise. Then \( F(A, \delta) \) is a complete \( \vee \)-semilattice and if \( \text{CSLAT} \) is the category of all complete \( \vee \)-semilattices with \( \vee \)-preserving maps as morphisms, then \( F : \text{Set}(Q) \to \text{CSLAT} \) is a covariant functor, such that for any morphism \( f : (A, \delta) \to (B, \gamma) \),

\[
s \in F(A, \delta), b \in B, \quad f_{\gamma}(s)(b) = F(f)(s)(b) = \bigvee_{x \in A} s(x) \otimes \gamma(f(x), b).
\]

In many papers (see, e.g.,[6, 5, 3]) for any \( Q \)-set \((A, \delta)\), the following extensional map \( \tilde{s} : s : A \to Q \) is a map \( F(A, \delta) \) was introduced, such that for any map \( s : A \to Q \), \( \tilde{s}(a) = \bigvee_{x \in A} \delta(a, x) \otimes s(x) \), for every \( a \in A, \alpha \in Q \). Then \( \tilde{s} \in F(A, \delta) \).

We show firstly that \( Q \)-sets can be considered to be spaces with specially defined fuzzy partitions.

**Proposition 2.1.** There exists a full and faithful functor \( I : \text{Set}(Q) \to \text{SpaceFP} \), \( I(X, \delta) = (X, \delta^X) \), which is also injective on objects.
In the following proposition we prove conversely, when a fuzzy partition can be defined by a similarity relation.

**Proposition 2.2.** Let \((X, \mathcal{A})\) be a space with a fuzzy partition, \(\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}\). Then the following statements are equivalent.

1. There exists a similarity relation \(\delta\) on \(X\), such that \(\mathcal{A} = \mathcal{G}_X \delta\).
2. The following condition holds:
   \[
   (\forall \lambda, \omega \in \Lambda, x \in \text{core}(A_\omega), z \in X) \quad A_\lambda(x) \otimes A_\omega(z) \leq A_\alpha(z),
   \]

3 **Fuzzy objects in spaces with fuzzy partitions**

As we mentioned in Introduction, \(Q\)-valued fuzzy objects in the category \(\text{Set}\) of classical sets or in the category \(\text{Set}(Q)\) of \(Q\)-sets, can be considered to be morphisms \(A \to Q\) in these categories, where \(A\) is an underlying object in that category and \(Q\) is a special object representing valued structure in that category. In the category \(\text{Set}\), \(Q\) is the underlying set of a lattice \(Q\) and in the category \(\text{Set}(Q)\), \(Q\) is the \(Q\)-set \((Q, \leftrightarrow)\). From that point of view, in the category \(\text{SpaceFP}\), we can also consider special morphisms \((A, \mathcal{A}) \to (Q, \mathcal{D})\), where \((Q, \mathcal{D})\) would be the space \(Q\) with some fuzzy partition \(\mathcal{D}\). And these special morphisms then could represent \(Q\)-valued fuzzy objects in the category \(\text{SpaceFP}\). Since \(\leftrightarrow\) is a similarity relation in the set \(Q\), for a construction of a fuzzy partition \(\mathcal{D}\) we can use the \(Q\)-set \((Q, \leftrightarrow)\) and the result of Proposition 2.1.

Let us mention that, using Proposition 2.1, for the equivalence relation \(\equiv_{\leftrightarrow}\) defined on \(Q\), we have \(Q/\equiv_{\leftrightarrow} = \{\{\alpha\} : \alpha \in Q\} \cong Q\). Let \(Q_\alpha\) be a fuzzy set in \(Q\) defined by

\[
Q_\alpha(\beta) = \alpha \leftrightarrow \beta.
\]

Then \(\text{core}(Q_\alpha) = \{\alpha\}\) and by \(\mathcal{D}\) we denote the fuzzy partition \(\mathcal{G}_{Q_\alpha \leftrightarrow}\) defined by \(Q_\alpha\)-set \((Q, \leftrightarrow)\), according to Proposition 2.1, i.e., \(\mathcal{D} = \{Q_\alpha : \alpha \in Q\}\). Hence, we can consider the pair \((Q, \mathcal{D})\) to be the set \(Q\) with the fuzzy partition \(\mathcal{D}\).

**Definition 3.1.** A \((Q\text{-valued})\) fuzzy object in a space with a fuzzy partition \((X, \mathcal{A})\) is a morphism \((f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{D})\) in the category \(\text{SpaceFP}\). By \((Q, \mathcal{D})^{(X, \mathcal{A})}\) we denote the set of all fuzzy objects in \((X, \mathcal{A})\).

In the following proposition we show that any fuzzy object \((f, \sigma)\) in a space with a fuzzy partition \((X, \mathcal{A})\) can be uniquely determined by either of the maps \(f : X \to Q\) or \(\sigma : \Lambda \to Q\).

**Proposition 3.1.** Let \((X, \mathcal{A}) \in |\text{SpaceFP}|\), \(\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}\).

1. Let \(\sigma : \Lambda \to Q\) be a map. Then the following statements are equivalent:
   a) There exists the unique map \(f : X \to Q\) such that \((f, \sigma) \in (Q, \mathcal{D})^{(X, \mathcal{A})}\).
   b) For all \(\lambda, \lambda' \in \Lambda\) and any \(x \in \text{core}(A_{\lambda'})\), \(A_{\lambda'}(x) \leq A_{\lambda}(x) \leftrightarrow \sigma(\lambda')\) holds.

2. Let \(f : X \to Q\) be a map. Then the following statements are equivalent.
   a) There exists the unique map \(\sigma : \Lambda \to Q\) such that \((f, \sigma) \in (Q, \mathcal{D})^{(X, \mathcal{A})}\).
   b) For any \(\lambda \in \Lambda\) and \(x \in \text{core}(A_\lambda)\), \(x' \in X\), \(A_{\lambda}(x') \leq f(x) \leftrightarrow f(x')\) holds.

We will use the following notation. If \(f : X \to Q\) is a map which satisfies the condition 2)(b) from previous proposition, by \([f]\) we denote the unique map \(\Lambda \to Q\) such that \((f, [f])\) is a fuzzy object in \((X, \mathcal{A})\). Similarly, if \(\sigma : \Lambda \to Q\) is a map satisfying the condition 1)(b), then by \([\sigma]\) we denote the unique map \(X \to Q\) such that \(([\sigma], \sigma)\) is a fuzzy object in \((X, \mathcal{A})\). In that case we say that \(f (\sigma\text{, respectively})\) defines a fuzzy object \((f, [f])\) \(((\sigma, \sigma), \text{respectively})\).

Let \((X, \mathcal{A})\) be a space with a fuzzy partition, \(\mathcal{D} = \{A_\lambda : \lambda \in \Lambda\}\). We set

\[
\mathcal{F}(X, \mathcal{A}) = \{(t, \rho) : (t, \rho)\ \text{is a fuzzy object in} \ (X, \mathcal{A})\},
\]

\[
\mathcal{F}_1(X, \mathcal{A}) = \{t| : X \to Q \text{ defines a fuzzy object in} \ (X, \mathcal{A})\},
\]

\[
\mathcal{F}_2(X, \mathcal{A}) = \{\tau| : \Lambda \to Q \text{ defines a fuzzy object in} \ (X, \mathcal{A})\}.
\]
Then according to previous Proposition 3.1, we have
\[ \mathcal{F}(X, \mathcal{A}) = \{(t, [\ell]) : t \in \mathcal{F}_1(X, \mathcal{A})\} = \{([\sigma], \sigma) : \sigma \in \mathcal{F}_2(X, \mathcal{A})\}. \] (3.4)

On a set \( \mathcal{F}(X, \mathcal{A}) \) an ordering can be defined such that \( (t, p) \leq (s, q) \Leftrightarrow t \leq s, p \leq q \). From the previous Proposition 3.1 it follows that \( (t, p) \leq (s, q) \) iff \( t \leq p \) or \( s \leq q \). Analogical ordering can be defined also on sets \( \mathcal{F}_1(A, \mathcal{A}) \) and \( \mathcal{F}_2(A, \mathcal{A}) \).

Moreover, we have

**Lemma 3.1.** \( (\mathcal{F}(X, \mathcal{A}), \leq) \) is a complete lattice.

In the following examples we show that the notion of a fuzzy object in a space with fuzzy partition extends classical notions of a fuzzy set in a set and also of an extensional map in a set with similarity relation.

**Example 3.1.** Let \( \mathcal{A} = \{x \in X\} \) and let \( (f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{C}) \) be a morphism in SpaceFP. Hence, \( f, \sigma : X \to Q \) are maps and we have \( \{f(x)\} = f(\text{core}(\{x\})) \subseteq \text{core}(Q_{\sigma(\mathcal{A})}) = \{\sigma(x)\} \). Therefore, \( f = \sigma \) and it is clear that the property from the definition of morphisms in SpaceFP holds automatically. Therefore, \( (f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{C}) \) is a morphism if \( f : X \to Q \) is a map. Hence, \( (Q, \mathcal{C})^{|X, \mathcal{A}|} = \{(f, f) : f \in Q^X\} \) and \( \mathcal{F}_1(X, \mathcal{A}) = Q^X \) is the set of all fuzzy sets in a set \( X \).

**Example 3.2.** Let \( (X, \delta) \in |\text{Set}(Q)| \) and let \( (f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{C}) \) be a morphism in SpaceFP. Since \( \mathcal{C} \) is the fuzzy partition defined by the similarity relation \( \leftrightarrow \) and, according to Proposition 2.1, \( I|\text{Set}(Q)| \) is a full subcategory in SpaceFP, it follows that \( f : (X, \delta) \to (Q, \leftrightarrow) \) is a morphism in \( \text{Set}(Q) \). Hence, \( (Q, \mathcal{C})^{|X, \mathcal{A}|} = \{(f, [f]) : f \text{ is an extensional map in } (X, \delta)\} \) and \( \mathcal{F}_1(X, \mathcal{A}) = (Q, \leftrightarrow)^{|X, \delta|} \).

**Example 3.3.** Let \( X \) be a set and let \( \mathcal{A} = \{A_h : \lambda \in \Lambda\} \), where \( A_h \subseteq X \) are crisp sets. Let \( (f, \sigma) : (X, \mathcal{A}) \to (Q, \mathcal{C}) \) be a morphism in SpaceFP. Then \( f(A_h) = f(\text{core}(A_h)) \subseteq \text{core}(Q_{\sigma(\mathcal{A})}) = \{\sigma(A_h)\} \) and it follows that \( f(A_h) = \{\sigma(A_h)\} \). The condition from the definition of morphisms in SpaceFP is satisfied automatically and we obtain that \( (Q, \mathcal{C})^{|X, \mathcal{A}|} \cong Q^\Lambda \), which represents special fuzzy sets, constant on elements of a partition \( \mathcal{A} \).

As we have mentioned, if \( (A, \delta) \) is a \( Q \)-set and \( s : A \to Q \) is a map, then \( s \) can be extended to an extensional set \( \tilde{s} : (A, \delta) \to (Q, \leftrightarrow) \), such that \( \tilde{s} \) is the smallest extensional set in \( (A, \delta) \), such that \( \tilde{s} \geq s \). In the next theorem we prove that analogical result holds also for spaces with fuzzy partitions. Let \( (X, \mathcal{A}) \) be a space with a fuzzy partition. We prove that any map \( t : X \to Q \) (or \( \sigma : A \to Q \), respectively) can be extended in the smallest way to the map \( \tilde{t} : X \to Q \) (or \( \tilde{\sigma} : A \to Q \), respectively), which defines a fuzzy object.

**Theorem 3.1.** Let \( (X, \mathcal{A}) \) be a space with a fuzzy partition. Then there exist maps
\[ Q^X \to \mathcal{F}_1(X, \mathcal{A}), \quad t \mapsto \tilde{t}, \]
\[ Q^\Lambda \to \mathcal{F}_2(X, \mathcal{A}), \quad \xi \mapsto \tilde{\xi}, \]
with the following properties.

1. There exist a similarity relation \( \rho_{X, \mathcal{A}} \) in \( \Lambda \) and a similarity relation \( \delta_{X, \mathcal{A}} \) in \( X \), such that
\[ x \in X, \quad \tilde{t}(x) = \bigvee_{z \in A} t(z) \otimes \delta_{X, \mathcal{A}}(z, x), \]
\[ \lambda \in \Lambda, \quad \tilde{\xi}(\lambda) = \bigvee_{\omega \in \Lambda} \xi(\omega) \otimes \rho_{X, \mathcal{A}}(\omega, \lambda), \]

2. \( \tilde{t} \) (\( \tilde{\xi} \), respectively) defines a fuzzy object in \( (X, \mathcal{A}) \) and \( \tilde{t} \geq t \ (\tilde{\xi} \geq \xi \), respectively),

3. if \( t \) (\( \xi \), respectively) defines a fuzzy object in \( (X, \mathcal{A}) \), then \( \tilde{t} = t \ (\tilde{\xi} = \xi \), respectively).
Proof. We show only how the corresponding similarity relations $\rho$ and $\delta$ are defined. Let $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ and let $t : X \to Q$, $\xi : \Lambda \to Q$ be maps. We define a fuzzy binary relation $\pi : \Lambda \times \Lambda \to Q$ by

$$\pi(\lambda, \omega) = \bigvee_{x \in \text{core}(A_\lambda)} A_\lambda(x) \lor \bigvee_{x \in \text{core}(A_\omega)} A_\omega(x).$$

Then $\pi$ is a symmetric and reflective fuzzy relation and we can consider the smallest fuzzy transitive closure $\rho = \rho_{X,\mathcal{A}}$ of $\pi$, i.e., $\rho_{X,\mathcal{A}}$ is the smallest fuzzy relation, such that for any $\lambda$, $\omega$, $\alpha \in \Lambda$, we have

$$\rho_{X,\mathcal{A}}(\lambda, \omega) \otimes \rho_{X,\mathcal{A}}(\omega, \alpha) \leq \rho_{X,\mathcal{A}}(\lambda, \alpha), \quad \pi(\lambda, \omega) \leq \rho_{X,\mathcal{A}}(\lambda, \omega).$$

In that case, $\rho_{X,\mathcal{A}}$ is a similarity relation in a set $X$. For details of the construction of the smallest fuzzy transitive closure see, e.g., [2].

Now, let a binary fuzzy relation $\delta_{X,\mathcal{A}}$ in $X$ be defined by

$$a, b \in X, \quad \delta_{X,\mathcal{A}}(a, b) := \rho_{X,\mathcal{A}}(\alpha, \beta) \iff a \in \text{core}(A_\alpha), b \in \text{core}(A_\beta).$$

Since $\rho_{X,\mathcal{A}}$ is a similarity relation in $\Lambda$, it follows that $\delta_{X,\mathcal{A}}$ is a similarity relation in a set $X$. \hfill \Box

In the proof of Theorem 3.1 we defined similarity relations $\rho_{X,\mathcal{A}}$ and $\delta_{X,\mathcal{A}}$. Corresponding $Q$-sets $(\Lambda, \rho_{X,\mathcal{A}})$ and $(X, \delta_{X,\mathcal{A}})$ are closely connected with the space with fuzzy partition $(X, \mathcal{A})$. In fact, we have

**Proposition 3.2.** There exist functors

$$\begin{array}{c}
\text{SpaceFP} \\ \xrightarrow{H} \\ \xrightarrow{G} \\ \text{Set}(Q),
\end{array}$$

such that for any space with a fuzzy partition $(X, \mathcal{A}) = \{A_\lambda : \lambda \in \Lambda\}$,

$$G(X, \mathcal{A}) = (\Lambda, \rho_{X,\mathcal{A}}), \quad H(X, \mathcal{A}) = (X, \delta_{X,\mathcal{A}}).$$

As we have mentioned in the introduction, spaces with fuzzy partitions are basic structures for fuzzy transforms (F-transforms). If $(X, \mathcal{A})$ is a space with a fuzzy partition $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$, then fuzzy transforms (upper and lower) are special maps $F^\uparrow, F^\downarrow : Q^X \to Q^\Lambda$, which fuzzify the precise values of independent variable by a closeness relation, and precise values of dependent variables as averages to an approximate values (see, e.g.,[12]). More precisely, if $f \in Q^X$, then

$$F^\uparrow(f)(\lambda) = \bigvee_{x \in X} f(x) \otimes A_\lambda(x), \quad F^\downarrow(f)(\lambda) = \bigwedge_{x \in X} (A_\lambda(x) \to f(x)).$$

In the next proposition we show that fuzzy objects in $(X, \mathcal{A})$ are, in some sense, fix points of these F-transforms.

**Proposition 3.3.** Let $(X, \mathcal{A})$ be a space with a fuzzy partition. Then for any $f \in F^\uparrow(X, \mathcal{A})$, we have

$$\forall \lambda \in \Lambda \quad F^\uparrow(f)(\lambda) = F^\downarrow(f)(\lambda) = f(z), \quad z \in \text{core}(A_\lambda).$$

4 Powerset objects in spaces with fuzzy partitions

In the next part we will deal with the powerset object

$$\mathcal{F}(X, \mathcal{A}) = ((Q, \mathcal{P})^{(X, \mathcal{A})}, \subseteq)$$

of a space with a fuzzy partition with ordering defined in Lemma 3.1. Our goal is to prove that, analogically as powerset objects of classical fuzzy sets or powerset objects of fuzzy objects in sets with similarity relations, $\mathcal{F}(X, \mathcal{A})$ satisfies conditions of a general definition of powerset objects, presented by Rodabaugh [14], i.e.,
Definition 4.1 (Rodabaugh [14]). Let \( K \) be a category and let CSLAT be the category of complete \( \lor \)-semilattices with \( \lor \)-preserving maps as morphisms. Then \( P = (P, \to, \leftarrow, V, \eta) \) is called CSLAT-powerset theory in \( K \), if

1. \( P : [K] \to |CSLAT| \) is an object-mapping,
2. for each \( f : A \to B \) in \( K \), there exists \( f_P^\to : P(A) \to P(B) \) in CSLAT,
3. for each \( f : A \to B \) in \( K \), there exists \( f_P^\leftarrow : P(B) \to P(A) \) in CSLAT,
4. \((f_P^\to, f_P^\leftarrow)\) is a Galois connection,
5. There exists a concrete functor \( V : K \to \text{Set} \), such that \( \eta \) determines in \( \text{Set} \) for each \( A \in K \) a mapping \( \eta_A : V(A) \to P(A) \),
6. For each \( f : A \to B \) in \( K \), \( f_P^\to \circ \eta_A = \eta_B \circ V(f) \).

We want to show firstly that the object functions \( \mathcal{F}, \mathcal{F}_1 \) and \( \mathcal{F}_2 \) defined by (1),(2) and (3), define functors \( \text{SpaceFP} \to \text{CSLAT} \).

Theorem 4.1 (Extension principle for fuzzy objects). For any \((X, \mathcal{A}) \in |\text{SpaceFP}|, \mathcal{F}_1(X, \mathcal{A}), \mathcal{F}_2(X, \mathcal{A}) \) and \( \mathcal{F}(X, \mathcal{A}) \) define object functions of functors \( \text{SpaceFP} \to \text{CSLAT} \).

The principal goal of the paper is to solve the question, if the powerset objects and powerset operators \( \mathcal{F}, \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of new fuzzy objects in the category \( \text{SpaceFP} \) presented above have similar properties to those of classical fuzzy objects \( s : X \to Q \) or to fuzzy objects in the category \( \text{Set}(Q) \). We want to show that all these fuzzy objects have powerset structures which are powerset theories in the category \( \text{SpaceFP} \), in the sense of Rodabaugh [14]. For classical Zadeh's powerset theory \( Z \) and classical powerset theory \( P \) in sets, there exists a strong relation between these two theories, which can be represented as some homomorphism \( P \to Z \). We show that analogously for these new powerset theories \( F \) there exist "new classical" powerset theories \( R \) and a homomorphism \( R \to F \).

We introduce firstly concrete functors \( T_1, T_2 \) and \( T \) in the category \( \text{SpaceFP} \), which are an analogy of the classical functor \( X \mapsto (2^X, \subseteq) \) in the category \( \text{Set} \). The functors \( T_1 : \text{SpaceFP} \to \text{CSLAT} \) are defined for objects \((X, \mathcal{A})\) and morphisms \((f, \sigma)\) by

\[
T_1(X, \mathcal{A}) = (2^X, \subseteq), \quad (\forall A \subseteq X)(f(A))^{+}_{T_1}(A) = f(A) \tag{4.5}
\]

\[
T_2(X, \mathcal{A}) = (2^X, \subseteq), \quad (\forall \Phi \subseteq A)(f(A))^{-}_{T_2}(\Phi) = \sigma(\Phi). \tag{4.6}
\]

The functor \( T \) will be defined lately.

Proposition 4.1. There exist the natural transformations:

\[
\eta_1 : T_1 \to \mathcal{F}_1, \quad \eta_2 : T_2 \to \mathcal{F}_2.
\]

We define the functor \( T : \text{SpaceFP} \to \text{CSLAT} \). Let \((X, \mathcal{A})\) be a space with a fuzzy partition, then we set

\[
T(X, \mathcal{A}) = \{(A, \Phi) : A \subseteq X, \Phi \subseteq \Lambda, [\eta_1, (X, \mathcal{A})](A) = [\eta_2, (X, \mathcal{A})](\Phi)\} = \{(A, \Phi) : A \subseteq X, \Phi \subseteq \Lambda, \eta_1(X, \mathcal{A})(A) = \eta_2(X, \mathcal{A})(\Phi)\} \subseteq T_1(X, \mathcal{A}) \times T_2(X, \mathcal{A}),
\]

where we use a notation from (4) and the set \( T(X, \mathcal{A}) \) is ordered point-wise by inclusion. Let \((f, \sigma) : (X, \mathcal{A}) \to (Y, \mathcal{B})\) be a morphism in \( \text{SpaceFP} \). Then a \( \lor \)-preserving map \( T(f, \sigma) = (f, \sigma)^{+}_{T} \) is defined by

\[
(A, \Phi) \in T(X, \mathcal{A}), \quad T(f, \sigma)(A, \Phi) = (f(A), \sigma(\Phi)).
\]

Theorem 4.2. There exists a natural transformation

\[
\eta : T \to \mathcal{F}.
\]
In the next theorem we show that powerset functors \( F, F_1 \) and \( F_2 \) have similar properties to standard powerset functors of fuzzy sets in classical sets and sets with similarity relations, respectively.

**Theorem 4.3** (Powerset theories in SpaceFP). Let \( V : CSLAT \rightarrow Set \) be the forgetful functor. The following statements then hold.

1. \( F_1 = (F_1 \rightarrow, V, T_1, \eta_1) \) is a CSLAT-powerset theory in the category SpaceFP.
2. \( F_2 = (F_2 \rightarrow, V, T_2, \eta_2) \) is a CSLAT-powerset theory in the category SpaceFP.
3. \( F = (F \rightarrow, V, T, \eta) \) is a CSLAT-powerset theory in the category SpaceFP.

**Example 4.1.** Let \((X, \mathcal{A})\) be a space with fuzzy partition from Example 3.1, i.e., \( \mathcal{A} = \{ \{x\} : x \in X\} \). In that case we have \( F(X, \mathcal{A}) = F_1(X, \mathcal{A}) = F_2(X, \mathcal{A}) = Q^X \). It is then clear that

\[
T(X, \mathcal{A}) = \{(A, A) : A \subseteq X\} \cong T_1(X, \mathcal{A}) \cong T_2(X, \mathcal{A}) \cong 2^X.
\]

and the powerset theory \( F \) is the classical Zadeh’s powerset theory.

5 Conclusions

We investigated **fuzzy objects** in the category SpaceFP of spaces with fuzzy partition, which could be a basic category for \( F \)-transforms and some other construction, as closure spaces or fuzzy approximation spaces. (\( Q \)-valued) Fuzzy objects in the category SpaceFP are morphisms \((X, \mathcal{A}) \rightarrow (Q, \mathcal{Q})\), where \( \mathcal{Q} \) is an appropriate partition in a complete residuated lattice \( Q \), derived from the biresiduation operation \( \leftrightarrow \) in \( Q \). We show that fuzzy objects in SpaceFP are natural generalizations of classical fuzzy sets in the category of sets and fuzzy objects in the category Set\((Q)\) of sets with similarity relations.

We introduce powerset objects functor \( F(X, \mathcal{A}) = ([Q, \mathcal{Q}]^X, \mathcal{A}, \leq) \) in the category SpaceFP and, as the main result of the paper, we show that these powerset objects define CSLAT-powerset theory in the sense of Rodabaugh [14]. This CSLAT-powerset theory then comprises CSLAT-powerset theories of classical fuzzy sets and fuzzy objects in the category Set\((Q)\).

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References


