Solution of First Order Linear Non Homogeneous Ordinary Differential Equation in Fuzzy Environment Based on Lagrange Multiplier Method

Sankar Prasad Mondal¹*, Tapan Kumar Roy¹

(1) Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah-711103, West Bengal, India

Abstract
In this paper the First Order Linear Non Homogeneous Ordinary Differential Equations (FOLNHODE) are described in fuzzy environment. Here coefficients and /or initial condition of FOLNHODE are taken as Generalized Triangular Fuzzy Numbers (GTFNs).The solution procedure of the FOLNHODE in fuzzy environment is developed by Lagrange multiplier Method. Finally a fish population problem is discussed in fuzzy environment.

Keywords: Fuzzy Differential Equation, Generalized triangular fuzzy number, Fish population model.

1 Introduction
It is observed that in recent years the topic of Fuzzy Differential Equations (FDEs) has been swiftly grown. In the year 1987, the term “fuzzy differential equation” was proposed by Kandel and Byatt [1]. To study FDE there have been many conceptions for the definition of fuzzy derivative. Chang and Zadeh [32] first introduced the concept of fuzzy derivative, later on it was followed up by Dobois and Prade [11] who used the extension principle in their study. Other methods have been discussed by Puri and Ralescu [23], Goetschel and Voxman [31], Seikkala [36] and Friedman et al. [24, 2] etc. Buckley and Feuring [16] have been comparing the different solutions through which one may obtain to the fuzzy differential equations using various derivatives of fuzzy function that have been presented in the different literature.

* Corresponding Author. Email address: sankar.res07@gmail.com
Bede introduced a strongly generalized differentiability of fuzzy functions in [5] and studied in [6, 7]. The initial value problem for fuzzy differential equation (FIVP) has been studied by Kaleva in [29, 30] and by Seikkala in [36]. Buckley and Feuring [16, 17] and Buckley et al. [18] gave a very basic formulation of a fuzzy first-order initial value problem. At first they found the crisp solution, then fuzzified it and then checked to see if it satisfies the fuzzy System of differential Equation. The initial condition of a FDE is taken as different type of fuzzy number. It was supposed that the initial conditions are triangular. It is seen in several paper like Chen and Chen [10], Mahapatra and Roy [22] that generalized fuzzy number are used for solving real life problem and still no one has used generalized fuzzy number for solving the FDE problem with fuzzy parameters.

Fuzzy differential equations play a significant role in the field of biology, engineering, physics and other sciences. For example, in population models [25], civil engineering [26], bioinformatics and computational biology [19], quantum optics and gravity [27], modeling hydraulic [3], HIV model [14], decay model [13], predator-prey model [28], population dynamics model [20], Friction model [8], Growth model [33]. Bacteria culture model [34], bank account and drug concentration problem [35]. First order linear fuzzy differential equations are among all the FDE which has many applications.

The structure of this paper is as follows: In first three sections, we introduce some concepts and introductory material to deal with the FDE. Solution procedure of FOLNHODE in fuzzy environments is discussed in section 4. In section 5 the procedure is applied in a fish population model. At the end in section 6 of the paper we present some conclusion and give some topics for future research.

2 Preliminary concept

Definition 2.1. Fuzzy Set: A fuzzy set \( \tilde{A} \) is defined by \( \tilde{A} = \{(x, \mu_A(x)) : x \in A, \mu_A(x) \in [0,1]\} \). In the pair \( (x, \mu_A(x)) \) the first element \( x \) belong to the classical set \( A \), the second element \( \mu_A(x) \), belong to the interval \([0, 1] \), called Membership function.

Definition 2.2. Height: The height \( h(\tilde{A}) \), of a fuzzy set \( \tilde{A} = (x, \mu_A(x)) : x \in X \), is the largest membership grade obtained by any element in that set i.e. \( h(\tilde{A}) = \text{sup} \mu_A(x) \).

Definition 2.3. Convex Fuzzy sets: A fuzzy set \( \tilde{A} = \{(x, \mu_A(x))\} \subseteq X \) is called convex fuzzy set if all \( A_\alpha \) for every \( \alpha \in [0,1] \) are convex sets i.e., for element \( x_1 \in A_\alpha \) and \( x_2 \in A_\alpha \) and \( \lambda x_1 + (1-\lambda)x_2 \in A_\alpha \) \( \forall \lambda \in [0,1] \). Otherwise the fuzzy set is called non-convex fuzzy set.

Definition 2.4. \( \alpha \)-Level or \( \alpha \)-cut of a fuzzy set: The \( \alpha \)-level set (or interval of confidence at level \( \alpha \) or \( \alpha \)-cut) of the fuzzy set \( \tilde{A} \) of \( X \) is a crisp set \( A_\alpha \) that contains all the elements of \( X \) that have membership values in \( A \) greater than or equal to \( \alpha \) i.e. \( \tilde{A} = \{x, \mu_A(x) \geq \alpha, x \in X, \alpha \in [0,1]\} \).

Definition 2.5. Fuzzy Number: [21] A fuzzy number is fuzzy set like \( u: R \rightarrow I = [0,1] \) which satisfies

1. \( u \) is upper semi-continuous.
2. \( u(x) = 0 \) outside the interval \([c, d]\)
3. There are real numbers \( a, b \) such \( c \leq a \leq b \leq d \) and
   
   (3.1) \( u(x) \) is monotonic increasing on \([c, a]\),
   (3.2) \( u(x) \) is monotonic decreasing on \([b, d]\),
   (3.3) \( u(x) = 1, a \leq x \leq b \)
Let $E^1$ be the set of all real fuzzy numbers which are normal, upper semi-continuous, convex and compactly supported fuzzy sets.

Otherwise, [12] A fuzzy number $u$ in a parametric form is a pair $(u_1, u_2)$ of function $u_1(r), u_2(r), 0 \leq r \leq 1,$ which satisfies the following requirements:

1) $u_1(r)$ is a bounded monotonic increasing left continuous function
2) $u_2(r)$ is a bounded monotonic decreasing left continuous function,
3) $u_1(r) \leq u_2(r), 0 \leq r \leq 1.$

A crisp number $x$ is simply represented by $(u_1(r), u_2(r)) = (x, x), 0 \leq r \leq 1.$

Definition 2.6. Generalized Fuzzy number (GFN): Generalized Fuzzy number $\tilde{A}$ as

$\tilde{A} = (a_1, a_2, a_3, a_4; \omega), \text{ where } 0 < \omega \leq 1,$ and $a_1, a_2, a_3, a_4 (a_1 < a_2 < a_3 < a_4)$ are real numbers. The generalized fuzzy number $\tilde{A}$ is a fuzzy subset of real line $R,$ whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

1) $\mu_{\tilde{A}}(x): R \to [0, 1]$
2) $\mu_{\tilde{A}}(x) = 0$ for $x \leq a_1$
3) $\mu_{\tilde{A}}(x)$ is strictly increasing function for $a_1 \leq x \leq a_2$
4) $\mu_{\tilde{A}}(x) = w$ for $a_2 \leq x \leq a_3$
5) $\mu_{\tilde{A}}(x)$ is strictly decreasing function for $a_3 \leq x \leq a_4$
6) $\mu_{\tilde{A}}(x) = 0$ for $a_4 \leq x$

Figure 1: Generalized Fuzzy Number

Definition 2.7. Generalized TFN: If $a_2 = a_3$ then $\tilde{A}$ is called a GTFN as $\tilde{A} = (a_1, a_2, a_4; \omega)$ or

$(a_1, a_3, a_4; \omega)$ with membership function $\mu_{\tilde{A}}(x) = \begin{cases} \omega \frac{x - a_1}{a_2 - a_1} & \text{if } a_1 \leq x \leq a_2 \\ \omega \frac{a_4 - x}{a_4 - a_2} & \text{if } a_2 \leq x \leq a_4 \\ 0 & \text{otherwise} \end{cases}$

Definition 2.8. TFN: If $a_2 = a_3, \omega = 1$ then $\tilde{A}$ is called a TFN as $\tilde{A} = (a_1, a_2, a_3)$ or $\tilde{A} = (a_1, a_3, a_4)$
Definition 2.9. Type of GTFN: If $\tilde{A} = (a_1, a_2, a_3, \omega)$ is a GTFN then it is called

1) symmetric GTFN if $a_2 - a_1 = a_3 - a_2$

2) Non symmetric type 1 ($l_s > r_s$) GTFN if $a_2 - a_1 > a_3 - a_2$

3) Non symmetric type 2 ($l_s < r_s$) GTFN if $a_2 - a_1 < a_3 - a_2
4) Left GTFN if $\tilde{A}$ is written as $\tilde{A}_{GTFN}^L = (a_1, a_2; \omega)$

![Diagram of Left GTFN]

Figure 6: Left GTFN

5) Right GTFN if $\tilde{A}$ is written as $\tilde{A}_{GTFN}^R = (a_2, a_3; \omega)$

![Diagram of Right GTFN]

Figure 7: Right GTFN

**Definition 2.10. Equality of two GTFN:** Two fuzzy number $\tilde{A} = (a_1, a_2, a_3; \omega_1)$ and $\tilde{B} = (b_1, b_2, b_3; \omega_2)$ are equal when $a_1 = b_1, a_2 = b_2, a_3 = b_3$ and $\omega_1 = \omega_2$.

### 3 Solution of system of differential equation by De Alemberts Method

Consider the system of first order differential equation

\[
\frac{dx}{dt} = a_1 x + b_1 y + f_1(t) \\
\frac{dy}{dt} = a_2 x + b_2 y + f_2(t)
\]  \hspace{1cm} (3.1)

Multiplying the second equation by some constant $\lambda$ and add termwise to the first equation we get

\[
\frac{d(x + \lambda y)}{dt} = (a_1 + \lambda a_2) x + (b_1 + \lambda b_2) y + f_1(t) + \lambda f_2(t)
\]

Or, \[
\frac{d(x + \lambda y)}{dt} = (a_1 + \lambda a_2) \left(x + \frac{b_1 + \lambda b_2}{a_1 + \lambda a_2} y\right) + f_1(t) + \lambda f_2(t)
\]  \hspace{1cm} (3.2)

Chose the number $\lambda$ so that \[
\frac{b_1 + \lambda b_2}{a_1 + \lambda a_2} = \lambda
\]  \hspace{1cm} (3.3)

Then (3.2) reduces to an equation linear in $x + \lambda y$

\[
\frac{d(x + \lambda y)}{dt} = (a_1 + \lambda a_2) (x + \lambda y) + f_1(t) + \lambda f_2(t)
\]

Which on integrating gives \[
x + \lambda y = e^{(a_1 + \lambda a_2) t} \left[C + \int [f_1(t) + \lambda f_2(t)] e^{-(a_1 + \lambda a_2) t} dt\right]
\]  \hspace{1cm} (3.4)
If equation (3.2) has distinct real roots \( \lambda_1 \) and \( \lambda_2 \), then we obtain two first integrals of system (3.1) from (3.4) and so the integration of the system will be completed.

4 Solution Procedure of 1st Order Linear Non Homogeneous FODE

The solution procedure of 1st order linear non homogeneous FODE of Type-I, Type-II and Type-III are described. Here fuzzy numbers are taken as GTFNs.

4.1. Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-I

Consider the initial value problem \( \frac{dx}{dt} = Kx + \epsilon \) \hspace{1cm} (4.5)

with Fuzzy Initial Condition (FIC) \( \tilde{x}(t_0) = \tilde{y}_0 = (y_1, y_2, y_3; \omega) \)

Let \( \tilde{x}(t) \) be a solution of FODE (4.5).

Let \( x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)] \) be the \( \alpha \)-cut of \( \tilde{x}(t) \)

and \( (\tilde{y}_0)_\alpha = [x_1(t_0, \alpha), x_2(t_0, \alpha)] = \left[ y_1 + \frac{a_1y_0}{\omega}, y_3 - \frac{a_2y_0}{\omega} \right] \) \( \forall \alpha \in [0, \omega] \), \( 0 < \omega \leq 1 \)

where \( y_0 = y_2 - y_1 \) and \( r_0 = y_3 - y_2 \)

Here we solve the given problem for \( k > 0 \) and \( k < 0 \) respectively.

Case 4.1.1. When \( k > 0 \)

The FODE (4.5) becomes a system of linear ODE

\[ \frac{dx_i(t, \infty)}{dt} = kx_i(t, \alpha) + \epsilon \] \hspace{1cm} (4.6)

with initial condition \( x_1(t_0, \alpha) = y_1 + \frac{a_1y_0}{\omega} \) and \( x_2(t_0, \alpha) = y_3 - \frac{a_2y_0}{\omega} \)

The solution of (4.6) is

\[ x_1(t, \alpha) = -\frac{\epsilon}{k} + \left( \frac{\epsilon}{k} + \left( y_1 + \frac{a_1y_0}{\omega} \right) \right) e^{k(t-t_0)} \] \hspace{1cm} (4.7)

and \[ x_2(t, \alpha) = -\frac{\epsilon}{k} + \left( \frac{\epsilon}{k} + \left( y_3 - \frac{a_2y_0}{\omega} \right) \right) e^{k(t-t_0)} \] \hspace{1cm} (4.8)

Now \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] = \frac{r_0}{\omega} e^{k(t-t_0)} > 0, \quad \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] = -\frac{r_0}{\omega} e^{k(t-t_0)} < 0 \)

and \[ x_1(t, \omega) = -\frac{\epsilon}{k} + \left( \frac{\epsilon}{k} + \frac{y_1}{\omega} \right) e^{k(t-t_0)} = x_2(t, \omega) \).

So the solution of FODE (4.5) is a generalized fuzzy number \( \tilde{x} \). The \( \alpha \)-cut of the solution is

\[ x(t, \alpha) = -\frac{\epsilon}{k} + \left( \frac{\epsilon}{k} + \left( y_1 + \frac{a_1y_0}{\omega}, y_3 - \frac{a_2y_0}{\omega} \right) \right) e^{k(t-t_0)} \]

Case 4.1.2. when \( k < 0 \)

Let \( k = -m \) where \( m \) is a positive real number.

Then the FODE (4.5) becomes a system of ODE as follows

\[ \frac{dx_1(t, \alpha)}{dt} = -mx_2(t, \alpha) + \epsilon \] \hspace{1cm} (4.9)

\[ \frac{dx_2(t, \alpha)}{dt} = -mx_1(t, \alpha) + \epsilon \] \hspace{1cm} (4.10)

with initial condition \( x_1(t_0, \alpha) = y_1 + \frac{a_1y_0}{\omega} \) and \( x_2(t_0, \alpha) = y_3 - \frac{a_2y_0}{\omega} \).

Multiplying equation (4.10) by \( \lambda \) and then adding with equation (4.9) we get

\[ \frac{d}{dt} \left[ x_1(t, \alpha) + \lambda x_2(t, \alpha) \right] = -m[x_2(t, \alpha) + \lambda x_1(t, \alpha)] + [\epsilon + \lambda \epsilon] \]

\[ = -m \lambda \left[ x_1(t, \alpha) + \frac{1}{\lambda} x_2(t, \alpha) \right] + (1 + \lambda) \epsilon \]

Chose the number \( \lambda = \frac{1}{\lambda} \) or, \( \lambda = \pm 1 \) \hspace{1cm} (4.11)

\[ = -m \lambda \left[ x_1(t, \alpha) + \frac{1}{\lambda} x_2(t, \alpha) \right] + (1 + \lambda) \epsilon \] \hspace{1cm} (4.12)
The solution of (4.11) is

\[ x_1(t, \alpha) + \lambda x_2(t, \alpha) = e^{-mt} \left\{ C + \int (1 + \lambda) \varepsilon e^{mt} dt \right\} \]

\[ = C e^{-mt} + \frac{(1 + \lambda)}{m} \varepsilon \] (4.13)

Therefore,

\[ x_1(t, \alpha) + x_2(t, \alpha) = c_1 e^{-mt} + \frac{2}{m} \varepsilon \] (4.14)

and

\[ x_1(t, \alpha) - x_2(t, \alpha) = c_2 e^{mt} \] (4.15)

From (4.14) and (4.15) we get

\[ x_1(t, \alpha) = \frac{1}{2} \left\{ c_1 e^{-mt} + c_2 e^{mt} \right\} + \frac{\varepsilon}{m} \] (4.16)

and

\[ x_2(t, \alpha) = \frac{1}{2} \left\{ c_1 e^{-mt} - c_2 e^{mt} \right\} + \frac{\varepsilon}{m} \] (4.17)

Using initial condition from equation (4.16) and (4.17) we get

\[ c_1 = \left\{ \gamma_1 + \gamma_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) - \frac{2 \varepsilon}{m} \right\} e^{m \lambda t_0} \quad \text{and} \quad c_2 = \left( \frac{\alpha}{\omega} - 1 \right) (l_{\gamma_0} + r_{\gamma_0}) e^{-m \lambda t_0} \]

Therefore from (4.16) and (4.17) the solution is

\[ x_1(t, \alpha) = \frac{\varepsilon}{m} + \frac{1}{2} \left( - \frac{2 \varepsilon}{m} + \gamma_1 + \gamma_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) (l_{\gamma_0} + r_{\gamma_0}) e^{m(t-t_0)} \]

And

\[ x_2(t, \alpha) = \frac{\varepsilon}{m} + \left( - \frac{\varepsilon}{m} + \gamma_2 \right) e^{-m(t-t_0)} = x_2(t, \omega) \]

Here

\[ \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] = \frac{1}{2 \omega} (l_{\gamma_0} - r_{\gamma_0}) e^{-m(t-t_0)} + \frac{1}{2 \omega} (l_{\gamma_0} + r_{\gamma_0}) e^{m(t-t_0)} \]

\[ \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] = - \frac{1}{2 \omega} (l_{\gamma_0} + r_{\gamma_0}) e^{m(t-t_0)} - \frac{1}{2 \omega} (l_{\gamma_0} + r_{\gamma_0}) e^{-m(t-t_0)} \]

and

\[ x_1(t, \omega) = \frac{\varepsilon}{m} + \left( - \frac{\varepsilon}{m} + \gamma_2 \right) e^{-m(t-t_0)} = x_2(t, \omega) \]

Here three cases arise.

**Case 1:** When \( l_{\gamma_0} = r_{\gamma_0} \) i.e., \( \overline{\gamma}_0 = (\gamma_1, \gamma_2, \gamma_3; \omega) \) is a symmetric GTFN

\[ \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] = \frac{1}{2 \omega} (l_{\gamma_0} + r_{\gamma_0}) e^{m(t-t_0)} > 0 \quad \text{and} \quad x_1(t, \omega) = x_2(t, \omega) \]

So the solution of the FODE (4.5) is a strong solution.

**Case 2:** When \( l_{\gamma_0} < r_{\gamma_0} \) i.e., \( \overline{\gamma}_0 = (\gamma_1, \gamma_2, \gamma_3; \omega) \) is a non symmetric GTFN

Here \( \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] < 0 \) and \( x_1(t, \omega) = x_2(t, \omega) \)

but \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] > 0 \) implies \( t > t_0 + \frac{1}{2m} \log \left[ \frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}} \right] \).

So the solution of the FODE (4.5) is a strong solution if \( t > t_0 + \frac{1}{2m} \log \left[ \frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}} \right] \).

**Case 3:** When \( l_{\gamma_0} > r_{\gamma_0} \) i.e., \( \overline{\gamma}_0 = (\gamma_1, \gamma_2, \gamma_3; \omega) \) is a non symmetric GTFN

Here \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] < 0 \) and \( x_1(t, \omega) = x_2(t, \omega) \)

but \( \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] < 0 \) implies \( t > t_0 + \frac{1}{2m} \log \left[ \frac{r_{\gamma_0} - l_{\gamma_0}}{l_{\gamma_0} + r_{\gamma_0}} \right] \).
So the solution of the FODE (4.5) is a strong solution if \( t > t_0 + \frac{1}{2m} \log \frac{\ln r_0 - \ln r_0}{\ln r_0 + \ln r_0} \).  

### 4.2. Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-II

Consider the initial value problem \( \frac{dx}{dt} = \tilde{k}x + \epsilon \)  \hspace{1cm} (4.18)  
with IC \( x(t_0) = \gamma \) and \( \tilde{k} \) is a GTFN.

Let \( \tilde{x}(t) \) be the solution of FODE (4.18)

Let \( x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)] \) be the \( \alpha \)-cut of the solution.

Here we solve the given problem for \( \tilde{k} > 0 \) and \( \tilde{k} < 0 \) respectively.

**Case 4.2.1: when \( \tilde{k} > 0 \):** Let \( \tilde{k} = (\beta_1, \beta_2, \beta_3; \lambda) \)

The \( \alpha \)-cut of \( \tilde{k} \) be

\[
(\tilde{k})_\alpha = [k_1(\alpha), k_2(\alpha)] = \left[ \beta_1 + \frac{\alpha l_k}{\lambda}, \beta_3 - \frac{\alpha r_k}{\lambda} \right] \quad \forall \alpha \in [0, \lambda], \quad 0 < \lambda \leq 1
\]

where \( l_k = \beta_2 - \beta_1 \) and \( r_k = \beta_3 - \beta_2 \).

The FODE (4.18) becomes a system of linear ODE

\[
\frac{dx_i(t, \alpha)}{dt} = k_i(\alpha)x_i(t, \alpha) + \epsilon \quad \text{for } i = 1, 2
\]

with IC \( x(t_0) = \gamma \).

The solution of (4.19)

\[
x_1(t, \alpha) = -\frac{\epsilon}{(\beta_1 + \frac{a t_k}{\lambda})} + \left\{ \gamma + \frac{\epsilon}{(\beta_1 + \frac{a t_k}{\lambda})} \right\} e^{(\beta_1 + \frac{a t_k}{\lambda})(t - t_0)}
\]

and

\[
x_2(t, \alpha) = -\frac{\epsilon}{(\beta_3 - \frac{a r_k}{\lambda})} + \left\{ \gamma + \frac{\epsilon}{(\beta_3 - \frac{a r_k}{\lambda})} \right\} e^{(\beta_3 - \frac{a r_k}{\lambda})(t - t_0)}.
\]

**Case 4.2.2: when \( \tilde{k} < 0 \):** Let \( \tilde{k} = -\tilde{m} \), where \( \tilde{m} = (\beta_1, \beta_2, \beta_3; \lambda) \) is a positive GTFN.

So \( (\tilde{m})_\alpha = [m_1(\alpha), m_2(\alpha)] = \left[ \beta_1 + \frac{a t_m}{\lambda}, \beta_3 - \frac{a r_m}{\lambda} \right] \forall \alpha \in [0, \lambda], 0 < \lambda \leq 1 \)

where \( t_m = \beta_2 - \beta_1 \) and \( r_m = \beta_3 - \beta_2 \).

Then the FODE (4.18) becomes a system of ODE as follows

\[
\frac{dx_1(t, \alpha)}{dt} = -m_2(\alpha)x_2(t, \alpha) + \epsilon
\]

\[
\frac{dx_2(t, \alpha)}{dt} = -m_1(\alpha)x_1(t, \alpha) + \epsilon
\]

with IC \( x(t_0) = \gamma \).

Multiplying equation (4.21) by \( \lambda \) and then adding with equation (4.20) we get

\[
\frac{d}{dt}[x_1(t, \alpha) + \lambda x_2(t, \alpha)] = -[m_2(\alpha)x_2(t, \alpha) + \lambda m_1(\alpha)x_1(t, \alpha)] + [\epsilon + \lambda \epsilon]
\]

\[
= -\lambda m_1(\alpha)[x_1(t, \alpha) + \frac{m_2(\alpha)}{\lambda m_1(\alpha)}x_2(t, \alpha)] + (1 + \lambda)\epsilon
\]

Chose the number \( \lambda = \frac{m_2(\alpha)}{\lambda m_1(\alpha)} \) or, \( \lambda = \pm \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} \)

(4.23)

The solution of (3) is

\[
x_1(t, \alpha) + \lambda x_2(t, \alpha) = e^{\lambda m_1(\alpha)\lambda}[C + \int_0^T (1 + \lambda)\epsilon e^{\lambda m_1(\alpha)t}dt]
\]

\[
= C e^{\lambda m_1(\alpha)\lambda} + \frac{(1+\lambda)}{\lambda m_1(\alpha)}\epsilon
\]

Therefore,
\[ x_1(t, \alpha) + \frac{m_2(\alpha)}{m_1(\alpha)} x_2(t, \alpha) = c_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t} + \left( \frac{1}{\sqrt{m_1(\alpha)m_2(\alpha)}} + \frac{1}{m_1(\alpha)} \right) \varepsilon \]

and

\[ x_1(t, \alpha) - \frac{m_2(\alpha)}{m_1(\alpha)} x_2(t, \alpha) = c_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}t} - \left( \frac{1}{\sqrt{m_1(\alpha)m_2(\alpha)}} - \frac{1}{m_1(\alpha)} \right) \varepsilon \]

Solving (4.25) and (4.26) we get

\[ x_1(t, \alpha) = \frac{1}{2} \left\{ c_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t} + c_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}t} \right\} + \frac{\varepsilon}{m_1(\alpha)} \]

and

\[ x_2(t, \alpha) = \frac{1}{2} \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}} \left\{ c_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t} - c_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}t} \right\} + \frac{\varepsilon}{m_2(\alpha)} \]

Now using initial condition in (4.27) and (4.28) we get

\[ c_1 = \left\{ \gamma \left( \frac{1}{\sqrt{m_1(\alpha)m_2(\alpha)}} - \frac{1}{m_1(\alpha)} \right) \right\} e^{\sqrt{m_1(\alpha)m_2(\alpha)}t_0} \]

and

\[ c_2 = \left\{ \gamma \left( 1 - \frac{m_2(\alpha)}{m_1(\alpha)} + \left( \frac{1}{\sqrt{m_1(\alpha)m_2(\alpha)}} - \frac{1}{m_1(\alpha)} \right) \right) e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t_0} \]

Putting this value in (4.27) and (4.28) we get

\[ x_1(t, \alpha) = \frac{1}{2} \left\{ \gamma \left( 1 - \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}} \right) - \varepsilon \left( \frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} - \frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} \sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})} \right) \right\} e^{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}(t-t_0)} \]

\[ + \frac{1}{2} \left\{ \gamma \left( 1 + \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}} \right) - \varepsilon \left( \frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} + \frac{1}{\beta_1 + \frac{\alpha l_m}{\lambda}} \sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})} \right) \right\} e^{-\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}(t-t_0)} \]

and

\[ x_2(t, \alpha) = -\frac{1}{2} \sqrt{\frac{\beta_1 + \frac{\alpha l_m}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}} \left\{ \gamma \left( 1 - \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}} \right) \right\} e^{\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}(t-t_0)} \]

\[ + \frac{1}{2} \sqrt{\frac{\beta_1 + \frac{\alpha l_m}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}} \left\{ \gamma \left( 1 + \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha l_m}{\lambda}} \right) \right\} e^{-\sqrt{(\beta_1 + \frac{\alpha l_m}{\lambda})(\beta_3 - \frac{\alpha r_m}{\lambda})}(t-t_0)} \]

\[ + \frac{\varepsilon}{\beta_1 + \frac{\alpha l_m}{\lambda}} \]
4.3. Solution Procedure of 1st Order Linear Non Homogeneous FODE of Type-III

Consider the initial value problem \( \frac{dx}{dt} = Rx + \epsilon \) (4.29)

With fuzzy IC \( \tilde{x}(t_0) = \tilde{y}_0 = (y_1, y_2, y_3; \omega) \) and \( \tilde{R} \) is a GTFN.

Let \( \tilde{x}(t) \) be the solution of FODE (4.29).

Let \( x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)] \) be the \( \alpha \)-cut of the solution.

\[
(\tilde{y}_0)_\alpha = [x_1(t_0, \alpha), x_2(t_0, \alpha)] = \left[ y_1 + \frac{\alpha y_0}{\omega}, y_3 - \frac{\alpha y_0}{\omega} \right] \quad \forall \ \alpha \in [0, \omega], \quad 0 < \omega \leq 1
\]

where \( l_y = y_2 - y_1 \) and \( r_y = y_3 - y_2 \)

Let \( \eta = \min(\lambda, \omega) \)

Here we solve the given problem for \( \tilde{R} > 0 \) and \( \tilde{R} < 0 \) respectively.

**Case I: when \( \tilde{R} > 0 \):** Let \( \tilde{R} = (\beta_1, \beta_2, \beta_3; \lambda) \)

The \( \alpha \)-cut of \( \tilde{R} \) be

\[
(\tilde{R})_\alpha = [k_1(\alpha), k_2(\alpha)] = \left[ \beta_1 + \frac{\alpha k_1}{\lambda}, \beta_3 - \frac{\alpha k_2}{\lambda} \right] \quad \forall \ \alpha \in [0, \lambda], \quad 0 < \lambda \leq 1
\]

where \( k_1 = \beta_2 - \beta_1 \) and \( k_2 = \beta_3 - \beta_2 \).

The FODE (4.29) becomes a system of linear ODE

\[
\frac{dx(t, \alpha)}{dt} = k_1 x_1(t, \alpha) + \epsilon \quad \text{for } i = 1, 2
\]

with initial condition \( x_1(t_0, \alpha) = y_1 + \frac{\alpha y_0}{\omega} \) and \( x_2(t_0, \alpha) = y_3 - \frac{\alpha y_0}{\omega} \)

Therefore the solution is of (3.3.1)

\[
x_1(t, \alpha) = -\frac{\epsilon}{(\beta_1 + \frac{\alpha k_1}{\lambda})} + \left[ y_1 + \frac{\alpha y_0}{\omega} \right] e^{(\beta_1 + \frac{\alpha k_1}{\lambda})(t-t_0)}
\]

And

\[
x_2(t, \alpha) = -\frac{\epsilon}{(\beta_3 - \frac{\alpha k_2}{\lambda})} + \left[ y_3 - \frac{\alpha y_0}{\omega} \right] e^{(\beta_3 - \frac{\alpha k_2}{\lambda})(t-t_0)}
\]

**Case II: when \( \tilde{R} < 0 \):** Let \( \tilde{R} = -\tilde{m} \) where \( \tilde{m} = (\beta_1, \beta_2, \beta_3; \lambda) \) is a positive GTFN.

Then \( (\tilde{m})_\alpha = \left[ \beta_1 + \frac{\alpha m_1}{\lambda}, \beta_3 - \frac{\alpha m_2}{\lambda} \right] \quad \forall \ \alpha \in [0, \lambda], \quad 0 < \lambda \leq 1
\]

Let \( \eta = \min(\lambda, \omega) \)

Then the FODE (4.29) becomes a system of ODE as follows

\[
\frac{dx_1(t, \alpha)}{dt} = -m_2(\alpha) x_2(t, \alpha) + \epsilon \quad (4.31)
\]

\[
\frac{dx_2(t, \alpha)}{dt} = -m_1(\alpha) x_1(t, \alpha) + \epsilon \quad (4.32)
\]

with IC \( x_1(t_0, \alpha) = y_1 + \frac{\alpha y_0}{\omega} \) and \( x_2(t_0, \alpha) = y_3 - \frac{\alpha y_0}{\omega} \).

Multiplying equation (4.32) by \( \lambda \) and then adding with equation (4.31) we get

\[
\frac{d}{dt} [x_1(t, \alpha) + \lambda x_2(t, \alpha)] = -[m_2(\alpha) x_2(t, \alpha) + \lambda m_1(\alpha) x_1(t, \alpha)] + [\epsilon + \lambda \epsilon]
\]

\[
= -\lambda m_1(\alpha) [x_1(t, \alpha) + \frac{m_2(\alpha)}{\lambda m_1(\alpha)} x_2(t, \alpha)] + (1 + \lambda) \epsilon
\]

Chose the number \( \lambda = \frac{m_2(\alpha)}{\lambda m_1(\alpha)} \) or, \( \lambda = \pm \frac{m_2(\alpha)}{m_1(\alpha)} \)

The solution of (4.33) is

\[
x_1(t, \alpha) + \lambda x_2(t, \alpha) = e^{-\lambda m_1(\alpha) t} \left[ C + \int (1 + \lambda) \epsilon e^{\lambda m_1(\alpha) t \lambda} dt \right]
\]
\[ C e^{-\frac{\lambda m_2(\alpha)t}{\lambda m_1(\alpha)}} + \frac{(1+\lambda)}{\lambda m_2(\alpha)} \epsilon \]

Therefore,

\[ x_1(t, \alpha) + \frac{m_2(\alpha)}{m_1(\alpha)} \epsilon x_2(t, \alpha) = c_1 e^{-\frac{\sqrt{m_1(\alpha)m_2(\alpha)}t}{\sqrt{m_1(\alpha)m_2(\alpha)}}} + \left( \frac{1}{\sqrt{m_1(\alpha) m_2(\alpha)}} + \frac{1}{m_1(\alpha)} \right) \epsilon \]

and

\[ x_1(t, \alpha) - \frac{m_2(\alpha)}{m_1(\alpha)} \epsilon x_2(t, \alpha) = c_2 e^{\frac{\sqrt{m_1(\alpha)m_2(\alpha)}t}{\sqrt{m_1(\alpha)m_2(\alpha)}}} - \left( \frac{1}{\sqrt{m_1(\alpha) m_2(\alpha)}} - \frac{1}{m_1(\alpha)} \right) \epsilon \]

Solving (4.36) and (4.37) we get

\[ x_1(t, \alpha) = \frac{1}{2} \{ c_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t} + c_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}t} \} + \frac{\epsilon}{m_1(\alpha)} \]

and

\[ x_2(t, \alpha) = \frac{1}{2} \{ c_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t} - c_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}t} \} + \frac{\epsilon}{m_2(\alpha)} \]

Using initial condition we get

\[ c_1 = \left\{ y_1 + \frac{\alpha l_0}{\omega} + \frac{m_2(\alpha)}{m_1(\alpha)} (y_3 - \frac{\alpha r_0}{\omega}) - \left( \frac{1}{\sqrt{m_1(\alpha) m_2(\alpha)}} + \frac{1}{m_1(\alpha)} \right) \epsilon \right\} e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t_0} \]

and

\[ c_2 = \left\{ y_1 + \frac{\alpha l_0}{\omega} - \frac{m_2(\alpha)}{m_1(\alpha)} (y_3 - \frac{\alpha r_0}{\omega}) + \left( \frac{1}{\sqrt{m_1(\alpha) m_2(\alpha)}} - \frac{1}{m_1(\alpha)} \right) \epsilon \right\} e^{\sqrt{m_1(\alpha)m_2(\alpha)}t_0} \]

Then from (4.38) and (4.39) we have

\[ x_1(t, \alpha) = \frac{1}{2} \left\{ y_1 + \frac{\alpha l_0}{\eta} - \frac{\beta_3 - \frac{\alpha r_m}{\eta}}{\sqrt{\beta_1 + \frac{\alpha l_m}{\eta}}} (y_3 - \frac{\alpha r_0}{\eta}) \right\} - \left( \frac{1}{\beta_1 + \frac{\alpha l_m}{\eta}} - \frac{1}{\beta_1 + \frac{\alpha l_m}{\eta}} \right) \epsilon \right\} e^{-\sqrt{\beta_1 + \frac{\alpha l_m}{\eta}} (y_3 - \frac{\alpha r_0}{\eta})(t - t_0)} \]

\[ + \frac{1}{2} \left\{ y_1 + \frac{\alpha l_0}{\eta} + \frac{\beta_3 - \frac{\alpha r_m}{\eta}}{\sqrt{\beta_1 + \frac{\alpha l_m}{\eta}}} (y_3 - \frac{\alpha r_0}{\omega}) \right\} - \left( \frac{1}{\beta_1 + \frac{\alpha l_m}{\eta}} + \frac{1}{\beta_1 + \frac{\alpha l_m}{\eta}} \right) \epsilon \right\} e^{-\sqrt{\beta_1 + \frac{\alpha l_m}{\eta}} (y_3 - \frac{\alpha r_0}{\eta})(t - t_0)} \]

\[ + \frac{\epsilon}{\beta_1 + \frac{\alpha l_m}{\eta}} \]
and
\[ x_2(t, \alpha) = \left\{ \begin{array}{ll}
-\frac{1}{2} \left( \frac{\beta_1 + \alpha l m}{\eta} \right) \left( \left( y_1 + \frac{\alpha l v_0}{\eta} \right) - \left( \frac{\beta_3 - \alpha r m}{\eta} \right) \left( y_3 - \frac{\alpha r v_2}{\eta} \right) \right) \\
-\frac{1}{2} \left( \frac{\beta_1 + \alpha l m}{\eta} \right) \left( \left( y_1 + \frac{\alpha l v_0}{\eta} \right) + \left( \frac{\beta_3 - \alpha r m}{\eta} \right) \left( y_3 - \frac{\alpha r v_2}{\eta} \right) \right) \\
+ \frac{1}{2} \left( \frac{\beta_1 + \alpha l m}{\eta} \right) \left( \left( y_1 + \frac{\alpha l v_0}{\eta} \right) + \left( \frac{\beta_3 - \alpha r m}{\eta} \right) \left( y_3 - \frac{\alpha r v_2}{\eta} \right) \right) \\
\end{array} \right. \]

\[ + \frac{e}{\left( \beta_3 - \frac{\alpha r m}{\eta} \right)} \]

5 Application

The population of fish in a pond is modeled by the differential equation \( \frac{dN}{dt} = P - kN \), where \( t \) is measured in years and \( P \) is a constant. If initially \( N(0) = N_0 \) number of fish in the pond then solve the above problem when

(i) \( \tilde{N}_0 = (4900, 5000, 5050; 0.8), k = 0.04 \) and \( P = 2000 \)

(ii) \( N_0 = 5000, \tilde{k} = (0.02, 0.03, 0.04, 0.06; 0.7) \) and \( P = 2000 \)

(iii) \( \tilde{N}_0 = (4850, 5000, 5200; 0.8), \tilde{k} = (0.02, 0.04, 0.05; 0.7) \) and \( P = 2000 \)

Solution: (i) Here \( (\tilde{N}_0)_{\alpha} = [4900 + 125\alpha, 5000 - 62.5\alpha] \).

Therefore solution of the model

\[ N_1(t, \alpha) = 50000 + (-45025 + 31.25\alpha)e^{-0.04t} + 75(1.25\alpha - 1)e^{0.04t} \]

and \( N_2(t, \alpha) = 50000 + (-45025 + 31.25\alpha)e^{-0.046t} + 75(1.25\alpha - 1)e^{0.04t} \)

| Table 1: Value of \( N_1(t, \alpha) \) and \( N_2(t, \alpha) \) for different \( \alpha \) and \( t=48 \) years |
|---|---|---|
| \( \alpha \) | \( N_1(t, \alpha) \) | \( N_2(t, \alpha) \) |
| 0 | 42887.4496 | 43910.5934 |
| 0.1 | 42951.8543 | 43847.1051 |
| 0.2 | 43016.2589 | 43783.6167 |
| 0.3 | 43080.6635 | 43720.1284 |
| 0.4 | 43145.0682 | 43656.6401 |
| 0.5 | 43209.4728 | 43593.1517 |
| 0.6 | 43273.8774 | 43529.6634 |
| 0.7 | 43338.2821 | 43466.1750 |
| 0.8 | 43402.6867 | 43402.6867 |
From above Table-1 we see that for this particular value of t=48 years, \( N_1(t, \alpha) \) is an increasing function, \( N_2(t, \alpha) \) is a decreasing function and \( N_1(t, 0.8) = N_2(t, 0.8) = 119.9997 \). So Solution of above model for particular value of t is a strong solution.

(ii) Here \( \dot{k}(\alpha) = [0.03 + 0.0142\alpha, 0.06 - 0.0285\alpha] \).

Therefore solution of the model is

\[
N_1(t, \alpha) = \frac{1}{2} \left[ 5000 \left( 1 - \frac{0.06 - 0.0285\alpha}{\sqrt{0.03 + 0.0142\alpha}} \right) - \frac{1}{\sqrt{(0.03 + 0.0142\alpha)(0.06 - 0.0285\alpha)}} \right] e^{\sqrt{(0.03 + 0.0142\alpha)(0.06 - 0.0285\alpha)}t} + \frac{2000}{0.03 + 0.0142\alpha} \]

and

\[
N_2(t, \alpha) = -\frac{1}{2} \left[ \frac{0.03 + 0.0142\alpha}{0.06 - 0.0285\alpha} \right] 5000 \left( 1 - \frac{0.06 - 0.0285\alpha}{\sqrt{0.03 + 0.0142\alpha}} \right) - \frac{1}{\sqrt{(0.03 + 0.0142\alpha)(0.06 - 0.0285\alpha)}} \left( \frac{1}{0.03 + 0.0142\alpha} - \frac{0.03 + 0.0142\alpha}{0.06 - 0.0285\alpha} \right) \right] e^{\sqrt{(0.03 + 0.0142\alpha)(0.06 - 0.0285\alpha)}t} + \frac{1}{2} \frac{0.03 + 0.0142\alpha}{0.06 - 0.0285\alpha} \left[ 5000 \left( 1 + \frac{1}{0.03 + 0.0142\alpha} + \frac{1}{\sqrt{(0.03 + 0.0142\alpha)(0.06 - 0.0285\alpha)}} \right) \right] e^{-\sqrt{(0.03 + 0.0142\alpha)(0.06 - 0.0285\alpha)}t} + \frac{2000}{0.06 - 0.0285\alpha}.

<table>
<thead>
<tr>
<th>\alpha</th>
<th>( N_1(t, \alpha) )</th>
<th>( N_2(t, \alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5884.6181</td>
<td>60694.2105</td>
</tr>
<tr>
<td>0.1</td>
<td>11039.2047</td>
<td>61362.5986</td>
</tr>
<tr>
<td>0.2</td>
<td>16086.4565</td>
<td>61552.5764</td>
</tr>
<tr>
<td>0.3</td>
<td>21012.4065</td>
<td>61376.2941</td>
</tr>
<tr>
<td>0.4</td>
<td>25803.6025</td>
<td>60914.7645</td>
</tr>
<tr>
<td>0.5</td>
<td>30447.1563</td>
<td>60231.0977</td>
</tr>
<tr>
<td>0.6</td>
<td>34930.7908</td>
<td>59378.3611</td>
</tr>
<tr>
<td>0.7</td>
<td>39242.8848</td>
<td>58404.7747</td>
</tr>
</tbody>
</table>

From above Table-2 we see that for this particular value of t=36 years, \( N_1(t, \alpha) \) is an increasing function, \( N_2(t, \alpha) \) is a decreasing function and \( N_1(t, 0.7) < N_2(t, 0.7) \). So Solution of above model for particular value of t is a strong solution.
(iii) Here

\((\hat{k})_\alpha = [0.02 + 0.0285\alpha, 0.05 - 0.0142\alpha]\) and \((\hat{N}_0)_\alpha = [4850 + 214.28\alpha, 5200 - 285.71\alpha]\)

Therefore solution of the model is

\[
N_1(t, \alpha) = \frac{1}{2} \left[ \left( (4850 + 214.28\alpha) - \sqrt{\frac{0.05 - 0.0142\alpha}{0.02 + 0.0285\alpha}} (5200 - 285.71\alpha) \right) - 2000 \left( \frac{1}{0.02 + 0.0285\alpha} - \frac{1}{\sqrt{(0.02 + 0.0285\alpha)(0.05 - 0.0142\alpha)}} \right) e^{\sqrt{(0.02 + 0.0285\alpha)(0.05 - 0.0142\alpha)} t} + \left( (4850 + 214.28\alpha) + \sqrt{\frac{0.05 - 0.0142\alpha}{0.02 + 0.0285\alpha}} (5200 - 285.71\alpha) \right) - 2000 \left( \frac{1}{0.02 + 0.0285\alpha} + \frac{1}{\sqrt{(0.02 + 0.0285\alpha)(0.05 - 0.0142\alpha)}} \right) e^{-\sqrt{(0.02 + 0.0285\alpha)(0.05 - 0.0142\alpha)} t} + \frac{2000}{0.02 + 0.0285\alpha} \right] 
\]

and

\[
N_2(t, \alpha) = \frac{1}{2} \sqrt{\frac{0.02 + 0.0285\alpha}{0.05 - 0.0142\alpha}} \left[ \left( (4850 + 214.28\alpha) - \sqrt{\frac{0.05 - 0.0142\alpha}{0.02 + 0.0285\alpha}} (5200 - 285.71\alpha) \right) - 2000 \left( \frac{1}{0.02 + 0.0285\alpha} \right) - \frac{1}{\sqrt{(0.02 + 0.0285\alpha)(0.05 - 0.0142\alpha)}} e^{\sqrt{(0.02 + 0.0285\alpha)(0.05 - 0.0142\alpha)} t} \right] + \left( (4850 + 214.28\alpha) + \sqrt{\frac{0.05 - 0.0142\alpha}{0.02 + 0.0285\alpha}} (5200 - 285.71\alpha) \right) - 2000 \left( \frac{1}{0.02 + 0.0285\alpha} \right) + \frac{1}{\sqrt{(0.02 + 0.0285\alpha)(0.05 - 0.0142\alpha)}} e^{-\sqrt{(0.02 + 0.0285\alpha)(0.05 - 0.0142\alpha)} t} + \frac{2000}{0.05 - 0.0142\alpha} \right] 
\]
Table 3: Value of $N_1(t, \alpha)$ and $N_2(t, \alpha)$ for different $\alpha$ and $t=24$ years

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$N_1(t, \alpha)$</th>
<th>$N_2(t, \alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21991.9444</td>
<td>44871.7242</td>
</tr>
<tr>
<td>0.1</td>
<td>23596.6949</td>
<td>43352.4427</td>
</tr>
<tr>
<td>0.2</td>
<td>25181.7015</td>
<td>41761.8778</td>
</tr>
<tr>
<td>0.3</td>
<td>26744.5135</td>
<td>40101.8966</td>
</tr>
<tr>
<td>0.4</td>
<td>28282.7157</td>
<td>38374.5032</td>
</tr>
<tr>
<td>0.5</td>
<td>29793.9341</td>
<td>36581.8356</td>
</tr>
<tr>
<td>0.6</td>
<td>31275.8400</td>
<td>34726.1618</td>
</tr>
<tr>
<td>0.7</td>
<td>32726.1549</td>
<td>32809.8760</td>
</tr>
</tbody>
</table>

From above Table 3 we see that for this particular value of $t=24$ years, $N_1(t, \alpha)$ is an increasing function, $N_2(t, \alpha)$ is a decreasing function and $N_1(t, 0.7) < N_2(t, 0.7)$. So Solution of above model for particular value of $t$ is a strong solution.

6 Conclusion

In this paper we have solved first order linear non homogeneous ordinary differential equation in fuzzy environment. The fuzzy number is taken as generalized triangular fuzzy number. In this paper we discussed three different cases: Initial value as fuzzy number, initial value and coefficients as fuzzy number, coefficients are fuzzy number. The solution procedure is developed by Lagrange Multiplier Method. An imprecise fish population model is discussed. For further work the same problem can be solved by using Generalized Trapezoidal Fuzzy Number and Generalized L-R type Fuzzy Number. Also we can follow the same method for first order linear system of homogeneous and non-homogeneous system of ordinary differential equation. This process can be followed for any economical or bio-mathematical model and in engineering sciences.

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