Notes on “Types of arcs in a fuzzy graph”

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Abstract
In this note, we show by an example that the theorem stating all paths in a complete fuzzy graph are strongest paths, in the paper titled “Types of arcs in a fuzzy graph” by Mathew and Sunitha is not always true and we provide the correct version of the theorem.

Keywords: strong arc, complete fuzzy graph, strong path, strongest path.

1 Introduction
Fuzzy graphs were introduced by Rosenfeld [5] in 1975. Fuzzy graph theory has numerous applications in modern science and technology especially in the fields of information theory, neural networks, expert systems, cluster analysis, medical diagnosis, control theory, etc. A fuzzy graph [5] is a triplet \( G : (V, \sigma, \mu) \) where \( V \) is the vertex set, \( \sigma \) is a fuzzy subset on \( V \) and \( \mu \) is a fuzzy relation on \( \sigma \) such that \( \mu(u, v) \leq \sigma(u) \wedge \sigma(v) \) \( \forall u, v \in V \). We assume that \( V \) is finite and non empty, \( \mu \) is reflexive and symmetric. In all the examples \( \sigma \) is chosen suitably. Also we denote the underlying crisp graph by \( G^* : (\sigma^*, \mu^*) \) where \( \sigma^* = \{ u \in V : \sigma(u) > 0 \} \) and \( \mu^* = \{ (u, v) \in V \times V : \mu(u, v) > 0 \} \). Here take \( \sigma^* = V \). A fuzzy graph \( G : (V, \sigma, \mu) \) is a complete fuzzy graph (CFG) if \( \mu(u, v) = \sigma(u) \wedge \sigma(v) \) for every \( u, v \in \sigma^* \). A path \( P \) of length \( n \) is a sequence of distinct nodes \( v_0, v_1, ..., v_n \) such that \( \mu(v_{i-1}, v_i) > 0 \), \( i = 1, 2, 3, ..., n \) and the degree of membership of a weakest arc is defined as its strength.
The strength of connectedness between two nodes $u$ and $v$ is defined as the maximum of the strengths of all paths between $u$ and $v$ and is denoted by $\text{CONN}_G(u,v)$. A $u-v$ path $P$ is called a strongest $u-v$ path if its strength equals $\text{CONN}_G(u,v)$. A fuzzy graph $G : (V, \sigma, \mu)$ is connected if for every $u,v$ in $\sigma^*$, $\text{CONN}_G(u,v) > 0$. A weakest node of $G : (V, \sigma, \mu)$ is a node with least membership value. Throughout in this paper, we assume that $G$ is connected. An arc of a fuzzy graph is called strong if its weight is at least as great as the connectedness of its end nodes when it is deleted and a $u-v$ path is called a strongest $u-v$ path if its strength equals $\text{CONN}_G(u,v)$.

2 Preliminaries and notations

Depending on the $\text{CONN}_G(u,v)$ of an arc $(u,v)$ in a fuzzy graph $G$ the following different types of arcs are defined [6].

**Definition 2.1.** An arc $(u,v)$ in $G$ in called $\alpha-$strong if $\mu(u,v) > \text{CONN}_{G-\{(u,v)\}}(u,v)$.

**Definition 2.2.** An arc $(u,v)$ in $G$ in called $\beta-$strong if $\mu(u,v) = \text{CONN}_{G-\{(u,v)\}}(u,v)$.

**Definition 2.3.** An arc $(u,v)$ in $G$ in called a $\delta-$arc if $\mu(u,v) < \text{CONN}_{G-\{(u,v)\}}(u,v)$.

**Definition 2.4.** A $\delta-$arc $(u,v)$ is called a $\delta^*-$arc if $\mu(u,v) > \mu(x,y)$ where $(x,y)$ is a weakest arc of $G$.

![Figure 1: Graph showing different types of arcs and strongest strong paths](image-url)

**Example 2.1.** Here arcs $(v,x)$ and $(v,w)$ are $\alpha-$strong arcs, $(u,v)$ and $(u,x)$ are $\beta-$strong arcs, and $(w,u)$ and $(w,x)$ are $\delta-$arcs. Arc $(w,x)$ is a $\delta^*-$arc. The path $u-x-v-w$ is a strongest strong path. The path $u-x-w$ is a strongest path but not a strong path since the arc $(x,w)$ is a $\delta^*-$arc. In the above example if we replace $\mu(w,x) = 0.4$ then arc $(w,x)$ will be a $\beta-$strong arc. Then the path $v-x-w$ will be a strong path which is not strongest since $\text{CONN}_{G}(v,w) = 0.6$ and strength of path $v-x-w$ is 0.4.

3 Strongest path in a complete fuzzy graph

Note that in a CFG all arcs are strong and hence all paths are strong paths [2, 6]. Theorem 11 of [6] states that in a CFG without $\alpha-$strong arcs, all paths are strongest paths. The following example shows that this is not always true.

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Figure 2: A Complete Fuzzy Graph without $\alpha$-strong arcs

**Example 3.1.** Here all arcs are $\beta$-strong arcs. But all paths are not strongest paths. Note that $\text{CONN}_G(v, x) = 0.6$ and the arc $v \rightarrow x$ and the path $v \rightarrow w \rightarrow x$ are the only strongest $v \rightarrow x$ paths. All other strong $v \rightarrow x$ paths are not strongest.

Based on the above observation we have the following theorem.

**Theorem 3.1.** Let $G : (V, \sigma, \mu)$ be a complete fuzzy graph and let $P$ be any $u \rightarrow v$ path. Then $P$ is a strongest $u \rightarrow v$ path if and only if either $u$ or $v$ is a weakest node in the path.

**Proof.** Consider a CFG $G : (V, \sigma, \mu)$ and $P$ any $u \rightarrow v$ path for some $u, v \in \sigma^*$. Since $G$ is complete, then

$$\text{CONN}_G(u, v) = \mu(u, v).$$

Let $u$ be a weakest node in $P$. Then by definition of CFG, $\mu(u, v) = \sigma(u)$. Let $(x, y)$ be an arc in the path $P$. Then by definition, $\mu(x, y) = \sigma(x) \land \sigma(y)$. Since $\sigma(x) \geq \sigma(u)$ and $\sigma(y) \geq \sigma(u)$ we have, $\mu(x, y) \geq \mu(u, v)$.

Then by definition, we have Strength of $P = \mu(u, v)$. From Eq. (3.1), we see that Strength of $P = \text{CONN}_G(u, v)$. By definition of strongest path, $P$ is a strongest $u \rightarrow v$ path. Proof is similar if $v$ is a weakest node in $P$.

Conversely, let $P$ be a strongest $u \rightarrow v$ path in $G$. Assume that neither $u$ nor $v$ is a weakest node in the path $P$. Let $x$ be a weakest node in $P$. Consider an arc $(x, y)$ in $P$, where $y$ is a node in $P$. Then by definition, $\mu(x, y) = \sigma(x)$. Since $\sigma(x) < \sigma(u)$ and $\sigma(x) < \sigma(u)$ we have, $\mu(x, y) < \mu(u, v)$. Then by definition of strength of a path, Strength of $P = \mu(x, y)$. From Eq. (3.1) we get Strength of $P \neq \text{CONN}_G(u, v)$. By definition, $P$ is not a strongest path, which is a contradiction. Hence either $u$ or $v$ is a weakest node in the path $P$.

**Example 3.2.** Note that, in Fig. 3, $u$ is the weakest node of the CFG. Here $\text{CONN}_G(u, x) = 0.4$. The $u \rightarrow x$ paths are
1. $P_1 : u - x$
2. $P_2 : u - v - x$
3. $P_3 : u - w - x$
4. $P_4 : u - v - w - x$
5. $P_5 : u - w - v - x$

Strength of each of these paths is 0.4 and hence all are strongest paths. Now consider any other node $v$ of the CFG in Fig 3. Note that $\text{CONN}_G(v, w) = 0.6$. The $v - w$ paths are,

1. $P_1 : v - w$
2. $P_2 : v - x - w$
3. $P_3 : v - u - w$
4. $P_4 : v - u - x - w$
5. $P_5 : v - x - u - w$

Here only $P_1$ and $P_2$ have strength 0.6 each and both are strongest paths. The strength of each of $P_3, P_4$ and $P_5$ is 0.4 and hence $P_3, P_4$ and $P_5$ are not strongest paths. Note that nodes $v$ and $w$ are not weakest nodes in $P_3, P_4$ and $P_5$.

4 Conclusion

In this paper, we have shown by an example that all paths in a complete fuzzy graph are not always strongest paths and also we have proved the necessary and sufficient condition for a path in a CFG to be a strongest path.

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