A study on parametric form of fuzzy line

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Abstract

In this paper, two parametric forms of fuzzy lines are proposed. Characteristics of the proposed fuzzy lines are investigated. Distance between a fuzzy point and a fuzzy line, a fuzzy half plane and angle bisectors of two fuzzy lines are also studied. Sup-min composition of fuzzy sets and concepts of same and inverse points of fuzzy geometry are applied to define all the ideas. All the defined ideas are supported by suitable numerical and pictorial illustrations.

Keywords: Fuzzy geometry, fuzzy numbers, fuzzy point, same and inverse points, fuzzy line, fuzzy distance.

1 Introduction

In the analysis of fuzzy geometrical shapes, very recently, Ghosh and Chakraborty [4, 6] defined few fundamental ideas of fuzzy plane geometry using newly introduced concepts of same and inverse points. Prior to the work by Ghosh and Chakraborty, some fuzzy geometrical ideas have been given by Buckley and Eslami [1, 2]. Rosenfeld [10] reported an updated overview, up to the year of 1998, on fuzzy geometry and topology of image subsets. Rosenfeld [7] considered fuzzy line as a fuzzy set consisting of collinear crisp points with varied membership values, i.e., a fuzzy set is a fuzzy line when its support set is a straight line. Chaudhuri [3] defined fuzzy line as a fuzzy set for which any α-level set is either empty or a straight line for all α in (0, 1]. These definitions of fuzzy line may not be suitable in fuzzy plane geometry. Since, according to [7], fuzzy line may not be fuzzily connected and its α-level sets can be union of disconnected line segments. Moreover, fuzzy lines in [3, 7] can have empty core also. Thus, these methods of defining fuzzy lines may be violating the corresponding well-known definitions for crisp lines. Reporting all the deficiencies of the existing literature, Chakraborty and Ghosh formulated fuzzy lines in detail [4]. In [4], four different forms of fuzzy lines have been proposed. In this work, we attempt to obtain parametric forms of fuzzy lines. Being a bi-infinite extension of fuzzy line segment [6], a fuzzy line may be visualized as a straight infinitely long hazy band consisting of a bunch of crisp lines with varied membership grades.
This consideration implies that the fuzzy line will have equally wide imprecise ranges on both the sides across its core line. But, in our opinion, for a general fuzzy line this visualization cannot always work. If the ranges of spreads are equal throughout and across the core line, then the fuzzy line may be said as a symmetric fuzzy line and otherwise it is a non-symmetric fuzzy line.

In the literature, Rosenfeld [9] and Pham [11] also introduced some concepts regarding fuzzy lines. The defined concepts of fuzzy lines in [11] and fuzzy half planes in [9] are particular cases of fuzzy lines defined here. In [1], it is defined that if \( h \) is the 'height' of the intersecting region of two fuzzy lines', then their degree of parallelness is 1 – \( h \), which implies that the fuzzy lines with same core are cent percent parallel (height of a fuzzy set is the largest membership grade in the fuzzy set). Owing to this fact, we propose another definition of parallel fuzzy lines. Along with the different properties and attributes of the fuzzy lines, how to get distance between a fuzzy point and fuzzy line is also proposed here. A definition of the angle bisector of two intersecting fuzzy lines is also given. The following is the organization of this paper.

Section 2 provides some basic definitions and terminologies which are used throughout this paper. A detailed investigation on fuzzy lines of parametric forms is made in the Section 3, which also includes definition of slope of a fuzzy line, parallel lines, angle bisectors of two fuzzy lines and a fuzzy half plane. In Section 4, conclusions of the proposed work is drawn.

2 Preliminaries

Basic definitions used here are taken from [1] and [6] with slight alteration. Small or capital letters with over tilde bar, i.e., \( \tilde{A}, \tilde{B}, \tilde{C}, \ldots \) and \( \tilde{a}, \tilde{b}, \tilde{c}, \ldots \) all represent fuzzy subsets of \( \mathbb{R}^n, n = 1, 2 \). Membership function of a fuzzy set \( \tilde{A} \) of \( \mathbb{R}^n \) is represented by \( \mu(\tilde{x} | \tilde{A}), \tilde{x} \in \mathbb{R}^n \) with \( \mu(\mathbb{R}^n) \subseteq [0, 1], n = 1, 2 \).

Definition 2.1. (\( \alpha \)-cut of a fuzzy set [6]). For a fuzzy set \( \tilde{A} \) of \( \mathbb{R}^n, n = 1, 2 \), its \( \alpha \)-cut is denoted by \( \tilde{A}(\alpha) \) and is defined by:

\[
\tilde{A}(\alpha) = \begin{cases} 
\{ x : \mu(\tilde{x} | \tilde{A}) \geq \alpha \} & \text{if } 0 < \alpha \leq 1 \\
\text{closure} \{ x : \mu(\tilde{x} | \tilde{A}) > 0 \} & \text{if } \alpha = 0.
\end{cases}
\]

The set \( \{ x : \mu(\tilde{x} | \tilde{A}) > 0 \} \) is called support of the fuzzy set \( \tilde{A} \). The set \( \tilde{A}(0) \) is often said as base of \( \tilde{A} \). The 1-cut, \( \tilde{A}(1) \) is called core of \( \tilde{A} \).

Definition 2.2. (Fuzzy number [1]). A fuzzy set \( \tilde{A} \) of \( \mathbb{R} \) is called a fuzzy number if its membership function \( \mu \) has the following properties:

(i) \( \mu(\tilde{x} | \tilde{A}) \) is upper semi-continuous,

(ii) \( \mu(\tilde{x} | \tilde{A}) = 0 \) outside some interval \([a, d]\), and

(iii) there exist real numbers \( b \) and \( c \) so that \( a \leq b \leq c \leq d \) and \( \mu(\tilde{x} | \tilde{A}) \) is increasing on \([a, b]\) and decreasing on \([c, d]\), and \( \mu(\tilde{x} | \tilde{A}) = 1 \) for each \( x \) in \([b, c]\).

Due to upper semi-continuity of \( \mu(\tilde{x} | \tilde{A}) \), for a fuzzy number \( \tilde{A} \), the set \( \{ x : \mu(\tilde{x} | \tilde{A}) \geq t \} \) is closed for all \( t \) in \( \mathbb{R} \). Therefore the set \( \tilde{A}(\alpha) \) is a closed and bounded interval of \( \mathbb{R} \) for all \( \alpha \) in \([0, 1]\).

The notation \( (a/b/d)_{LR} \) is used to represent the above defined fuzzy number when \( b = c \), where \( L \) and \( R \) are reference functions of a fuzzy set (see [5]). A fuzzy number is called triangular fuzzy number if the functions \( L \) and \( R \), in \( (a/b/d)_{LR} \), are linear and more explicitly \( L(x) = R(x) = \max \{0, 1 - |x| \} \). Let us recall here that any fuzzy number can be represented by a \( LR \)-type (see Theorem 4.5 of [12]). We denote a triangular fuzzy number by \( (a/b/d) \). A perception of fuzzy number along a line is given below.

Definition 2.3. (Fuzzy number along a line [6]). In defining fuzzy number, conventionally, real line \( (\mathbb{R}) \) is taken as universal set. Instead of real line as universal set, any line of \( \mathbb{R}^2 \) plane may be taken as follows. Let in \( \mathbb{R}^2 \), \( x \)-axis
represents the real line and \( \tilde{p} \) be a fuzzy number. In x-axis, \( \tilde{p} \) can be represented by \( \mu((x,0)|\tilde{p}) = \mu(x|\tilde{p}) \forall x \in \mathbb{R} \). More explicitly:

\[
\mu((x,y)|\tilde{p}) = \begin{cases} 
\mu(x|\tilde{p}) & \text{if } y = 0 \\
0 & \text{elsewhere.}
\end{cases}
\]

Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the transformation which includes rotation of axes with angle \( \theta \) and translation of origin to \( (\frac{ac}{a^2+b^2}, \frac{bc}{a^2+b^2}) \), which is the point of intersection of \( ax + by = c \) and its perpendicular line through origin. This \( T \) can be expressed by \( T(x,y) = (x \cos \theta - y \sin \theta + \frac{ac}{a^2+b^2}, x \sin \theta + y \cos \theta + \frac{bc}{a^2+b^2}) \), which is a bijective transformation and transforms x-axis to \( ax + by = c \). Now, \( \tilde{p} \) may be considered as a fuzzy number on the line \( ax + by = c \) defined in the following way:

\[
\mu((u,v)|\tilde{p}) = \begin{cases} 
\mu((x,0)|\tilde{p}) & \text{if } (u,v) = T(x,0), \text{ } au + bv = c \\
0 & \text{elsewhere.}
\end{cases}
\]

It is to observe that for each and every fuzzy number \( \tilde{p} \) on the line \( ax + by = c \), there always exists unique fuzzy number in the real line and vice versa.

**Definition 2.4.** (Fuzzy points [1]). A fuzzy point at \( (a,b) \) in \( \mathbb{R}^2 \), written as \( \tilde{P}(a,b) \), is defined by its membership function:

(i) \( \mu((x,y)|\tilde{P}(a,b)) \) is upper semi-continuous,

(ii) \( \mu((x,y)|\tilde{P}(a,b)) = 1 \) if and only if \( (x,y) = (a,b) \), and

(iii) \( \tilde{P}(a,b)(\alpha) \) is a compact, convex subset of \( \mathbb{R}^2 \), for all \( \alpha \) in \([0,1]\). Often the notations \( \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \ldots \) are used to represent fuzzy points.

**Definition 2.5.** (Same and inverse points with respect to fuzzy points [6]). Let \( (x_1,y_1) \) and \( (x_2,y_2) \) be two points on support of two continuous fuzzy points \( \tilde{P}(a,b) \) and \( \tilde{P}(c,d) \) respectively. Let \( L_1 \) be a line joining \( (x_1,y_1) \) and \( (a,b) \). As \( \tilde{P}(a,b) \) is a fuzzy point, along \( L_1 \) there exists a fuzzy number, \( \tilde{r}_1 \) say, on the support of \( \tilde{P}(a,b) \). Membership function of this fuzzy number \( \tilde{r}_1 \) can be written as: \( \mu((x,y)|\tilde{r}_1) = \mu((x,y)|\tilde{P}(a,b)) \) for \( (x,y) \) in \( L_1 \), and 0 otherwise. Similarly, along a line, \( L_2 \) say, joining \( (x_2,y_2) \) and \( (c,d) \), there exists a fuzzy number, \( \tilde{r}_2 \) say, on the support of \( \tilde{P}(c,d) \). Now \( (x_1,y_1), (x_2,y_2) \) are said to be same points with respect to \( \tilde{P}(a,b) \) and \( \tilde{P}(c,d) \) if:

(i) \( (x_1,y_1) \) and \( (x_2,y_2) \) are same points with respect to \( \tilde{r}_1 \) and \( \tilde{r}_2 \), and

(ii) \( L_1, L_2 \) have equal angle with the line joining \( (a,b) \) and \( (c,d) \).

The points \( (x_1,y_1), (x_2,y_2) \) are said to be inverse points if \( (x_1,y_1), (x_2,y_2) \) are same point with respect to \( \tilde{P}(a,b) \) and \( \tilde{P}(c,d) \).

**Definition 2.6.** (Fuzzy distance [6]). Fuzzy distance \( \tilde{D} \) between two fuzzy points \( \tilde{P}_1 \) and \( \tilde{P}_2 \) is defined by its membership function as: \( \mu(d|\tilde{D}) = \sup \{ \alpha : d = \tilde{d}(u,v) \}, u \in \tilde{P}_1(0) \text{ and } v \in \tilde{P}_2(0) \) are inverse points with membership value \( \alpha \}, \tilde{d}(\_\_\_) \) is the Euclidean distance metric.

**Definition 2.7.** (Fuzzy line segment [6]). Fuzzy line segment \( \tilde{L}_{\tilde{P}_1\tilde{P}_2} \) joining two fuzzy points \( \tilde{P}_1 \), \( \tilde{P}_2 \) is defined by its membership function as: \( \mu((x,y)|\tilde{L}_{\tilde{P}_1\tilde{P}_2}) = \sup \{ \alpha : (x,y) \text{ lies on the line joining same points } (x_1,y_1) \in \tilde{P}_1(0), (x_2,y_2) \in \tilde{P}_2(0) \text{ and } \mu((x_1,y_1)|\tilde{P}_1) = \mu((x_2,y_2)|\tilde{P}_2) = \alpha } \}.

**Definition 2.8.** (Angle between two fuzzy line segments [6]). Let \( \tilde{L}_1, \tilde{L}_2, \tilde{L}_3 \) are three continuous fuzzy points. The angle between \( \tilde{L}_{\tilde{P}_1\tilde{P}_2}, \tilde{L}_{\tilde{P}_2\tilde{P}_3} \) is denoted by \( \Theta \) and is defined by: \( \mu(\Theta|\tilde{D}) = \sup \{ \alpha : \text{ where } \theta \text{ is angle between the line segments } \tilde{T}_{uv} \text{ and } \tilde{T}_{vw}, \text{ where } u, v \text{ and } v, w \text{ are same points of membership value } \alpha; u \in \tilde{P}_1(0), v \in \tilde{P}_2(0), w \in \tilde{P}_3(0) \} \).

Now let us move to study fuzzy lines and their characteristics.
3 Fuzzy lines

Definition 3.1. (Fuzzy line passing through \(n \geq 2\) collinear fuzzy points [4]). Suppose \(\tilde{P}_1(a_1, b_1), \tilde{P}_2(a_2, b_2), \ldots, \tilde{P}_n(a_n, b_n)\) are \(n\) collinear fuzzy points, i.e., \((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\) lie on the same line \(l\) say. Suppose the points \((a_i, b_i)\) are ordered on the line \(l\) from left to right for \(i = 1, 2, \ldots, n\). To construct a fuzzy line, \(\tilde{L}\) say, passing through all of these \(n\) fuzzy points, it may be considered that: at infinite distance apart, there exist two fuzzy points having core on \(l\), one is on the left of \(\tilde{P}_1(a_1, b_1)\) and another is on the right of \(\tilde{P}_n(a_n, b_n)\). Let these two points be \(\tilde{P}_{1\infty}, \tilde{P}_{n\infty}\) respectively; \(\tilde{T}_{1\infty}\) and \(\tilde{T}_{n\infty}\) be the line segments joining \(\tilde{P}_{1\infty}, \tilde{P}_1\) and \(\tilde{P}_n, \tilde{P}_{n\infty}\) respectively. The line \(\tilde{L}\) may be defined as:

\[
\tilde{L} = \tilde{T}_{1\infty} \cup \tilde{T}_{P_1P_2} \cup \ldots \cup \tilde{T}_{P_{n-1}P_n} \cup \tilde{T}_{n\infty}.
\]

Since support of a fuzzy point must be a compact set, i.e., closed and bounded set in \(\mathbb{R}^2\), all the lines joining same points of \(\tilde{P}_1\) and \(\tilde{P}_{1\infty}\) must be parallel to the line \(l\). Similarly all the lines joining same points of \(\tilde{P}_n\) and \(\tilde{P}_{n\infty}\) always be parallel to the line \(l\). Because, otherwise, the sets \(\tilde{T}_{1\infty}(0)\) and \(\tilde{T}_{n\infty}(0)\) will be unbounded and hence they cannot represent supports of fuzzy points. Therefore, slope of the fuzzy half lines (or infinite tails) \(\tilde{T}_{1\infty}\) and \(\tilde{T}_{n\infty}\) must be equal to slope of the core line \(l\). Thus the fuzzy line may have figure like Fig. 1, where a fuzzy line passing through five collinear fuzzy points \(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_5\) is depicted.

![Figure 1: Fuzzy line passing through five collinear fuzzy points](image)

In the Definition 3.1, a fuzzy line is perceived as union of crisp lines or line segments with different membership values. However there are other ways too. For instance, fuzzy lines may be imagined as collection of crisp points of different membership values or as group of fuzzy points. Basically whatever be the way has been followed to define fuzzy line, it must be visualized as a straight infinitely long hazy band having one crisp straight line on its core with a smooth transition of membership values between the neighboring points of that crisp line.

Theorem 3.1. Let \(\tilde{L}\) be a fuzzy line and \(l\) be a line perpendicular to \(\tilde{L}(1)\). Along \(l\), there always exists a fuzzy number on \(\tilde{L}\).

Proof. Let \(\tilde{L} = \{(x, \mu(x|\tilde{L}) : x \in \mathbb{R}^2\}\) be the fuzzy line. The line \(l\) can also be presented by \(\{(x, \mu(x|l)) : x \in \mathbb{R}^2\}\) where \(\mu(x|l) = 1\) for \(x \in l\) and zero otherwise. Obviously, the set \(\{(x, \mu(x|\tilde{L}) : x \in \mathbb{R}^2\}\) \(\cap\) \(\{(x, \mu(x|l)) : x \in \mathbb{R}^2\}\) is a fuzzy set, whose \(\mu\) is evaluated by \(t\)-norm ‘min’. Let us denote this fuzzy set as \(A\). We have to prove \(A\) is a fuzzy...
number along \(l\). Let \(0 < \alpha \leq 1\). The set \(\tilde{A}(\alpha)\) is a line segment or union of line segments which are subsets of \(l\). We will prove that it is exactly a line segment and cannot be union of line segments. Let \(A_\alpha\) and \(B_\alpha\) be the points of intersection of \(l\) with \(f(x, y; \alpha) = 0\) and \(h(x, y; \alpha) = 0\) respectively, where \(f(x, y; \alpha) = 0\) and \(h(x, y; \alpha) = 0\) are boundary of the \(\alpha\)-cut of \(\tilde{L}\). We will show \(\tilde{A}(\alpha) = \overline{A_\alpha B_\alpha}\). Let \(\exists K \in \overline{A_\alpha B_\alpha}\) but \(K \notin \tilde{A}(\alpha)\). So \(\mu(\tilde{K}) < \alpha\). Therefore, \(K \notin \tilde{L}(\alpha)\). But this is not possible, because all \(\alpha\)-level sets of \(\tilde{L}\) are closed convex sets and for \(0 < \alpha < \beta \leq 1\) we get \(\tilde{A}(\beta) \subseteq \tilde{A}(\alpha)\). Thus, membership function of \(\tilde{A}\) is upper semi-continuous and \(\tilde{A}(0)\) is a compact set.

If \(G\) be the point of intersection of \(l\) and \(\tilde{L}(1)\), then \(\mu(G|\tilde{A}) = \min\{\mu(G|l), \mu(G|\tilde{L})\} = 1\). Hence the fuzzy set \(\tilde{A}\) is a fuzzy number along the line \(l\).

Theorem 3.1 implicitly describes a fuzzy line as a collection fuzzy numbers along the lines perpendicular to the core line \(l\). We notice from the Fig. 1 that along the line \(CD\), which is perpendicular to \(l \equiv PQ\), the fuzzy number \((F/G/H)_{LR}\) exists in the support of the fuzzy line \(\tilde{L}\). Varying \(G\) on the line \(PQ\), we will have several such fuzzy numbers. The set of points alike to \(F\) will determine a curve \(y = p(x)\). Similarly, the set of points alike to \(H\) will determine a curve \(y = q(x)\). Thus a fuzzy line is bounded by two boundary curves and there must be one crisp line between those curves. Let the crisp line is \(y = mx + c\). Then the fuzzy line \(\tilde{L}\) can be expressed by an equational form \((y - p(x))/y - mx - c/y - q(x))_{LR} = 0\), where \(L: [0, \infty] \to [0, 1]\) and \(R: [0, \infty] \to [0, 1]\) are suitable reference functions, i.e., they are symmetric with respect to ‘0’ and non-increasing functions with \(L(0) = R(0) = 1\).

Here the notation \((y - p(x))/y - mx - c/y - q(x))_{LR} = 0\) means that membership grade of the points on \(\tilde{L}(0)\) gradually increases (following the function \(L\)) from 0 to 1 as we move from \(y = p(x)\) to \(y = mx + c\) and gradually decreases (following the function \(R\)) from 1 to 0 as we move from \(y = mx + c\) to \(y = q(x)\).

Theorem 3.2. Let \(\tilde{L}\) be a fuzzy line and \(l\) be a line perpendicular to \(\tilde{L}(1)\). Along \(l\), there always exists a fuzzy number on \(\tilde{L}\).

Theorems 3.1 and 3.2 helps to think fuzzy line \(\tilde{L}\) as a set of fuzzy numbers at each points of \(\tilde{L}(1)\) or as a set of fuzzy points at each point of \(\tilde{L}(1)\). These perceptions lead to define parametric forms of fuzzy lines.

Definition 3.2. (Parametric form).

\(\text{(Method 1)}\) Let \(\tilde{L}\) be a fuzzy line, \((x, y)\) be a point on \(\tilde{L}(1)\), \(\tilde{N}(x, y)\) be a fuzzy number along the line which is perpendicular to \(L(1)\) and passing through \((x, y)\). Then \(\tilde{L}\) may be expressed by \(\tilde{L} = \{(u, \tilde{N}(u)) : u \in \tilde{L}(1)\}\).

\(\text{(Method 2)}\) Let \(\tilde{L}\) be a fuzzy line, \(\tilde{P}(x, y)\) be a fuzzy point at \((x, y) \in \tilde{L}(1)\). Then \(\tilde{L}\) may be expressed by \(\tilde{L} = \{(u, \tilde{P}(u)) : u \in \tilde{L}(1)\}\).

Example 3.1. (Parametric form). Let us consider the fuzzy line \(\tilde{L}\) passing through the fuzzy points \(\tilde{P}_1(-1, 3)\) and \(\tilde{P}_2(0, 2)\), where \(\tilde{P}_1(-1, 3)(0) = \{(x, y) : (x + 1)^2 + (y - 3)^2 \leq 2\}\), \(\tilde{P}_2(0, 2) = \{(x, y) : x^2 + (y - 2)^2 \leq 2\}\), and their membership functions are right circular cone whose vertices are \((-1, 3)\) and \((0, 2)\) respectively.

\(\text{(Method 1)}\) Here \(\tilde{L}(1) = \{(x, y) : x + y = 2\}\), i.e., parametrically, \(\tilde{L}(1) = \{(x, 2 - x) : x \in \mathbb{R}\}\). Considered fuzzy line \(\tilde{L}\) can be expressed in parametric form as:

\[
\tilde{L} = \left\{ \left( \frac{2-k}{2}, \frac{2+k}{2} \right), \tilde{N}\left( \frac{2-k}{2}, \frac{2+k}{2} \right) : k \in \mathbb{R} \right\},
\]

where \(\tilde{N}(\frac{2-k}{2}, \frac{2+k}{2}) = \left(\frac{-k}{2}, \frac{k}{2}\right) / \left(\frac{2-k}{2}, \frac{2+k}{2}\right) / \left(\frac{2-k}{2}, \frac{2+k}{2}\right)\) is a fuzzy number along \(y = x\).

If we let \(\tilde{A} = \tilde{N}(1, 1) = \{(0, 0)/(1, 1)/(2, 2)\}\) be the fuzzy number along the line \(y = x = 0\), then membership function of \(\tilde{L}\) can be generated by placing \(\tilde{A}\) on the line \(y + x = 2\) with support on the line segment joining \((0, 0)\) and \((2, 2)\), and moving the triangular fuzzy number \(\tilde{N}(1, 1)\) along the line \(y + x = 2\).
(Method 2) If we let \( \widetilde{P}(1,1) \) be a fuzzy point with membership function as right circular cone, and support as \( \{(x,y):(x-1)^2+(y-1)^2 \leq 2\} \). The fuzzy line \( L \) can be generated by translating \( \mu((\cdot)\widetilde{P}) \) along \( x+y=2 \). Parametrically \( L \) can be written as,

\[
L = \{(u,k-u),\widetilde{P}(u,k-u) : k \in \mathbb{R}\},
\]

where graph of \( \mu(\cdot)\widetilde{P}(u,k-u) \) is a right circular point with base \( \{(x,y):(x-u)^2+(y-k+u)^2 \leq 1\} \).

From the Theorem 3.1 it is obtained that along a line \( f \), which is perpendicular to \( \widetilde{L} \), a fuzzy number exists in \( \widetilde{L}(0) \). However, spread of all of those fuzzy numbers may not be equal in general. If spreads on both the sides across the core are equal, then the fuzzy lines may be called as symmetric and otherwise non-symmetric.

**Definition 3.3. (Symmetric and non-symmetric fuzzy lines).** If all the fuzzy numbers, situated on a fuzzy line \( \widetilde{L} \) and along the perpendicular lines of \( \widetilde{L}(1) \), have same spread, then \( \widetilde{L} \) is said to be a symmetric fuzzy line and otherwise non-symmetric.

**Note 3.1.** Symmetric fuzzy line may be visualized by pulling a fuzzy number (point) continuously along the line \( \widetilde{L}(1) \) with the restriction that core of the fuzzy number (point) always retained on \( \widetilde{L}(1) \). Thus, evaluating membership value of the points on a symmetric fuzzy line is quite easier, since once a fuzzy number along a perpendicular line on support of the fuzzy line becomes known to us, the entire fuzzy line will be known.

We note that if a fuzzy line \( \widetilde{L} \) is perceived as group of fuzzy points, then boundary of supports of fuzzy points \( \widetilde{P} \) which belong to \( L \) touch the boundary of \( \widetilde{L}(0) \). For example, if a fuzzy line \( \widetilde{L}(0) \) passing through \( \widetilde{P}_1 \) and \( \widetilde{P}_2 \) is considered, the fuzzy points lie on \( \widetilde{L}(0) \) are \( \widetilde{P} = \lambda \widetilde{P}_1 + (1-\lambda)\widetilde{P}_2 \) \( (0 \leq \lambda \leq 1) \) whose supports touch the boundary curves of \( \widetilde{L}(0) \). However, there exist several fuzzy points whose core lie on \( \widetilde{L}(1) \) and support of them are subset of \( \widetilde{L}(0) \). We may expect that all of these fuzzy points partially contained in the fuzzy line \( \widetilde{L} \). Let us now find containment of any fuzzy point on a fuzzy line.

**Definition 3.4. (Containment of a fuzzy point in a fuzzy line (Method 1)).** Let \( \widetilde{P} \) be a fuzzy point and \( \widetilde{L} \) be a fuzzy line. \( \widetilde{P} \) is said to lie on \( \widetilde{L} \) if

(i) \( \widetilde{P}(0) \) touches the boundaries of \( \widetilde{L}(0) \),

(ii) surface of \( z = \mu((x,y)|\widetilde{P}) \) touches the surface of \( z = \mu((x,y)|\widetilde{L}) \).

According to [1], a fuzzy point \( \widetilde{P} \) is contained in a fuzzy line \( \widetilde{L} \) if \( \mu((x,y)|\widetilde{P}) \leq \mu((x,y)|\widetilde{L}) \) for all \( (x,y) \in \mathbb{R}^2 \). This definition informs that if \( \widetilde{P}(0) \not\subset \widetilde{L}(0) \) then \( \widetilde{P} \) is not fuzzily contained in \( \widetilde{L} \) even if \( \widetilde{P}(1) \in \widetilde{L}(1) \). Thus the condition \( \widetilde{P} \subset \widetilde{L} \) only can say enough about containment of fuzzy point in fuzzy environment instead, a measurement of belongingness of \( \widetilde{P} \) in \( \widetilde{L} \), only when \( \widetilde{P}(1) \in \widetilde{L}(1) \), would be more reasonable. In the next definition, a method of this measurement has been proposed.

**Definition 3.5. (Containment of a fuzzy point in a fuzzy line (Method 2)).** Let \( \widetilde{P} \) be a fuzzy point and \( \widetilde{L} \equiv \left\{ f(x,y) / f(x,y) - mx - c / h(x,y) \right\}_{LR} = 0 \). If \( \widetilde{P}(1) \in \widetilde{L}(1) \), then the fuzzy point \( \widetilde{P} \) must be fuzzily contained in \( \widetilde{L} \) with some membership value, \( \beta \) say. This \( \beta \) may be obtained by:

\[
\beta = \begin{cases} 
1 & \text{if } \widetilde{P} \leq \widetilde{L} \text{ or } \widetilde{P}(0) \not\subset \widetilde{L}(0) \\
\beta_1 & \text{if } \widetilde{P}(0) \text{ exceeds } \widetilde{L}(0) \text{ on the side of } f \\
\beta_2 & \text{if } \widetilde{P}(0) \text{ exceeds } \widetilde{L}(0) \text{ on the side of } h \\
\min\{\beta_1,\beta_2\} & \text{if } \widetilde{P}(0) \text{ exceeds } \widetilde{L}(0) \text{ on the either sides of } \widetilde{L}(1) 
\end{cases}
\]

where \( \beta_1 = \sup_{(x,y)|f(x,y)=0} \mu((x,y)|\widetilde{P}) \) and \( \beta_2 = \sup_{(x,y)|h(x,y)=0} \mu((x,y)|\widetilde{P}) \).

If \( \widetilde{P}(1) \not\in \widetilde{L}(1) \), then \( \widetilde{P} \) cannot be fuzzily contained in \( \widetilde{L} \) and we define \( \beta = 0 \) in this situation.
Note 3.2. For any point \((x, y) \in \tilde{P}(0) \cap \tilde{L}(0), \mu\left((x, y)\right) \geq \beta\).

Up to this point, we observe that fuzzy lines are perceived as one of the following way:

(i) collection of crisp points,

(ii) collection of fuzzy points, or

(iii) collection of crisp line segments or half-lines or lines.

However any of the three considerations depends on the other two, since fuzzy points or crisp lines are collection of crisp points and intersection of two curves is a point or set of points. Unique characterization of fuzzy lines, the identification of a fuzzy set to be a fuzzy line is given in the following theorem.

Theorem 3.3. (Characterization theorem). A fuzzy set is a fuzzy line when and only when its core is a crisp straight line and along any line perpendicular to the core lines there must exists a fuzzy number on the support of the fuzzy line.

Proof. Forward part of the theorem is true from the Theorem 3.1. As we observe that, under the assumption, the considered fuzzy set will be a fuzzy line in the parametric form (Method 1), so the converse is true. Hence the result follows.

Let us now define slope and intercepts of a fuzzy line.

Definition 3.6. (Slope of a fuzzy line). Let \(\tilde{L}\) be a fuzzy line and \(\tilde{I}\) be a line segment in \(\tilde{L}(0)\). Let us consider that membership value of \(\tilde{I}\) on \(\tilde{L}\) as \(\inf\{\mu(x|\tilde{L}) : x \in \tilde{I}\}\). Slope of the fuzzy line may defined by its membership function as:

\[\mu(m|\tilde{m}) = \sup\{\mu(\tilde{I}|\tilde{L}) : \text{slope of} \tilde{I} = m\}\]

Example 3.2. (Slope of a fuzzy line). Let \(\tilde{L}\) be the fuzzy line passing through the fuzzy points \(\tilde{P}_1(2, 3)\) and \(\tilde{P}_2(7, 9)\). Supports of \(\tilde{P}_1(2, 3)\) and \(\tilde{P}_2(7, 9)\) are \(\{(x, y) : (x - 2)^2 + (y - 3)^2 \leq 1\}\) and \(\{(x, y) : (x - 7)^2 + (y - 9)^2 \leq 1\}\) respectively.

Membership functions of \(\tilde{P}_1(2, 3)\) and \(\tilde{P}_2(7, 9)\) have shapes right circular cones with vertices at \((2, 3)\) and \((7, 9)\) respectively.

Fuzzy line \(\tilde{L}\) is determined by \(\tilde{L}_{1\alpha} \cup \tilde{L}_{2\alpha} \cup \tilde{L}_{3\alpha}\).

Any half line in the infinite fuzzy half lines \(\tilde{L}_{1\alpha}\) and \(\tilde{L}_{2\alpha}\) have slope \(1.2\).

Same points of \(\tilde{P}_1\) and \(\tilde{P}_2\) with membership value \(\alpha \in [0, 1]\) are \(S_{\theta\alpha}^{1} = (2 + \frac{1-\alpha}{2} \cos \theta, 3 + \frac{1-\alpha}{2} \sin \theta)\) and \(S_{\theta\alpha}^{2} = (7 + (1 - \alpha) \cos \theta, 9 + (1 - \alpha) \sin \theta)\) respectively, for each \(\theta \in [0, 2\pi]\).

Slope of the line segment \(S_{\theta\alpha}^{1}S_{\theta\alpha}^{2}\) is \(6 + \frac{1-\alpha}{2} \sin \theta \over 5 + \frac{1-\alpha}{2} \cos \theta\).

If \(\tilde{m}\) be the slope of the fuzzy line \(\tilde{L}\), then according to the Definition 3.6, membership value of \(\sup_{\theta \in [0, 2\pi]} \frac{6 + \frac{1-\alpha}{2} \sin \theta}{5 + \frac{1-\alpha}{2} \cos \theta}\) on \(\tilde{m}\) is \(\alpha\).

Therefore, core of \(\tilde{m}\) is 1.2 and for each \(\alpha \in [0, 1]\), \(\tilde{m}(\alpha) = \frac{6 - \frac{1}{2} \sin b(\alpha) + 6 + \frac{1}{2} \sin b(\alpha)}{3 + \frac{1}{2} \cos b(\alpha)}\) where

\[b(\alpha) = 2\tan^{-1} \frac{\sqrt{244 - (1 - \alpha)^2} + 12}{(1 - \alpha) - 10}\]

In the next y-intercept of a fuzzy line is defined and similarly x-intercept can be defined.

Definition 3.7. (y-intercept of a fuzzy line). y-intercept of a fuzzy line \(\tilde{L}\), \(\tilde{c}\) say, may be defined by its membership function as:

\[\mu(\tilde{c} | \tilde{c}) = \begin{cases} 
\mu((c, 0) | \tilde{L}) & \text{if } (c, 0) \in \tilde{L}(0) \cap (y-axis) \\
0 & \text{otherwise,}
\end{cases}\]

i.e., \(\tilde{c} = \tilde{L} \cap (y-axis)\).
Example 3.3. (y-intercept of a fuzzy line) Let us evaluate y-intercept of the fuzzy line $\tilde{L}$ passing through $\tilde{P}_1(2,3)$ and $\tilde{P}_2(7,9)$ in the Example 3.2. According to the Definition 3.7. y-intercept of $L$ is the triangular fuzzy number \( \left( \frac{2}{3}, \frac{4}{3}, \frac{5}{3} \right) \) and y-intercept of $L$ is equal to $\tilde{P}_2(7,9)$. Clearly, if $\tilde{L}_1$ and $\tilde{L}_2$, are not parallel, and therefore, according to Definition 3.9, $\tilde{L}_1$ and $\tilde{L}_2$ are non-symmetric, $\tilde{L}_1(0) \cap \tilde{L}_2(0) \neq \emptyset$, and hence two non-symmetric fuzzy lines cannot be parallel. According to [1], if $L_1$ is equal to $L_2$ according to a rigid transformation, then also their degree of parallelness is not ‘1’. We may define parallel fuzzy lines and degree of parallelness of fuzzy lines as follows.

Definition 3.8. (Distance between a fuzzy point and a fuzzy line). Let $\tilde{P}$ be a fuzzy point and $\tilde{L}$ be a fuzzy line. Distance between $\tilde{P}$ and $\tilde{L}$ may be defined as:

$$ D = \bigcup_{\alpha \in [0,1]} \left( \{ (d_l, \alpha) : d_l = \inf A_\alpha \} \cup \{ (d_u, \alpha) : d_u = \sup A_\alpha \} \right), $$

where $A_\alpha = \{ (d(x,y) : x \in \tilde{L}(\alpha) \text{ and } y \in \tilde{P}(\alpha) \}$. Here the tuple $(d_l, \alpha)$ or $(d_u, \alpha)$ mean that ‘the distance $d_l$ or $d_u$ adjoined with membership value $\alpha$’.

Theorem 3.4. $\tilde{D}$, the distance between a fuzzy point and a fuzzy line is a fuzzy number.

Proof. Similar to Theorem 4.1 in [6].

Example 3.4. Let $\tilde{L}$ be the line $(y = 1/3 - 2/y = 3) = 0$. Let $\tilde{P}(1,6)$ be a fuzzy point whose graph of $\mu$ is a right elliptical cone with base $\{(x,y) : \frac{(x-1)^2}{(1/2)^2} + \frac{(y-6)^2}{(1/3)^2} \leq 1 \}$, i.e.,

$$ \mu((x,y)|\tilde{P}(1,6)) = \begin{cases} 1 - \sqrt{\frac{(x-1)^2}{(1/2)^2} + \frac{(y-6)^2}{(1/3)^2}} & \text{if } \frac{(x-1)^2}{(1/2)^2} + \frac{(y-6)^2}{(1/3)^2} \leq 1, \\ 0 & \text{elsewhere.} \end{cases} $$

Therefore $\tilde{L}(\alpha) = \{ (x,y) : 1 + \alpha \leq y \leq 3 - \alpha \}$ and $\tilde{P}(1,6)(\alpha) = \{ (x,y) : \frac{(x-1)^2}{(1/2)^2} + \frac{(y-6)^2}{(1/3)^2} \leq (1 - \alpha)^2 \}$. So, $A_\alpha = [\frac{8-4\alpha}{3}, \frac{16-4\alpha}{3}]$ and hence $\mu \left( \frac{8-4\alpha}{3} | \tilde{D} \right) = \mu \left( \frac{16-4\alpha}{3} | \tilde{D} \right) = \alpha$. Precisely $\tilde{D}$ is,

$$ \mu(x|\tilde{D}) = \begin{cases} \frac{3x-8}{4} & \text{if } \frac{4}{3} \leq x \leq 4, \\ \frac{16-3x}{4} & \text{if } 4 \leq x \leq \frac{16}{3}, \\ 0 & \text{otherwise.} \end{cases} $$

In [1], parallelness of two fuzzy lines $\tilde{L}_1$ and $\tilde{L}_2$ is defined by height of the fuzzy set $\tilde{L}_1 \cap \tilde{L}_2$. Clearly, if $\tilde{L}_1$ and $\tilde{L}_2$ are non-symmetric, $\tilde{L}_1(0) \cap \tilde{L}_2(0) \neq \emptyset$, and hence two non-symmetric fuzzy lines cannot be parallel. According to [1], if $L_1$ is equal to $L_2$ according to a rigid transformation, then also their degree of parallelness is not ‘1’. We may define parallel fuzzy lines and degree of parallelness of fuzzy lines as follows.

Definition 3.9. (Parallel fuzzy lines). Let $\tilde{L}_1$ and $\tilde{L}_2$ be two fuzzy lines. These fuzzy lines are said to be fuzzily parallel if

(i) $\tilde{L}_1(1)$ and $\tilde{L}_2(1)$ are parallel,

(ii) the two boundary curves of $\tilde{L}_1(\alpha)$ are parallel (Two curves are said to be parallel if any of them can be obtained by applying a rigid translation of another) to the respective two boundary curves of $\tilde{L}_2(\alpha)$, for each $\alpha \in [0,1]$. It is to note that if $\tilde{L}_1$ and $\tilde{L}_2$ are parallel, then $\tilde{m}_1 = \tilde{m}_2$, where $\tilde{m}_1$ and $\tilde{m}_2$ are slope of $\tilde{L}_1$ and $\tilde{L}_2$ respectively. But, often it happens that for two fuzzy lines $\tilde{L}_1$ and $\tilde{L}_2$, $\tilde{L}_1(1)$ and $\tilde{L}_2(1)$ are parallel but boundary curves of their $\alpha$-cuts are not parallel, and therefore, according to Definition 3.9, $\tilde{L}_1$ and $\tilde{L}_2$ are not fuzzily parallel. But this is not desirable and we feel that if $\tilde{L}_1(1)$ and $\tilde{L}_2(1)$ are parallel then there always exist some degree of possibility that $\tilde{L}_1$ and $\tilde{L}_2$ are fuzzily parallel. This degree of parallelness ($p$) of two fuzzy lines may be defined as $1 - \text{height of the difference of fuzzy numbers } \tilde{m}_1$ and $\tilde{m}_2$, i.e.,

$$ p = 1 - \sup_{x \in \mathbb{R}} \left| \mu(x|\tilde{m}_1) - \mu(x|\tilde{m}_2) \right|. $$
Note 3.3. This measurement of $p$ is applicable when $\tilde{L}_1(1)$ and $\tilde{L}_2(1)$ are parallel, i.e., $\tilde{m}_1(1) = \tilde{m}_2(1)$. When $\tilde{m}_1(1) \neq \tilde{m}_2(1)$, then $p = 0$. If $p = 0$, $L_1$ and $L_2$ said to be intersecting.

Note 3.4. If two fuzzy lines are intersecting then their intersection may not be a fuzzy point. It is easily followed from the fact that support of the fuzzy lines may not be convex and hence their intersection may be non-convex.

![Figure 2: Angle bisectors (by Method 1) of two fuzzy lines](image)

**Definition 3.10.** (Angle between two fuzzy lines and angle bisectors). Let $\tilde{L}_1$ and $\tilde{L}_2$ be two fuzzy lines. If their angle of elevations with x-axis are $\theta_1$ and $\theta_2$, then angle between them is $\theta_2 - \theta_1$.

Let us suppose $\tilde{L}_1$ and $\tilde{L}_2$ are intersecting, then their exist angle bisectors of them. Angle bisectors of $\tilde{L}_1$ and $\tilde{L}_2$ can be defined in two different ways – (i) by their intersecting set and angle of bisector of their core lines or (ii) by combining the angle bisectors of the intersecting lines or line segments lie on $\tilde{L}_1(0)$ and $\tilde{L}_2(0)$.

(Method 1) Let angle bisectors of $\tilde{L}_1(1)$ and $\tilde{L}_2(1)$ be $l_1$ and $l_2$; intersecting set of $\tilde{L}_1$ and $\tilde{L}_2$ be $\tilde{A}$. The angle bisectors of the fuzzy lines may be defined as: $\forall \{A_1 : \tilde{A}_1(1) \in l_1\}$ and $\forall \{A_2 : \tilde{A}_2(1) \in l_2\}$, where $\tilde{A}$, $\tilde{A}_1$ and $\tilde{A}_2$ are fuzzy points having same base according to a rigid translation.

In the Fig. 2, angle bisectors of two fuzzy lines $\tilde{L}_1$ and $\tilde{L}_2$ obtained by the Method 1 is illustrated. The region between the neighboring dotted curves of the lines $PQ$ and $KL$ are supports of $\tilde{L}_1$ and $\tilde{L}_2$. The lines $PQ$ and $KL$ are respectively $L_1(1)$ and $L_2(1)$. Core lines of the angle bisectors are $MN$ and $ST$ respectively. The intersecting set of $\tilde{L}_1$ and $\tilde{L}_2$ is the fuzzy set $\tilde{A}$. Shape of the support and membership function of $\tilde{A}$, $\tilde{A}_1$ and $\tilde{A}_2$ are same and $\tilde{A}_1$, $\tilde{A}_2$ are obtained by some rigid translation of $\tilde{A}$ with the restriction that $A_1(1) \in ST$ and $A_2(1) \in PQ$.

**Note 3.5.** The angle bisectors obtained by Method 1 are symmetric fuzzy lines.
(Method 2) The fuzzy lines $\tilde{L}_1$ and $\tilde{L}_2$ can be represented by: $\tilde{L}_1 = \bigvee \{T_1 : T_1 \text{ is a line segment or half line in } \tilde{L}_1(0)\}$, and $\tilde{L}_2 = \bigvee \{T_2 : T_2 \text{ is a line segment or half line in } \tilde{L}_2(0)\}$. Their angle bisector, $\tilde{L}_B$, may be defined by:

$$\tilde{L}_B = \bigvee \{I : I \text{ is an angle bisector of two line segments or half lines } T_1 \in \tilde{L}_1(0) \text{ and } T_2 \in \tilde{L}_2(0)\}.$$

**Note 3.6.** Core of the fuzzy line $\tilde{L}_B$ in Method 2 always contains two intersecting lines.

**Note 3.7.** The angle bisectors by Method 1 and Method 2 are not identical.

**Definition 3.11.** (Fuzzy half plane). Let $\tilde{L} : (f(x,y)/g(x,y)h(x,y))_{LR} = 0$ be a fuzzy line where $g(x,y) \equiv ax + by + c = 0$. Let $(x_1,y_1)$ and $(x_2,y_2)$ be two points such that $f(x_1,y_1) = 0$ and $h(x_2,y_2) = 0$. Without loss of generality, let us assume $ax_1 + by_1 + c \leq 0$ and $ax_2 + by_2 + c \geq 0$. The line $ax + by + c = 0$ separates $\mathbb{R}^2$ plane into two parts. Analogously we may say that the fuzzy line $L$ fuzzily separates whole $\mathbb{R}^2$ plane into two parts, $H_1$ and $H_2$ say. Any of those two parts may be termed as fuzzy half plane. Thus, fuzzy (closed) half plane may be defined by its membership function as follows:

$$\mu((x,y)|H_1) = \begin{cases} 
\mu((x,y)|\tilde{L}) & \text{if } ax + by + c < 0 \\
1 & \text{if } ax + by + c \geq 0
\end{cases}$$

$$\mu((x,y)|H_2) = \begin{cases} 
\mu((x,y)|\tilde{L}) & \text{if } ax + by + c > 0 \\
1 & \text{if } ax + by + c \leq 0.
\end{cases}$$

**Note 3.8.** Two fuzzy points $\tilde{P}_1$ and $\tilde{P}_2$ are said to be lie on different sides of the fuzzy line $\tilde{L}$, if $\tilde{P}_1 \in H_1$ and $\tilde{P}_2 \in H_2$ or $\tilde{P}_2 \in H_1$ and $\tilde{P}_1 \in H_2$.

Analogous to the conventional geometry, intersection of two (closed) half planes is a straight line, in fuzzy geometry we obtain that intersection of two closed fuzzy half planes $H_1 \cap H_2$ is the fuzzy line $\tilde{L}$.

## 4 Conclusion

In this paper we have attempted to formulate parametric form of fuzzy lines. From the present investigation, we obtain that the formulation of fuzzy lines can be made in four different ways—first, group of crisp points with varied membership values, second, group of line segments or half lines with different membership values, third, group of fuzzy points, and last one is the group of fuzzy numbers along perpendicular lines on the core of the fuzzy line. From the proposed formulations of fuzzy lines, it is observed that $\alpha$-cut of fuzzy lines are closed, connected and arc-wise connected subset of $\mathbb{R}^2$ but not necessarily convex. Fuzzy line is always normalized and more precisely its core contains a crisp line. Membership function of fuzzy line is upper semi-continuous—it follows from the fact that $\alpha$-cut being closed. Along with the construction procedure of membership function of fuzzy lines, we have attempted to introduce some concepts of fuzzy half planes, parallel and intersecting fuzzy lines, distance between a fuzzy point and a fuzzy line, and angle bisector of two intersecting fuzzy lines. An equational form of fuzzy line is also proposed. Given definition of fuzzy half plane can be used to obtain a mathematical form of fuzzy half plane. Also, the definition of containment may be applied to test in which fuzzy half plane of a fuzzy line a given fuzzy fuzzy point lies. Application on fuzzy linear and affine spaces [8] of the proposed analysis of fuzzy lines may be considered as future research. Proposed fuzzy half plane may be applied in fuzzy optimization problem in future.

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