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## Solution of second order linear fuzzy difference equation by Lagrange's multiplier method

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### Abstract

In this paper we execute the solution procedure for second order linear fuzzy difference equation by Lagrange's multiplier method. In crisp sense the difference equation are easy to solve, but when we take in fuzzy sense it forms a system of difference equation which is not so easy to solve. By the help of Lagrange's multiplier we can solved it easily. The results are illustrated by two different numerical examples and followed by two applications.

**Keywords:** Fuzzy number, Second order fuzzy difference equation, Lagrange's multiplier.

### 1 Introduction

#### 1.1. Fuzzy sets, fuzzy number and generalized fuzzy number

In 1965, Lotfi A. Zadeh [1], a professor of electrical engineering at the University of California in Berkeley, published the papers on his new theory of Fuzzy Sets and Systems. After then various researcher contribute in that topic. Since the 1980s, this mathematical theory of the concept "un-sharp amounts" has been applied with great success in many different fields of research. Since Chang and Zadeh [2] introduced the concept of fuzzy numbers in 1972. A lot of mathematicians have been studying them (one-dimension or  $n$ -dimension fuzzy numbers, see for example [3,4,5,6]) and try to modeled with it. With the development of theories and applications of fuzzy numbers, this concept becomes more important for modeling.

Most of the researcher take a fuzzy number with maximum gradation one. Now it is not necessary that the maximum gradation is one. It is difference for one to one. So we can make it generalized. That is the gradation is greater than zero and less than or equal to one. Many researchers use the concepts of generalized fuzzy number for their research such as [7,8,9,10].

#### 1.2. Necessity for difference equation and fuzzy difference

In modeling of real natural phenomena, difference equations also play an important role in many areas of discipline, exemplary in economics, biomathematics, science and engineering. Many experts in such areas

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widely use difference equations in order to make some problems under study more comprehensible. In many cases, information about the physical phenomena related is always immanent with uncertainty.

### 1.3. Difference equation

Recently the study of the qualitative behavior of difference equation and difference equation system is a which topic of a great interest. A difference equation is an equation specifying the change in a variable between two periods. By using the difference equation we can study the concerning factors, which cause the change in the value of the given functions in different time periods. It is known that difference equation appears naturally as discrete analogous having many applications in computer science, control engineering, ecology, population dynamics, queuing problems, statistical problems, stochastic time series, geometry, psychology, sociology physics economics engineering etc. The theory of difference equations developed greatly during last three decades. The theory of difference equation occupies an important position in different fields. No doubt the theory of difference equation plays an important role in mathematics as well as its applications. Moreover the data are absorbed in relation with a real world phenomenon that can be described by difference equations.

### 1.4. Fuzzy difference equation

A difference fuzzy equation is a fuzzy difference equation when (i) initial condition is fuzzy number, (ii) coefficients is fuzzy number, (iii) initial conditions and coefficients are both fuzzy numbers. Fuzzy difference equation growing rapidly developed for the many years. Now the problem is that the solution procedure of difference equation and fuzzy difference are not same. To study the behavior and solutions of a fuzzy difference equation we need to study the concepts of fuzzy difference, since the fuzzy difference is not same as crisp difference. We can show that every fuzzy difference can be converted to system of fuzzy difference equations.

### 1.5. Review on fuzzy difference equation

There exist several research papers where difference equation is solved with fuzzy environment. Now we concentrate some published papers:

In [11] Deeba et.al. solve a fuzzy difference equation with an application. The model of CO<sub>2</sub> level in blood is modeled with fuzzy difference equation and solved by Deeba et.al. by [12]. Lakshmikantham and Vatsala [13] discuss the basic theory of fuzzy difference equations. Papaschinopoulos et.al. [14,15]. Papaschinopoulos and Schinas [16] discuss the behavior and solution of some different type of fuzzy difference equations. Papaschinopoulos and Stefanidou [17] give a description on boundedness and asymptotic behavior of the solutions of a fuzzy difference equation. Umekkan et.al. [18] give an application of finance in fuzzy difference equation. Stefanidou et.al. [19] give a brief discussion on an exponential type fuzzy difference equation. The asymptotic behavior of a second order fuzzy difference equation is delivered by Din [20]. The behavior of solutions to a fuzzy non linear difference equation are treated by Zhang et.al. [21]. Memarbashi and Ghasemabadi [22] solved fuzzy difference equation of Volterra type. A fuzzy difference equation of rational form were solved by Stefanidou and Papaschinopoulos [23]. The application of fuzzy difference equation in finance is considered by Konstantinos et.al. [24].

### 1.6. Motivation

Second order fuzzy difference equation is very important for modeling. Now if it is in fuzzy environment then the solution procedure is different. After taking  $\alpha$ -cut of a fuzzy difference equation it turns to system of fuzzy difference equation. Sometimes it is complicated to solve. But using Lagrange's multiplier it is very easy to solve them.

### 1.7. Novelties

In spite the above mention developments few developments can still be done by our self which are:

- (i) Difference equation is defined in fuzzy environment i.e., only initial conditions are fuzzy number, only coefficients are fuzzy number and both initial condition and coefficients are fuzzy number.
- (ii) Difference equation is solved with fuzzy initial condition i.e, initial condition is fuzzy number.
- (iii) The solutions are found by Lagrange's multiplier method.
- (iv) Proposed method is illustrated by two numerical example and two applications.

### 1.8. Structure of the paper

The structure of the paper is as follows: In first section we introduce the previous work on fuzzy set theory and fuzzy difference equation. Second section goes to preliminary concept. We define difference equation in third section. In fourth section we define fuzzy difference equation. In fifth section we solve second order fuzzy difference equation with initial value as fuzzy number by Lagrange's multiplier method. In sixth section the numerical examples are illustrated. The applications are given in seventh section. The conclusion is done in eighth section.

## 2 Preliminaries

### Definition 2.1

**Fuzzy Set:** A fuzzy set  $\tilde{A}$  is defined by  $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0,1]\}$ . In the pair  $(x, \mu_{\tilde{A}}(x))$  the first element  $x$  belong to the classical set  $A$ , the second element  $\mu_{\tilde{A}}(x)$ , belong to the interval  $[0, 1]$ , called Membership function.

### Definition 2.2

**$\alpha$ -cut of a fuzzy set:** The  $\alpha$ -level set (or interval of confidence at level  $\alpha$  or  $\alpha$ -cut) of the fuzzy set  $\tilde{A}$  of  $X$  is a crisp set  $A_\alpha$  that contains all the elements of  $X$  that have membership values in  $\tilde{A}$  greater than or equal to  $\alpha$  i.e.  $\tilde{A} = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X, \alpha \in [0,1]\}$ .

### Definition 2.3

**Fuzzy Number:** [25] A fuzzy number is fuzzy set like  $u: R \rightarrow I = [0,1]$  which satisfies

- (1)  $u$  is upper semi-continuous.
- (2)  $u(x) = 0$  outside the interval  $[c, d]$
- (3) There are real numbers  $a, b$  such  $c \leq a \leq b \leq d$  and
  - (3.1)  $u(x)$  is monotonic increasing on  $[c, a]$ ,
  - (3.2)  $u(x)$  is monotonic decreasing on  $[b, d]$ ,
  - (3.3)  $u(x) = 1, a \leq x \leq b$

Let  $E^1$  be the set of all real fuzzy numbers which are normal, upper semi-continuous, convex and compactly supported fuzzy sets.

### Definition 2.4

**Fuzzy Number (Parametric form):** [26] A fuzzy number  $u$  in a parametric form is a pair  $(u_1, u_2)$  of function  $u_1(r), u_2(r), 0 \leq r \leq 1$ , which satisfies the following requirments:

- (1)  $u_1(r)$  is a bounded monotonic increasing left continuous function,
- (2)  $u_2(r)$  is a bounded monotonic decreasing left continuous function,
- (3)  $u_1(r) \leq u_2(r), 0 \leq r \leq 1$ .

A crisp number  $x$  is simply represented by  $(u_1(r), u_2(r)) = (x, x), 0 \leq r \leq 1$ . By appropriate

definitions, the fuzzy number space  $\{(u_1(r), u_2(r))\}$  becomes a convex cone  $E^1$  which could be embedded isomorphically and isometrically into a Banach space [27, 28].

**Definition 2.5**

[29] Let  $x = (x_1(r), x_2(r)), y = (y_1(r), y_2(r)) \in E^1, 0 \leq r \leq 1$  and arbitrary  $k \in R$ .

Then

- (1)  $x = y$  iff  $x_1(r) = y_1(r)$  and  $x_2(r) = y_2(r)$ ,
- (2)  $x + y = (x_1(r) + y_1(r), x_2(r) + y_2(r))$ ,
- (3)  $x - y = (x_1(r) - y_2(r), x_2(r) - y_1(r))$ ,
- (4)  $kx = \begin{cases} (kx_1(r), kx_2(r)), & k \geq 0 \\ (kx_2(r), kx_1(r)), & k < 0 \end{cases}$

**Definition 2.6**

[30] For arbitrary  $u = (u_1, u_2), v = (v_1, v_2) \in E^1$ , the quantity

$$D(u, v) = \left( \int_0^1 (u_1 - v_1)^2 + \int_0^1 (u_2 - v_2)^2 \right)^{\frac{1}{2}}$$

Is the distance between fuzzy numbers  $u$  and  $v$ .

**Definition 2.7**

**Triangular Fuzzy Number:** A Triangular fuzzy number (TFN) denoted by  $\tilde{A}$  is defined as  $(a, b, c)$  where the membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \frac{x - a}{b - a}, & a \leq x \leq b \\ 1, & x = b \\ \frac{c - x}{c - b}, & b \leq x \leq c \\ 0, & x \geq c \end{cases}$$

**Definition 2.8**

**$\alpha$ -cut of a fuzzy set  $\tilde{A}$ :** The  $\alpha$ -cut of  $\tilde{A} = (a, b, c)$  is given by

$$A_\alpha = [a + \alpha(b - a), c - \alpha(c - b)], \forall \alpha \in [0, 1]$$

**Theorem 2.1**

Let  $I$  be an interval of real numbers, and let  $f: I \times I \rightarrow I$  be a continuous function.

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), n = 0, 1, \dots, \tag{2.1}$$

Where the initial values  $x_{-1}, x_0 \in I$ .

and  $f$  satisfies the following conditions:

- (1) There exist positive number  $a$  and  $b$  with  $a < b$  such that  $a \leq f(x, y) \leq b$  for all  $x, y \in [a, b]$ .
- (2)  $f(x, y)$  is increasing in  $x \in [a, b]$  for each  $y \in [a, b]$ , and  $f(x, y)$  is decreasing in  $y \in [a, b]$  for each  $x \in [a, b]$ .
- (3) (2.1) has no solutions of prime period two in  $[a, b]$ .

Then there exists exactly one equilibrium solution  $\bar{x}$  (1) which lies in  $[a, b]$ . Moreover, every solution of (2.1) with initial conditions  $x_{-1}, x_0 \in [a, b]$  converges to  $\bar{x}$ .

### 3 Difference equation

A  $q$ -th order linear difference equation (synonymously, a linear recurrence relation) is a set of equations of the form

$$x_n - (a_{n-1}x_{n-1} + a_{n-2}x_{n-2} + \dots + a_{n-q}x_{n-q}) = r_n \quad (3.2)$$

For  $n = q, q + 1, \dots$

If  $r_n = 0$ , for all  $n$ , the equation is said to be a homogeneous difference equation otherwise it is non homogeneous difference equation. The term  $r_n$  is called the forcing factor.

Now if  $a_i$  ( $i = 1, 2, \dots, n$ ) do not depend on  $n$  then the equation said to have constant, coefficients.

### 4 Fuzzy difference equation

Consider the second order non homogeneous difference equation as

$$a_0u_{n+2} + a_1u_{n+1} + a_2u_n = f(n) \quad (4.3)$$

With initial condition  $u_{n=0} = u_0$  and  $u_{n=1} = u_1$

The above second order difference equation is called fuzzy difference equation if any of one case is followed by the above difference equation:

- (1) The initial condition or conditions are fuzzy number (Type I)
- (2) The coefficient or coefficients are fuzzy number (Type II)
- (3) The initial condition or conditions and coefficient or coefficients are fuzzy numbers (Type III)

### 5 Exact solution of second order difference equation with fuzzy initial value

#### 5.1. Formulation of second order difference equation with fuzzy initial value

Consider the second order non homogeneous difference equation as

$$a_0u_{n+2} + a_1u_{n+1} + a_2u_n = f(n) \quad (5.4)$$

With initial condition  $\tilde{u}_{n=0} = \tilde{u}_0$  and  $\tilde{u}_{n=1} = \tilde{u}_1$

Different cases can be arising. If we take different sign of  $a_0, a_1$  and  $a_2$  then the different cases are as follows:

- (i)  $a_0 > 0, a_1 > 0, a_2 > 0$
- (ii)  $a_0 > 0, a_1 < 0, a_2 > 0$
- (iii)  $a_0 > 0, a_1 > 0, a_2 < 0$
- (iv)  $a_0 > 0, a_1 < 0, a_2 < 0$
- (v)  $a_0 < 0, a_1 > 0, a_2 > 0$
- (vi)  $a_0 < 0, a_1 < 0, a_2 > 0$
- (vii)  $a_0 < 0, a_1 > 0, a_2 < 0$
- (viii)  $a_0 < 0, a_1 < 0, a_2 < 0$

**Note:** In crisp sense the above all cases are equal. But in fuzzy sense they are not same.

For proving these we take the cases (iv) and (viii).

**Case (iv):** For case (iv) if we take the  $\alpha$ -cut the we get

$$a_0[u_{n+2,2}(\alpha), u_{n+2,1}(\alpha)] + a_1[u_{n+1,1}(\alpha), u_{n+1,2}(\alpha)] + a_2[u_{n,1}(\alpha), u_{n,2}(\alpha)] = [f(n), f(n)]$$

i.e.,

$$\begin{cases} a_0 u_{n+2,2}(\alpha) + a_1 u_{n+1,1}(\alpha) + a_2 u_{n,1}(\alpha) = f(n) \\ a_0 u_{n+2,1}(\alpha) + a_1 u_{n+1,2}(\alpha) + a_2 u_{n,2}(\alpha) = f(n) \end{cases} \quad (5.5)$$

**Case (viii):**

For case (viii) if we take the  $\alpha$ -cut the we get

$$a_0 [u_{n+2,1}(\alpha), u_{n+2,2}(\alpha)] + a_1 [u_{n+1,2}(\alpha), u_{n+1,1}(\alpha)] + a_2 [u_{n,2}(\alpha), u_{n,1}(\alpha)] = [f(n), f(n)]$$

i.e.,

$$\begin{cases} a_0 u_{n+2,1}(\alpha) + a_1 u_{n+1,2}(\alpha) + a_2 u_{n,2}(\alpha) = f(n) \\ a_0 u_{n+2,2}(\alpha) + a_1 u_{n+1,1}(\alpha) + a_2 u_{n,1}(\alpha) = f(n) \end{cases} \quad (5.6)$$

Clearly (5.5) and (5.6) are not same.

**5.2. Exact solution of second order difference equation with fuzzy initial value by Lagranges multiplier method**

There can be various cases. For simplicity of the paper we take only one case

Now consider a simple case:  $a_0 > 0$  and  $a_1, a_2 < 0$

Now taking the  $\alpha$ -cut of the above equation we get

$$a_0 [u_{n+2,1}(\alpha), u_{n+2,2}(\alpha)] + a_1 [u_{n+1,2}(\alpha), u_{n+1,1}(\alpha)] + a_2 [u_{n,2}(\alpha), u_{n,1}(\alpha)] = [f(n), f(n)]$$

Or,

$$a_0 u_{n+2,1}(\alpha) + a_1 u_{n+1,2}(\alpha) + a_2 u_{n,2}(\alpha) = f(n) \quad (5.7)$$

and

$$a_0 u_{n+2,2}(\alpha) + a_1 u_{n+1,1}(\alpha) + a_2 u_{n,1}(\alpha) = f(n) \quad (5.8)$$

With initial value

$$u_{0,1}(\alpha) = u_{01}(\alpha)$$

$$u_{0,2}(\alpha) = u_{02}(\alpha)$$

$$u_{1,1}(\alpha) = u_{11}(\alpha)$$

$$u_{1,2}(\alpha) = u_{12}(\alpha)$$

The above equations are system of difference equation

Multiplying the equation (5.8) by  $\lambda$  and adding with (5.7) we get

$$a_0 [u_{n+2,1}(\alpha) + \lambda u_{n+2,2}(\alpha)] + a_1 [u_{n+1,2}(\alpha) + \lambda u_{n+1,1}(\alpha)] + a_2 [u_{n,2}(\alpha) + \lambda u_{n,1}(\alpha)] = [f(n) + \lambda f(n)]$$

$$\text{Or, } a_0 [u_{n+2,1}(\alpha) + \lambda u_{n+2,2}(\alpha)] + \frac{a_1}{\lambda} [u_{n+1,1}(\alpha) + \frac{1}{\lambda} u_{n+1,2}(\alpha)] + \frac{a_2}{\lambda} [u_{n,1}(\alpha) + \frac{1}{\lambda} u_{n,2}(\alpha)] = [1 + \lambda]f(n) \quad (5.9)$$

Take  $\lambda = \frac{1}{\lambda}$  i.e.,  $\lambda = \pm 1$

When  $\lambda = 1$  the above equation becomes

$$a_0 [u_{n+2,1}(\alpha) + u_{n+2,2}(\alpha)] + a_1 [u_{n+1,1}(\alpha) + u_{n+1,2}(\alpha)] + a_2 [u_{n,1}(\alpha) + u_{n,2}(\alpha)] = 2f(n) \quad (5.10)$$

Or,

$$a_0 w_{n+2}(\alpha) + a_1 w_{n+1}(\alpha) + a_2 w_n(\alpha) = 2f(n)$$

Where,  $w_n(\alpha) = u_{n,1}(\alpha) + u_{n,2}(\alpha)$

When  $\lambda = -1$  the above equation becomes

$$a_0 [u_{n+2,1}(\alpha) - 1u_{n+2,2}(\alpha)] - a_1 [u_{n+1,1}(\alpha) - u_{n+1,2}(\alpha)] - a_2 [u_{n,1}(\alpha) - u_{n,2}(\alpha)] = 2f(n)$$

Or,

$$a_0x_{n+2}(\alpha) + a_1x_{n+1}(\alpha) + a_2x_n(\alpha) = 0 \tag{5.11}$$

Where,  $x_n(\alpha) = u_{n,1}(\alpha) - u_{n,2}(\alpha)$

Clearly it is easy to solve (5.10) and (5.11).

After solving the difference equation we can find  $w_n(\alpha)$  and  $x_n(\alpha)$ .

Then we can write

$$u_{n,1}(\alpha) = \frac{w_n(\alpha)+x_n(\alpha)}{2} \text{ and } u_{n,2}(\alpha) = \frac{w_n(\alpha)-x_n(\alpha)}{2}$$

The others cases can be done in similar way.

## 6 Numerical example

### Example 6.1

Let the difference equation is  $y_{n+2} - 4y_{n+1} + 3y_n = 0$  with fuzzy initial condition  $y_0 = (0,1,2)$  and  $y_1 = (4,6,8)$ . Then find  $y_n$ .

**Solution:** Taking  $\alpha$  cut of the difference equation

$$(y_{n+2,1}(\alpha), y_{n+2,2}(\alpha)) - 4(y_{n+1,2}(\alpha), y_{n+1,1}(\alpha)) + 3(y_{n,1}(\alpha), y_{n,2}(\alpha)) = (0,0) \tag{6.12}$$

i.e.,

$$y_{n+2,1}(\alpha) - 4y_{n+1,2}(\alpha) + 3y_{n,1}(\alpha) = 0 \tag{6.13}$$

$$y_{n+2,2}(\alpha) - 4y_{n+1,1}(\alpha) + 3y_{n,2}(\alpha) = 0 \tag{6.14}$$

Multiplying by  $\lambda$  in equation (6.14) and adding with equation (6.13) we have

$$(y_{n+2,1}(\alpha) - 4y_{n+1,2}(\alpha) + 3y_{n,1}(\alpha)) + \lambda(y_{n+2,2}(\alpha) - 4y_{n+1,2}(\alpha) + 3y_{n,2}(\alpha)) = 0$$

Or,

$$(y_{n+2,1}(\alpha) + \lambda y_{n+2,2}(\alpha)) - 4\lambda \left(\frac{1}{\lambda} y_{n+1,1}(\alpha) + y_{n+1,2}(\alpha)\right) + 3(y_{n,1}(\alpha) + \lambda y_{n,2}(\alpha)) = 0 \tag{6.15}$$

Taking  $\lambda = \frac{1}{\lambda}$  or,  $\lambda = \pm 1$

If  $\lambda = 1$  the above equation implies

$$(y_{n+2,1}(\alpha) + y_{n+2,2}(\alpha)) - 4(y_{n+1,1}(\alpha) + y_{n+1,2}(\alpha)) + 3(y_{n,1}(\alpha) + y_{n,2}(\alpha)) = 0 \tag{6.16}$$

Let  $y_{n,1}(\alpha) + y_{n,2}(\alpha) = x_n(\alpha)$

Then obviously  $y_{n+2,1}(\alpha) + y_{n+2,2}(\alpha) = x_{n+2}(\alpha)$  and  $y_{n+1,1}(\alpha) + y_{n+1,2}(\alpha) = x_{n+1}(\alpha)$

Therefore the equation (6.16) becomes

$$x_{n+2}(\alpha) - 4x_{n+1}(\alpha) + 3x_n(\alpha) = 0 \tag{6.17}$$

Put  $\lambda = -1$  in equation (6.15)

$$(y_{n+2,1}(\alpha) - y_{n+2,2}(\alpha)) + 4(-y_{n+1,1}(\alpha) + y_{n+1,2}(\alpha)) + 3(y_{n,1}(\alpha) - y_{n,2}(\alpha)) = 0 \tag{6.18}$$

Let  $y_{n,1}(\alpha) - y_{n,2}(\alpha) = u_n(\alpha)$

Then obviously  $y_{n+2,1}(\alpha) - y_{n+2,2}(\alpha) = u_{n+2}(\alpha)$  and  $y_{n+1,1}(\alpha) - y_{n+1,2}(\alpha) = u_{n+1}(\alpha)$

Therefore the equation (6.18) becomes

$$u_{n+2}(\alpha) - 4u_{n+1}(\alpha) + 3u_n(\alpha) = 0 \tag{6.19}$$

Solving (6.17) and (6.19) we can find  $x_n(\alpha)$  and  $u_n(\alpha)$

Now we find the value of  $y_{n,1}(\alpha), y_{n,2}(\alpha)$  from the above relation

$$\begin{cases} y_{n,1}(\alpha) = \frac{x_n(\alpha)+u_n(\alpha)}{2} \\ y_{n,2}(\alpha) = \frac{x_n(\alpha)-u_n(\alpha)}{2} \end{cases} \tag{6.20}$$

We have system of fuzzy difference equation (6.17) and (6.19) whose general solution is

$$x_n(\alpha) = c_1(\alpha)(1)^n + c_2(\alpha)(3)^n$$

$$u_n(\alpha) = c_3(\alpha)(1)^n + c_4(\alpha)(3)^n$$

Therefore the general solution is

$$y_{n,1}(\alpha) = \frac{1}{2}\{c_1(\alpha) + c_3(\alpha)\}(1)^n + \frac{1}{2}\{c_2(\alpha) + c_4(\alpha)\}(3)^n$$

$$y_{n,2}(\alpha) = \frac{1}{2}\{c_1(\alpha) - c_3(\alpha)\}(1)^n + \frac{1}{2}\{c_2(\alpha) - c_4(\alpha)\}(3)^n$$

Therefore using initial condition we get from (6.17) and (6.19) we get

$$x_{n,1}(\alpha) = (2\alpha - 1)(1)^n + (1 - \alpha)(3)^n$$

and

$$u_{n,2}(\alpha) = (2 + 4\alpha)(1)^n + (2 - 2\alpha)(3)^n$$

Therefore the general solution of the main equation is written as

$$y_{n,1}(\alpha) = \left(\frac{-4 + \alpha}{2}\right)(1)^n + \left(\frac{4 + \alpha}{2}\right)(3)^n$$

$$y_{n,2}(\alpha) = \left(\frac{-2 - \alpha}{2}\right)(1)^n + \left(\frac{6 - \alpha}{2}\right)(3)^n$$

Table 1: Solution for different values of  $n$

$n = 3$			$n = 5$		
$\alpha$	$y_{n,1}(\alpha)$	$y_{n,2}(\alpha)$	$\alpha$	$y_{n,1}(\alpha)$	$y_{n,2}(\alpha)$
0	52.0000	80.0000	0	484.0000	728.0000
0.1	53.4000	78.6000	0.1	496.2000	715.8000
0.2	54.8000	77.2000	0.2	508.4000	703.6000
0.3	56.2000	75.8000	0.3	520.6000	691.4000
0.4	57.6000	74.4000	0.4	532.8000	679.2000
0.5	59.0000	73.0000	0.5	545.0000	667.0000
0.6	60.4000	71.6000	0.6	557.2000	654.8000
0.7	61.8000	70.2000	0.7	569.4000	642.6000
0.8	63.2000	68.8000	0.8	581.6000	630.4000
0.9	64.6000	67.4000	0.9	593.8000	618.2000
1	66.0000	66.0000	1	606.0000	606.0000
$n = 6$			$n = 8$		
$\alpha$	$y_{n,1}(\alpha)$	$y_{n,2}(\alpha)$	$\alpha$	$y_{n,1}(\alpha)$	$y_{n,2}(\alpha)$
0	1456.0000	2186.0000	0	13120.0000	19682.0000
0.1	1492.5000	2149.5000	0.1	13448.1000	19353.9000
0.2	1529.0000	2113.0000	0.2	13776.2000	19025.8000
0.3	1565.5000	2076.5000	0.3	14104.3000	18697.7000
0.4	1602.0000	2040.0000	0.4	14432.4000	18369.6000
0.5	1638.5000	2003.5000	0.5	14760.5000	18041.5000
0.6	1675.0000	1967.0000	0.6	15088.6000	17713.4000
0.7	1711.5000	1930.5000	0.7	15416.7000	17385.3000
0.8	1748.0000	1894.0000	0.8	15744.8000	17057.2000
0.9	1784.5000	1857.5000	0.9	16072.9000	16729.1000
1	1821.0000	1821.0000	1	16401.0000	16401.0000



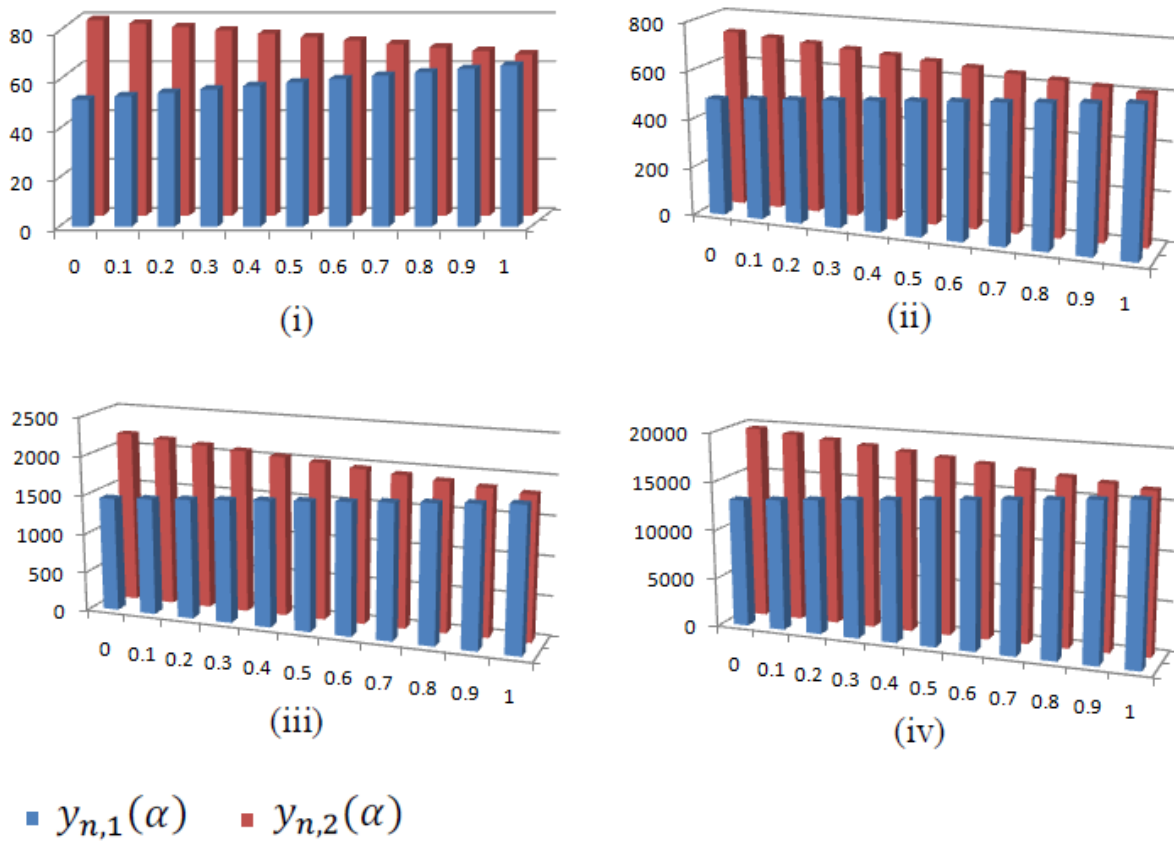


Figure 1: Figure of  $y_{n,1}(\alpha)$  and  $y_{n,2}(\alpha)$  for different values of  $n$

**Description and discussion on the table:** (i), (ii), (iii) and (iv) are the figure for  $y_{n,1}(\alpha)$  and  $y_{n,2}(\alpha)$  for  $n = 3, 5, 6$  and  $8$  respectively. Clearly for all fixed  $n$ ,  $y_{n,1}(\alpha)$  is increasing and  $y_{n,2}(\alpha)$  is a decreasing function as  $\alpha$  goes from 0 to 1. Hence the solution is a strong solution.

**Example 6.2**

Let the difference equation is  $y_{n+2} - 4y_n = 0$  with initial condition  $y_0 = (3, 5, 7)$  and  $y_1 = (10, 12, 15)$ . Then find  $y_n$ .

**Solution:** Taking  $\alpha$  cut of the difference equation

$$(y_{n+2,1}(\alpha), y_{n+2,2}(\alpha)) - 4(y_{n,2}(\alpha), y_{n,1}(\alpha)) = (0, 0) \tag{6.21}$$

which implies

$$y_{n+2,1}(\alpha) - 4y_{n,2}(\alpha) = 0 \tag{6.22}$$

and

$$y_{n+2,2}(\alpha) - 4y_{n,1}(\alpha) = 0 \tag{6.23}$$

Multiplying by  $\lambda$  in equation (6.23) and adding with (6.22) we get

$$(y_{n+2,1}(\alpha) + \lambda y_{n+2,2}(\alpha)) + (-4y_{n,2}(\alpha) - 4\lambda y_{n,1}(\alpha)) = 0$$

Or,

$$(y_{n+2,1}(\alpha) + \lambda y_{n+2,2}(\alpha)) + \lambda \left( \frac{-4}{\lambda} y_{n,2}(\alpha) - y_{n,1}(\alpha) \right) = 0 \tag{6.24}$$

Taking  $\lambda = \frac{1}{\lambda}$  i.e.,  $\lambda = +1, -1$

Now putting  $\lambda = 1$  in (6.24) we get

$$(y_{n+2,1}(\alpha) + y_{n+2,2}(\alpha)) - 4(y_{n,1}(\alpha) + y_{n,2}(\alpha)) = 0 \tag{6.25}$$

Putting  $\lambda = -1$  in (6.24) we get

$$(y_{n+2,1}(\alpha) - y_{n+2,2}(\alpha)) + (y_{n,1}(\alpha) - y_{n,2}(\alpha)) = 0 \tag{6.26}$$

Let  $u_n(\alpha) = y_{n,1}(\alpha) + y_{n,2}(\alpha)$  and  $x_n(\alpha) = y_{n,1}(\alpha) - y_{n,2}(\alpha)$  then (6.25) and (6.26) becomes

$$u_{n+2}(\alpha) - 4u_n(\alpha) = 0 \tag{6.27}$$

$$x_{n+2}(\alpha) + 4x_n(\alpha) = 0 \tag{6.28}$$

The general solution of (6.27) and (6.28) are

$$u_n(\alpha) = d_1(\alpha) \cos \frac{n\pi}{2} + d_2(\alpha) \sin \frac{n\pi}{2} \text{ and } x_n(\alpha) = d_3(\alpha) \cos \frac{n\pi}{2} + d_4(\alpha) \sin \frac{n\pi}{2}$$

Therefore the general solution of main equation is written as

$$y_{n,1}(\alpha) = \frac{u_n(\alpha) + x_n(\alpha)}{2} = 2^{n-1} \left[ \{d_1(\alpha) + d_3(\alpha)\} \cos \frac{n\pi}{2} + \{d_2(\alpha) + d_4(\alpha)\} \sin \frac{n\pi}{2} \right]$$

and

$$y_{n,2}(\alpha) = \frac{u_n(\alpha) - x_n(\alpha)}{2} = 2^{n-1} \left[ \{d_1(\alpha) - d_3(\alpha)\} \cos \frac{n\pi}{2} + \{d_2(\alpha) - d_4(\alpha)\} \sin \frac{n\pi}{2} \right]$$

Therefore the solution is

$$y_{n,1}(\alpha) = (3 + 2\alpha) \cos \frac{n\pi}{2} + (10 + 2\alpha) \sin \frac{n\pi}{2}$$

and

$$y_{n,2}(\alpha) = (7 - 2\alpha) \cos \frac{n\pi}{2} + (15 - 3\alpha) \sin \frac{n\pi}{2}$$

**Table 2:** Solution for different values of  $n$

$n = 3$			$n = 4$		
$\alpha$	$y_{n,1}(\alpha)$	$y_{n,2}(\alpha)$	$\alpha$	$y_{n,1}(\alpha)$	$y_{n,2}(\alpha)$
0	1.1927	4.2500	0	8.7381	12.3979
0.1	1.3544	4.1059	0.1	8.8731	12.1670
0.2	1.5160	3.9619	0.2	9.0082	11.9360
0.3	1.6777	3.8178	0.3	9.1432	11.7051
0.4	1.8394	3.6737	0.4	9.2783	11.4742
0.5	2.0010	3.5297	0.5	9.4133	11.2432
0.6	2.1627	3.3856	0.6	9.5484	11.0123
0.7	2.3244	3.2415	0.7	9.6834	10.7813
0.8	2.4860	3.0975	0.8	9.8184	10.5504
0.9	2.6477	2.9534	0.9	9.9535	10.3195
1	2.8094	2.8094	1	10.0885	10.0885
$n = 5$			$n = 6$		
$\alpha$	$y_{n,1}(\alpha)$	$y_{n,2}(\alpha)$	$\alpha$	$y_{n,1}(\alpha)$	$y_{n,2}(\alpha)$
0	-9.0233	-15.3603	0	-0.6519	1.3672
0.1	-9.3060	-15.0093	0.1	-0.5336	1.2836
0.2	-9.5887	-14.6583	0.2	-0.4153	1.2000
0.3	-9.8714	-14.3073	0.3	-0.2970	1.1163
0.4	-10.1540	-13.9563	0.4	-0.1787	1.0327
0.5	-10.4367	-13.6052	0.5	-0.0605	0.9491
0.6	-10.7194	-13.2542	0.6	0.0578	0.8655
0.7	-11.0021	-12.9032	0.7	0.1761	0.7818
0.8	-11.2848	-12.5522	0.8	0.2944	0.6982
0.9	-11.5675	-12.2012	0.9	0.4127	0.6146
1	-11.8502	-11.8502	1	0.5310	0.5310

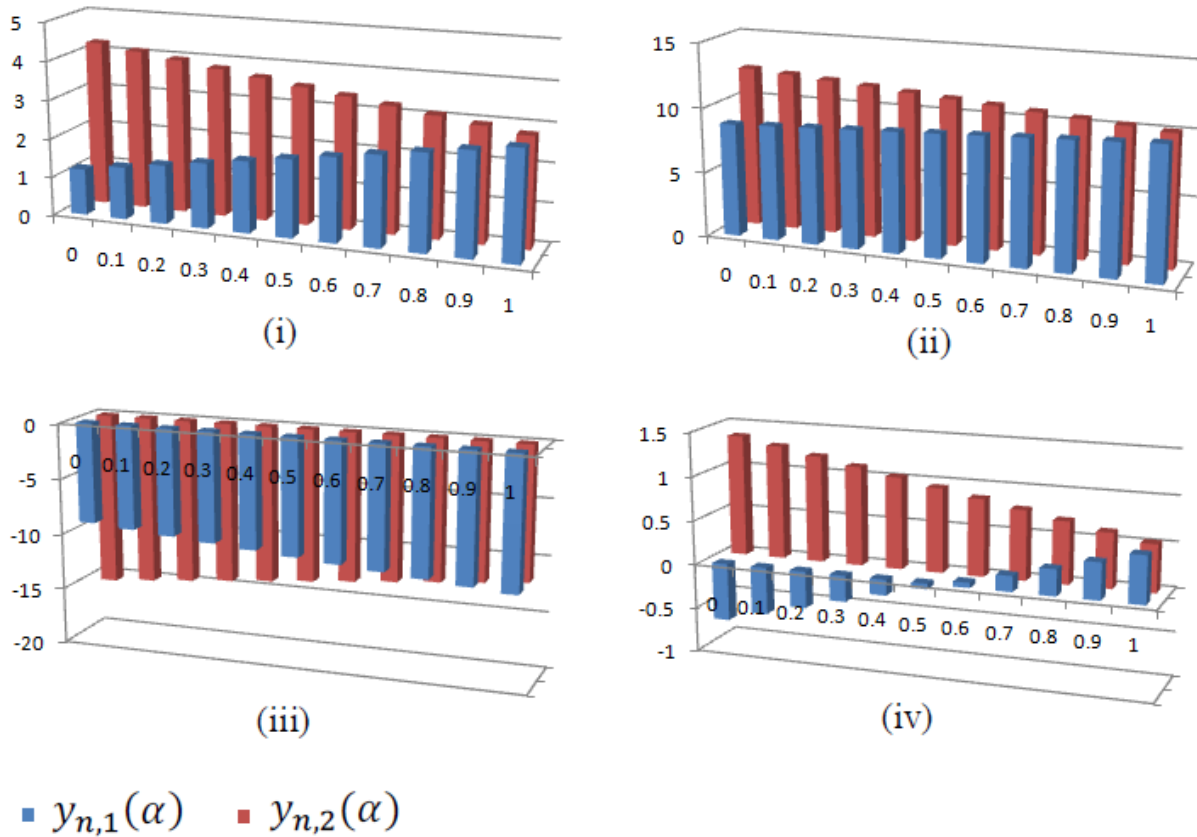


Figure 2: Figure of  $y_{n,1}(\alpha)$  and  $y_{n,2}(\alpha)$  for different values of  $n$

**Description and discussion on the table:** (i), (ii), (iii) and (iv) are the figure for  $y_{n,1}(\alpha)$  and  $y_{n,2}(\alpha)$  for  $n = 3, 4, 5$  and  $6$  respectively. Clearly for all fixed  $n$ ,  $y_{n,1}(\alpha)$  is increasing and  $y_{n,2}(\alpha)$  is a decreasing function as  $\alpha$  goes from 0 to 1. Hence the solution is a strong solution.

## 7 Application

**Application 1:** In a new colony of geese there are 10 pairs of bird home of which procreate eggs in their first year. In each next year, pair of birds which are in there are in second or latter year have, on average 4 eggs (2 male or female). Assuming no death is in the problem.

The difference equation which describe the geese population is

$$u_{t+1} = u_t + 2u_{t-1}$$

Where  $u_t$  represents the geese population (in pairs at the beginning of the  $t$ -th year)

**Application 2:** A pair of hares required a maturation period of one month before they can make off spring. Each pair of mature hares presents at end of one month produce two new pairs by the end of next month. If  $u_t$  indicate the number of pairs alive at the end of  $n$ th month and no hares die,  $u_t$  satisfies the difference equation  $u_t = u_{t-1} + 2u_{t-2}$

Solve the both problem (Application 1 and Application 2) with initial condition as.  $u_1 = \tilde{u}_1 = (0, 2, 4)$  and  $u_2 = \tilde{u}_2 = (6, 8, 10)$ .

**Solution:** Before going to the both both application we first solve the general solution of

$$u_{t+1} = u_t + 2u_{t-1} \text{ with initial condition } u_1 = \tilde{u}_1, u_2 = \tilde{u}_2 \quad (7.29)$$

Taking  $\alpha$  cut of the difference equation is

$$(u_{t+1,1}(\alpha), u_{t+1,2}(\alpha)) = (u_{t,1}(\alpha), u_{t,2}(\alpha)) + 2(u_{t-1,1}(\alpha), u_{t-1,2}(\alpha))$$

i.e.,

$$u_{t+1,1}(\alpha) - u_{t,1}(\alpha) - 2u_{t-1,1}(\alpha) = 0 \quad (7.30)$$

$$u_{t+1,2}(\alpha) - u_{t,2}(\alpha) - 2u_{t-1,2}(\alpha) = 0 \quad (7.31)$$

We have from equation (7.30)

$$u_{t+1,1}(\alpha) - u_{t,1}(\alpha) - 2u_{t-1,1}(\alpha) = 0$$

The auxiliary equation is  $m^2 - m - m = 0$ , whose roots are  $m = -1, 2$  (Distinct real roots)

So, the general solution is

$$u_{t,1}(\alpha) = c_1(-1)^t + c_2(2)^t$$

Using initial condition we have

$$u_{11}(\alpha) = c_1 + c_2$$

$$u_{21}(\alpha) = -c_1 + 2c_2$$

$$u_{t,1}(\alpha) = \frac{2u_{11}(\alpha) - u_{21}(\alpha)}{3}(-1)^t + \frac{u_{11}(\alpha) + u_{21}(\alpha)}{3}(2)^t$$

From equation (7.31)

$$u_{t+1,2}(\alpha) - u_{t,2}(\alpha) - 2u_{t-1,2}(\alpha) = 0$$

The auxiliary equation is  $m^2 - m - m = 0$ , whose roots are  $m = -1, 2$  (Distinct real roots)

The general solution is written as

$$u_{t,2}(\alpha) = c_3(-1)^t + c_4(2)^t$$

Using initial condition

$$u_{12}(\alpha) = c_3 + c_4$$

$$u_{22}(\alpha) = -c_3 + 2c_4$$

Solving we get

$$u_{t,2}(\alpha) = \frac{2u_{12}(\alpha) - u_{22}(\alpha)}{3}(-1)^t + \frac{u_{12}(\alpha) + u_{22}(\alpha)}{3}(2)^t$$

The required solution of fuzzy difference equation is

$$u_{t,1}(\alpha) = \frac{2u_{11}(\alpha) - u_{21}(\alpha)}{3}(-1)^t + \frac{u_{11}(\alpha) + u_{21}(\alpha)}{3}(2)^t$$

$$u_{t,2}(\alpha) = \frac{2u_{12}(\alpha) - u_{22}(\alpha)}{3}(-1)^t + \frac{u_{12}(\alpha) + u_{22}(\alpha)}{3}(2)^t$$

**Numerical:** The solutions are

$$u_{t,1}(\alpha) = \alpha(-1)^t + (4 + \alpha)(2)^t$$

$$u_{t,2}(\alpha) = (2 - \alpha)(-1)^t + (6 - \alpha)(2)^t$$

Table 3: Solution for different values of  $t$

$t = 4$			$t = 6$		
$\alpha$	$u_{t,1}(\alpha)$	$u_{t,2}(\alpha)$	$\alpha$	$u_{t,1}(\alpha)$	$u_{t,2}(\alpha)$
0	64.0000	98.0000	0	256.0000	386.0000
0.1	65.7000	96.3000	0.1	262.5000	379.5000
0.2	67.4000	94.6000	0.2	269.0000	373.0000
0.3	69.1000	92.9000	0.3	275.5000	366.5000
0.4	70.8000	91.2000	0.4	282.0000	360.0000
0.5	72.5000	89.5000	0.5	288.5000	353.5000
0.6	74.2000	87.8000	0.6	295.0000	347.0000
0.7	75.9000	86.1000	0.7	301.5000	340.5000
0.8	77.6000	84.4000	0.8	308.0000	334.0000
0.9	79.3000	82.7000	0.9	314.5000	327.5000
1	81.0000	81.0000	1	321.0000	321.0000
$t = 8$			$t = 10$		
$\alpha$	$u_{t,1}(\alpha)$	$u_{t,2}(\alpha)$	$\alpha$	$u_{t,1}(\alpha)$	$u_{t,2}(\alpha)$
0	1024.0000	1538.0000	0	4096.0000	6146.0000
0.1	1049.7000	1512.3000	0.1	4198.5000	6043.5000
0.2	1075.4000	1486.6000	0.2	4301.0000	5941.0000
0.3	1101.1000	1460.9000	0.3	4403.5000	5838.5000
0.4	1126.8000	1435.2000	0.4	4506.0000	5736.0000
0.5	1152.5000	1409.5000	0.5	4608.5000	5633.5000
0.6	1178.2000	1383.8000	0.6	4711.0000	5531.0000
0.7	1203.9000	1358.1000	0.7	4813.5000	5428.5000
0.8	1229.6000	1332.4000	0.8	4916.0000	5326.0000
0.9	1255.3000	1306.7000	0.9	5018.5000	5223.5000
1	1281.0000	1281.0000	1	5121.0000	5121.0000

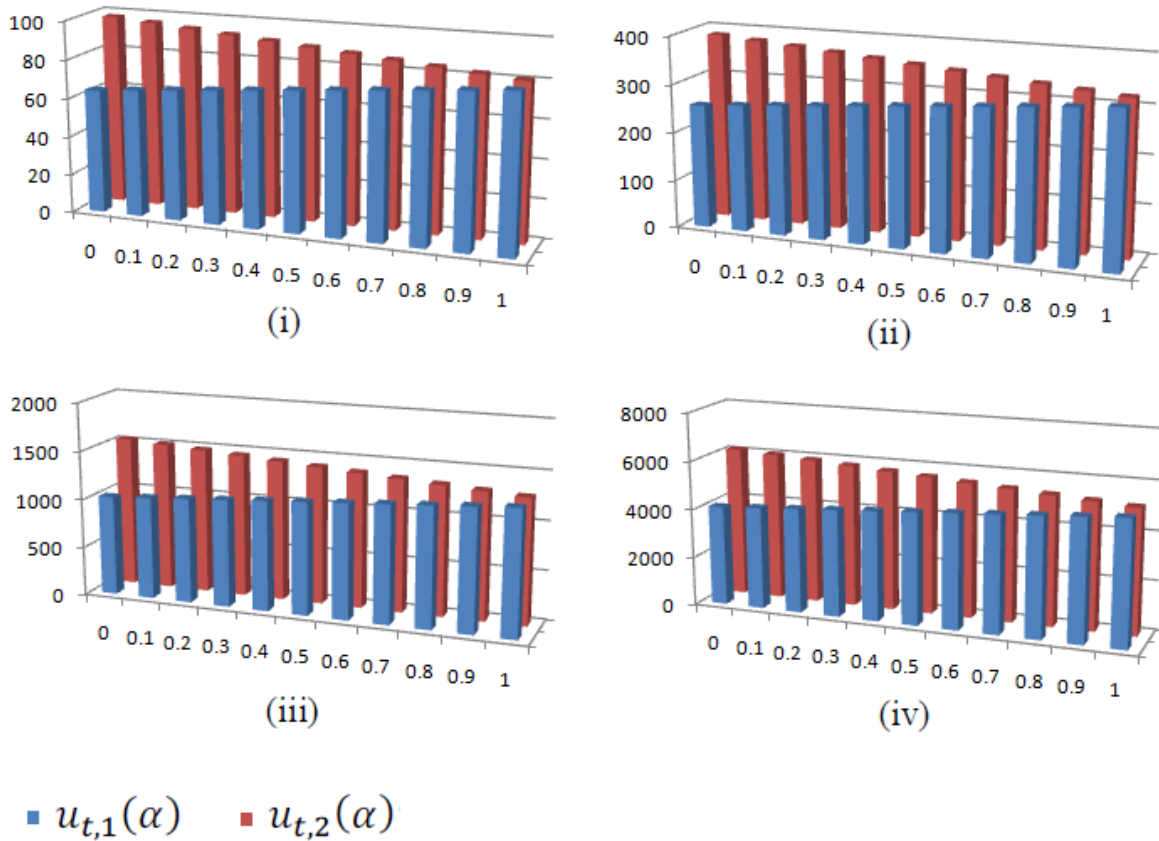


Figure 3: Figure of  $u_{t,1}(\alpha)$  and  $u_{t,2}(\alpha)$  for different values of  $t$

**Description and discussion the figure and table:** (i), (ii), (iii) and (iv) are the figure for  $u_{t,1}(\alpha)$  and  $u_{t,2}(\alpha)$  for  $t = 4, 6, 8$  and  $10$  respectively. Clearly for all fixed  $t$ ,  $u_{t,1}(\alpha)$  is increasing and  $u_{t,2}(\alpha)$  is a decreasing function as  $\alpha$  goes from 0 to 1. Hence the solution is a strong solution.

### 8 Conclusion

Difference equations are more important for modeling when it is applicable in various problems in different fields. It is also important if it is study in fuzzy environment. When it can be study in fuzzy environment the behavior of it changes. In this paper we execute the solution procedure for second order linear fuzzy difference equation by Lagrange’s multiplier method. In crisp sense the difference equation are easy to solve, but when we take in fuzzy sense it transforms a system of difference equation. Using Lagrange’s multiplier we convert the system of difference equation to another two difference equation which is easy to solve. The problems are illustrated by two different numerical examples and followed by two applications. The theory and results are surely very important for researcher.

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