On the solution of second-order system under uncertainty

M. Taheri Lari*
Department of Management, Mashhad Branch, Islamic Azad University, Mashhad, Iran.

Abstract
Recently, Zhang et. al [15] proposed two interesting results which assure the global existence of solutions for fuzzy second-order differential equations with initial conditions under generalized H-differentiability. However, proof of Theorem 3.2 [15] needs some improvements. This paper provide these improvements.

Keywords: Global existence of solutions, Fuzzy second-order differential equations, Generalized H-differentiability.

1 Main results
Recently, investigating the solutions of fuzzy differential equations is considered under generalized H-differentiability in several papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 1, 2, 13, 14]. So, considering the global existence of solutions is necessary. Zhand et. al [15] have obtained two results which assure the existence of solutions of fuzzy second-order differential equations of the form

\[
\begin{aligned}
x''(t) &= f(t,x,x'), \\
x(t_0) &= k_1, \\
x'(t_0) &= k_2,
\end{aligned}
\]

under generalized H-differentiability where \(f \in C[\mathcal{F} \times R_\mathcal{F} \times R_\mathcal{F}](\mathbb{R}_\mathcal{F} \text{ is called the space of fuzzy numbers}).\) However, one of the proposed result (Theorem 3.2 in [15]) needs some improvements. Firstly, let us consider the mentioned result.

**Theorem 1.1. (Theorem 3.2 in [15]).** Assume that

1. \(f(t,x,x')\) is locally Lipschitzian in \(x,x'\) for \((t,x,0) \in \mathcal{F} \times R_\mathcal{F} \times R_\mathcal{F} (\mathcal{F} = [t_0, \infty)).\)
2. \(g \in C[\mathcal{F} \times [0, \infty] \times [0, \infty]],\) \(g(t,u,v)\) is nondecreasing in \(u,v \geq 0\) for each \(t \in J,\) and the maximal solution \(r(t,t_0,u_0,v_0)\) of the scalar initial value problem

\[
u'' = g(t,u,u'), \quad u(t_0) = u_0, \quad u'(t_0) = u'_0,
\]

exists throughout \(J.\)
3. \(D(f(t,x,x'),0) \leq g(t,D(x,0),D(x,0)), \forall (t,x,x') \in \mathcal{F} \times R_\mathcal{F} \times R_\mathcal{F}.\)
4. \(D(x(t,t_0,x_0',0),0) \leq r(t,t_0,x_0,x_0'), D(x_0,0) \leq u_0 \) and \(D(x_0',0) \leq u'_0,\) then the largest interval of existence of any solution \(x(t,t_0,u_0,v_0)\) of (1.1) with \(D(x,0) \leq u_0\) is \(\mathcal{F}.\) In addition, if \(r(t,t_0,u_0,u_0')\) is bounded on \(\mathcal{F},\) then \(\lim_{t \to +\infty} x(t,t_0,x_0',0)\) exists in \((\mathbb{R}_\mathcal{F},D).\)

*Corresponding author. Email address: masoudtaherilari@yahoo.com, Tel: +98 9153049276.
In order to prove Theorem 1.1, the authors in [15] have used the Dini derivative. However, the first part of theorem should modify as follows (notice that we used the same notations and also just provide some parts that should be improved. The reminder of proof is completely similar to the original).

Let \( m(t) = D(x(t, t_0, x_0, x'_0), \hat{0}) \), \( t_0 \leq t < \beta \). Then, we have two following cases:

1. Assume that \( x' \) is \((i)\)-differentiable, then there exist \( H \)-difference \( x'(t + h, t_0, x_0, x'_0) \ominus x'(t, t_0, x_0, x'_0) \), \( x'(t, t_0, x_0, x'_0) \ominus x'(t - h, t_0, x_0, x'_0) \) for \( h > 0 \) sufficiently small and also we have

\[
D_+ m'(t) = \liminf_{h \to 0} \frac{D(x'(t + h, t_0, x_0, x'_0), \hat{0}) - D(x'(t, t_0, x_0, x'_0), \hat{0})}{h} 
\leq \liminf_{h \to 0} \frac{D(x'(t + h, t_0, x_0, x'_0), x'(t, t_0, x_0, x'_0))}{h} 
= \liminf_{h \to 0} D \left( \frac{x'(t + h, t_0, x_0, x'_0) \ominus x'(t, t_0, x_0, x'_0)}{h}, \hat{0} \right) 
= D(x''(t, t_0, x_0, x'_0), \hat{0}) 
= D(f(t, x(t, t_0, x_0, x'_0), x'(t, t_0, x_0, x'_0)), \hat{0}) 
\leq g(t, m(t), m'(t)), \quad t \leq t_0 < \beta. 
\]

Also, we have

\[
D_- m'(t) = \liminf_{h \to 0} \frac{D(x'(t, t_0, x_0, x'_0), \hat{0}) - D(x'(t - h, t_0, x_0, x'_0), \hat{0})}{h} 
\leq \liminf_{h \to 0} \frac{D(x'(t, t_0, x_0, x'_0), x'(t - h, t_0, x_0, x'_0))}{h} 
= \liminf_{h \to 0} D \left( \frac{x'(t, t_0, x_0, x'_0) \ominus x'(t - h, t_0, x_0, x'_0)}{h}, \hat{0} \right) 
= D(x''(t, t_0, x_0, x'_0), \hat{0}) 
= D(f(t, x(t, t_0, x_0, x'_0), x'(t - h, t_0, x_0, x'_0)), \hat{0}) 
\leq g(t, m(t), m'(t)), \quad t \leq t_0 < \beta. 
\]

One can continue according to the proof given in [15].

2. Assume that \( x' \) is \((ii)\)-differentiable, then there exist \( H \)-difference \( x'(t, t_0, x_0, x'_0) \ominus x'(t + h, t_0, x_0, x'_0) \), \( x'(t - h, t_0, x_0, x'_0) \ominus x'(t, t_0, x_0, x'_0) \) for \( h > 0 \) sufficiently small and also we have

\[
D_+ m'(t) = \liminf_{h \to 0} \frac{D(x'(t, t_0, x_0, x'_0), \hat{0}) - D(x'(t + h, t_0, x_0, x'_0), \hat{0})}{-h} 
\leq \liminf_{h \to 0} \frac{D(x'(t, t_0, x_0, x'_0), x'(t + h, t_0, x_0, x'_0))}{-h} 
= \liminf_{h \to 0} D \left( \frac{x'(t, t_0, x_0, x'_0) \ominus x'(t + h, t_0, x_0, x'_0)}{-h}, \hat{0} \right) 
= D(x''(t, t_0, x_0, x'_0), \hat{0}) 
= D(f(t, x(t, t_0, x_0, x'_0), x'(t + h, t_0, x_0, x'_0)), \hat{0}) 
\leq g(t, m(t), m'(t)), \quad t \leq t_0 < \beta. 
\]
Also, we have

\[
D_ + m'(t) = \liminf_{h \to +0} \frac{D(x'(t-h,t_0,x_0,x_0'), \mathbb{0}) - D(x'(t,t_0,x_0,x_0'), \mathbb{0})}{-h} \\
\leq \liminf_{h \to +0} \frac{D(x'(t-h,t_0,x_0,x_0'), x'(t-t_0,x_0,x_0'), \mathbb{0})}{-h} \\
= \liminf_{h \to +0} D\left(\frac{x'(t-h,t_0,x_0,x_0')}{-h}, \mathbb{0}\right) \\
= D(x''(t,t_0,x_0,x_0'), \mathbb{0}) \\
\leq g(t, m(t), m'(t)), \ t \leq t_0 < \beta.
\]

One can continue according to the proof given in [15].

Please note that, for the case of (iii)- or (iv)-differentiability, the proofs reduce to the crisp cases. Since such type of differentiability valid for real-valued functions [6]. So we omit the proof of these last parts.

At the end, we emphasize that the results are interesting, but some improvement are done.

References

http://dx.doi.org/10.1016/j.camwa.2004.03.009

http://dx.doi.org/10.1016/j.ins.2008.11.030


http://dx.doi.org/10.1016/j.ins.2008.11.004

http://dx.doi.org/10.1016/j.fss.2004.08.001

http://dx.doi.org/10.1016/S0165-0114(98)00141-9


http://dx.doi.org/10.1016/j.na.2005.05.020
http://dx.doi.org/10.1016/0165-0114(90)90010-4


http://dx.doi.org/10.1016/j.nahs.2009.06.013


http://dx.doi.org/10.1016/j.na.2005.05.068

http://dx.doi.org/10.1016/j.camwa.2010.06.038