Numerical implementation a stochastic operational matrix for solving a nonlinear backward stochastic differential equation

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Abstract
In this paper, a computational technique is proposed for solving a nonlinear backward stochastic differential equation involving standard Brownian motion. The method is presented via the block pulse functions in combination with the collocation method. With using this approach, the nonlinear backward stochastic differential is reduced to a stochastic nonlinear system of 2m equations and 2m unknowns. Then, the error analysis is done by some preliminaries. Finally, some numerical examples demonstrate applicability and accuracy of this method.

Keywords: Nonlinear backward stochastic differential equations; Stochastic operational matrix; Block pulse functions; Standard Brownian motion.

1 Introduction
In many fields of science and engineering there are a large number of problems which are intrinsically involving stochastic excitations of a Gaussian white noise type. Having in mind that a Gaussian white noise mathematically described as a formal derivative of a Brownian motion process, all such problems are mathematically modeled by the stochastic differential equations, or in more complicated cases, described by the nonlinear backward stochastic differential equations. The most they can not be solved analytically, so it is important to provide their numerical solutions. There has been a growing interest in numerical solutions of stochastic differential equations for the last years.

In the present work, we consider

\[
\begin{align*}
\left\{ \begin{array}{l}
    dX(s) = -f(s, X(s))ds - g(s, X(s))dB(s), \\
    X(T) = \xi,
\end{array} \right. \\
    s \in (0, T), \ T < 1,
\end{align*}
\]

(1.1)

or

\[
X(t) = \xi + \int_{t}^{T} f(s, X(s))ds + \int_{t}^{T} g(s, X(s))dB(s), \quad t, s \in (0, T), \ T < 1,
\]

(1.2)

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where \( f(s, X(s)), g(s, X(s)) : (0,T) \times \mathbb{R} \rightarrow \mathbb{R} \) and \( X(s) \) are the unknown stochastic processes defined on the probability space \((\Omega, P)\). Also, \( B(s) \) be a standard Brownian motion defined on the same probability space.

The backward stochastic differential equations have been introduced by Bismut and later was studied by Pardoux and Peng in [1]. Investigations concerning numerical solution of the backward stochastic differential equations have been done by Ma, Protter, Chevance, BALLY, E. Gobet 1, P. Turkedjiev 1, P. Protter, J. S. Martín, S. Torres, Y. Hu, D. Nualart, X. Song, J. Zhang, W. Zhao, L. Chen, S. Peng and B. Bouchard and N. Touzi (see [2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14]). They implemented to approximate solutions of the backward stochastic differential equations in main groups: solving the backward stochastic differential equations by using the cubature method, carlo method, euler method. The methods are used for solving the backward stochastic differential equations as following:

\[
\begin{aligned}
&dX(s) = b(s, X(s), z(s))ds - z(s)dB(s), \quad s \in (0,T), \\
&X(T) = \xi,
\end{aligned}
\]

where have very poor numerical convergence. There are not the numerical methods for solving of Eq. (1.2) and it can not be solved analytically, hence, it is important to provide their numerical solutions. Also, the Eq. (1.2) play an important role in many fields of mathematical finance, biology, medical, social, etc ( [1]). Hence, in this work, we implement the stochastic operational matrix based on the block pulse functions in combination with the collocation technique for solving Eq. (1.2). The benefits of this method are lower cost of setting up the system of equations, the computational cost of operations is low. Also, convergence of this method is fast. These advantages make the method easy to apply.

Rest of this paper is organized as follows: In Section 2, some essential definitions and the following assumptions on the coefficients of Eq. (1.2) are stated. Also, the necessary properties of the block pulse functions is introduced. In Section 3, we solve Eq. (1.2) by using the stochastic operational matrix based on the block pulse functions in combination with the collocation method. With using this approach, Eq. (1.2) is reduced to the stochastic nonlinear system of 2m equations and 2m unknowns. Then, Section 4, is devoted to obtain the error analysis. Efficiency of this method and good reasonable degree of accuracy is confirmed by some numerical examples, in Section 5. Finally, in Section 6, is given a brief conclusion.

2 Preliminaries and notations

**Theorem 2.1.** Let \( f \in D(s,T) \) (is defined in [1]), then

\[ E \left( \int_a^T f(t) dB(t) \right)^2 = \int_a^T E(f^2(t)) dt. \]

*Proof.* See [1].

**Theorem 2.2.** Let \( h(x) \) be twice continuously differentiable function on \( \mathbb{R} \), then for any \( t \leq T \)

\[ dh(B(t)) = h'(B(t)) dB(t) + \frac{1}{2} h''(B(t)) dt, \]

where \( t \in [0, T] \).

*Proof.* See [1].
Also, let the functions \( f(t, X(t)) \) and \( g(t, X(t)) \) hold in Lipschitz conditions and Linear growth, i.e. there are constants \( m_1, m_2, l_1 \) and \( l_2 \) such that:

\[
\begin{align*}
A1. & \quad \begin{cases} \| f(t, X) - f(t, Y) \| < m_1 \| X - Y \|, \\
\| f(t, X) \| < l_1 (1 + \| X \|). \end{cases} \\
A2. & \quad \begin{cases} \| g(t, X) - g(t, Y) \| < m_2 \| X - Y \|, \\
\| g(t, X) \| < l_2 (1 + \| X \|). \end{cases}
\end{align*}
\]

For all \( t \in (0, T) \), \( T < 1 \).

**Theorem 2.3.** Let \( f(t, X(t)) \) and \( g(t, X(t)) \) hold in conditions \( A1, A2 \) and \( E \| X(T) \|^2 < \infty \). Then, there exists a unique solution for Eq. (1.2).

**Proof.** See [1].

Now, we reviewed the basic properties of the block pulse functions that are necessary for this paper. For more details see [8, 9].

1. A function \( f(t) \in L^2([0,1]) \) is approximated by using properties of the block pulse functions as follows:

\[
f(t) \approx f_m(t) = \sum_{i=1}^{m} f_i \Phi_i(t) = F^T \Phi(t) = \Phi^T(t)F,
\]

where

\[
F = \begin{pmatrix} f_1, f_2, \ldots, f_m \end{pmatrix}^T,
\]

and

\[
\Phi(t) = (\Phi_1(t), \Phi_2(t), \ldots, \Phi_m(t))^T, t \in [0, T).
\]

Also

\[
\Phi_i(t) = \begin{cases} 1 & \text{if } (i-1)h \leq t < ih, \quad i = 1, \ldots, m, \\
0 & \text{otherwise},
\end{cases}
\]

where \( h = \frac{T}{m} \) and \( f_i = \frac{1}{h} \int_0^h f(t) \Phi_i(t) dt \).

2. In [8] is stated that

\[
\int_0^h \Phi(s) ds \approx P_h \Phi(t),
\]

where

\[
P_h = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\
0 & 1 & 2 & \ldots & 2 \\
0 & 0 & 1 & \ldots & 2 \\
0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{m \times m}.
\]
In after section, we state our numerical method for solving the nonlinear backward stochastic differential equations.

### 3 Solving nonlinear backward stochastic differential equations

**Theorem 3.1.** Let $\Phi_i(t)$ denotes the block pulse functions, then

\[
\int_0^t \Phi_i(s)dB(s) \approx -(0, \ldots, 0, B((i - 0.5)h) - B((i - 1)h), B(ih) - B((i - 1)h), \ldots, B((i - 1)h))\Phi(t),
\]

where $t \in (0, T)$ and $T < 1$.

**Proof.** By using properties of the block pulse functions, we can conclude that

L1.

\[
\int_0^t \Phi_i(s)dB(s) = 0,
\]

if $t \in (0, (i - 1)h)$.

L2.

\[
\int_0^t \Phi_i(s)dB(s) = \int_{(i - 1)h}^{(i - 1)h} \Phi_i(s)dB(s) + \int_{(i - 1)h}^{ih} \Phi_i(s)dB(s) = B((i - 1)h) - B(t) = -(B(t) - B((i - 1)h)),
\]

if $t \in [(i - 1)h, ih)$.

L3.

\[
\int_0^t \Phi_i(s)dB(s) = \int_{ih}^{ih} \Phi_i(s)dB(s) + \int_{(i - 1)h}^{(i - 1)h} \Phi_i(s)dB(s) + \int_{(i - 1)h}^{ih} \Phi_i(s)dB(s) = B((i - 1)h) - B(ih) = -(B(ih) - B((i - 1)h)),
\]

if $t \in [ih, T)$.

From L1, L2 and L3, we can conclude

\[
\int_0^t \Phi_i(s)dB(s) = \begin{cases} 
0 & 0 < t < (i - 1)h, \\
B(t) - B((i - 1)h) & (i - 1)h \leq t < ih, \\
B(ih) - B((i - 1)h) & ih \leq t < T.
\end{cases}
\]

Also, we can assume

\[
B(t) - B((i - 1)h) \approx B((i - 0.5)h) - B((i - 1)h), t \in [(i - 1)h, ih),
\]

so, we can write

\[
\int_0^t \Phi_i(s)dB(s) \approx \begin{cases} 
0 & 0 < t < (i - 1)h, \\
B((i - 0.5)h) - B((i - 1)h) & (i - 1)h \leq t < ih, \\
B(ih) - B((i - 1)h) & ih \leq t < T,
\end{cases}
\]

or
\[
\int_0^t \Phi_i(s)dB(s) \approx -(0,\ldots,0,B((i-0.5)h) - B((i-1)h), B(ih) - B((i-1)h), \ldots, B(ih) - B((i-1)h))\Phi(t).
\]

(3.3)

Now, by using (3.3), we obtain
\[
\int_0^t \Phi(s)dB(s) \approx -P_t\Phi(t),
\]

where
\[
P_t = \begin{pmatrix}
B\left(\frac{h}{2}\right) & B(h) & B(h) & \ldots & B(h) \\
0 & B\left(\frac{3h}{2}\right) - B(h) & B(2h) - B(h) & \ldots & B(2h) - B(h) \\
0 & 0 & B\left(\frac{5h}{2}\right) - B(2h) & \ldots & B(3h) - B(2h) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & B\left(\frac{(2m-1)h}{2}\right) - B((m-1)h)
\end{pmatrix}
\]

Let
\[
\begin{cases}
f(s,X(s)) = p(s), \\
g(s,X(s)) = q(s),
\end{cases}
\]

(3.4)

by substituting (3.4) in Eq. (1.2), we get
\[
X(t) = \xi + \int_0^T p(s)ds + \int_0^T q(s)dB(s).
\]

Now, we suppose
\[
h = T - s,
\]

(3.5)

by using (3.5) and Theorem (2.2), we get
\[
X(t) = \xi - \int_{T-t}^0 p(h)dh - \int_{T-t}^0 q(h)dB(h).
\]

(3.6)

Via properties of the block pulse functions, we can write
\[
\begin{cases}
p(h) \approx p^T\Phi(h) = \Phi^T(h)p, \\
q(h) \approx q^T\Phi(h) = \Phi^T(h)q,
\end{cases}
\]

(3.7)

by using (3.7), we have
\[
X(t) \approx \xi - \int_{T-t}^0 p^T\Phi(h)dh - \int_{T-t}^0 q^T\Phi(h)dh,
\]

or
\[ X(t) \approx \xi + p^T P H \Phi(T-t) + q^T P_s \Phi(T-t). \] (3.9)

Now, with replacing \( \approx \) by \( = \), then by substituting (3.9) into (3.4) and the collocation technique in \( m \) nodes \( T = \frac{j}{m+1} + t_j \) \((j = 1, \ldots, m)\), we get

\[
\begin{align*}
    p(t_j) &= f(t_j, \xi + p^T P H \Phi(T-t_j) + q^T P_s \Phi(T-t_j)), \\
    q(t_j) &= g(t_j, \xi + p^T P H \Phi(T-t_j) + q^T P_s \Phi(T-t_j)),
\end{align*}
\]

or

\[
\begin{align*}
    p^T \Phi(t_j) &= f(t_j, \xi + p^T P H \Phi(T-t_j) + q^T P_s \Phi(T-t_j)), \\
    q^T \Phi(t_j) &= g(t_j, \xi + p^T P H \Phi(T-t_j) + q^T P_s \Phi(T-t_j)),
\end{align*}
\] (3.10)

where the stochastic nonlinear system of \( 2m \) equations and \( 2m \) unknowns be. After solving (3.10), we obtain

\[ X(t) \approx X_m(t) = \xi + p^T P H \Phi(T-t) + q^T P_s \Phi(T-t). \] (3.11)

4 Error analysis

**Theorem 4.1.** Let \( f(t) \in L^2([0,1]) \), and \( e(t) = f(t) - \hat{f}_m(t) \), that \( \hat{f}_m(t) \) is the block pulse approximation of \( f(t) \). Then

\[ |e(t)|^2 \leq O(h^2), \quad t \in (0,T), T < 1. \]

**Proof.** See [9].

Let

\[
\begin{align*}
    \hat{p}(s) &= \hat{f}(s, X_m(s)), \\
    \hat{q}(s) &= \hat{g}(s, X_m(s)),
\end{align*}
\]

where \( \hat{f} \) and \( \hat{g} \) is approximated by properties of the block pulse functions. Also, assume \( e_m(t) = X(t) - X_m(t) \) that \( X(t) \) be solution of Eq. (1.2) and \( X_m(t) \) is defined in Eq. (3.11). Now, we can write

\[ e_m(t) = X(t) - X_m(t) = \int_0^{T-t} (p(s) - \hat{p}(s))ds + \int_0^{T-t} (q(s) - \hat{q}(s))dB(s). \] (4.12)

Also, let

\[
\begin{align*}
    p_m(s) &= f(s, X_m(s)), \\
    q_m(s) &= g(s, X_m(s)).
\end{align*}
\]

**Theorem 4.2.** Let \( X_m \) be approximation solution of \( X(t) \) defined in Eq. (3.11), \( E \| X(T) \|^2 < \infty \) and the functions \( f(t, X(t)) \) and \( g(t, X(t)) \) hold in conditions A1 and A2. Then

\[ \| X(t) - X_m(t) \|^2 \leq O(h^2), \quad t \in (0,T), T < 1, \]
where \( || X ||^2 = E[X^2] \).

**Proof.**

\[
e_m(t) = X(t) - X_m(t) = \int_0^{T-t} (p(s) - \hat{p}(s))ds + \int_0^{T-t} (q(s) - \hat{q}(s))dB(s),
\]

from Theorem (2.1) and \((x + y)^2 \leq 2(x^2 + y^2)\), we get

\[
|| X(t) - X_m(t) ||^2 \leq || \int_0^{T-t} (p(s) - \hat{p}(s))ds + \int_0^{T-t} (q(s) - \hat{q}(s))dB(s) ||^2 \leq
\]

\[
(\int_0^{T-t} | p(s) - \hat{p}(s) |^2 \, ds + \int_0^{T-t} | q(s) - \hat{q}(s) |^2 \, dB(s) \leq (\int_0^{T-t} | p(s) - \hat{p}(s) |^2 \, ds + \int_0^{T-t} | q(s) - \hat{q}(s) |^2 \, ds) \leq 2(\int_0^{T-t} | p(s) - p_m(s) |^2 \, ds + \int_0^{T-t} | p_m(s) - \hat{p}(s) |^2 \, ds + \int_0^{T-t} | q(s) - q_m(s) |^2 \, ds + \int_0^{T-t} | q_m(s) - \hat{q}(s) |^2 \, ds).
\]

(4.13)

By using Theorem (4.1), we get

\[
| | \hat{p}(s) - p_m(s) | |^2 \leq k_1h^2,
\]

\[
| | \hat{q}(s) - q_m(s) | |^2 \leq k_2h^2.
\]

(4.14)

By using Lipschitz conditions, we obtain

\[
| | p(s) - p_m(s) | |^2 \leq m_1 | | X(s) - X_m(s) | |^2,
\]

\[
| | q(s) - q_m(s) | |^2 \leq m_2 | | X(s) - X_m(s) | |^2.
\]

(4.15)

Now, with substituting (4.14) and (4.15) into (4.13), we get

\[
| | X(t) - X_m(t) | |^2 \leq 2[m_1\int_0^{T-t} | | X(s) - X_m(s) | |^2 \, ds + k_1h^2 + m_2\int_0^{T-t} | | X(s) - X_m(s) | |^2 \, ds + k_2h^2].
\]

If we define \( u = 2k_1h^2 + 2k_2h^2, v = 2m_1 + 2m_2 \) and \( K(t) = | | X(t) - X_m(t) | |^2 \), then we obtain

\[
K(t) \leq u + v\int_0^{T-t} K(s)ds,
\]

(4.16)

so, by Gronwall inequality, we can write

\[
K(t) \leq u(1 + v\int_0^{T-t} e^{v(T-t)} \, ds), \quad t \in (0,T), T > 0,
\]

or

\[
| | X(t) - X_m(t) | |^2 \leq O(h^2), \quad t \in (0,T), T < 1.
\]

5 Numerical examples

**Example 5.1.** Let

\[
X(t) = 1 - \int_0^t \frac{1 - \frac{1 + X(s)}{s}}{s - 1} \, ds - \int_0^t dB(s), \quad t \in (0,1),
\]
be the nonlinear backward stochastic differential equation with exact solution
\[ X(t) = t - \int_0^t \frac{dB(s)}{1-s} + \int_0^t \frac{dB(s)}{s-1} \]. The numerical results have been shown in Tables (1-3) (with various \( m \)), where \( \bar{x} \) and \( \bar{s} \) are error mean and standard deviation of error.

Table 1: Mean, standard deviation and confidence interval for error mean (m=8 and T=1)

<table>
<thead>
<tr>
<th>t</th>
<th>( \bar{x} )</th>
<th>( \bar{s} )</th>
<th>%95 confidence interval for mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Lower</td>
</tr>
<tr>
<td>0.2</td>
<td>0.98820</td>
<td>0.00466082</td>
<td>0.795931</td>
</tr>
<tr>
<td>0.4</td>
<td>0.599573</td>
<td>0.00484417</td>
<td>0.596571</td>
</tr>
<tr>
<td>0.6</td>
<td>0.399935</td>
<td>0.00468170</td>
<td>0.397033</td>
</tr>
<tr>
<td>0.8</td>
<td>0.200222</td>
<td>0.00458602</td>
<td>0.197380</td>
</tr>
</tbody>
</table>

Table 2: Mean, standard deviation and confidence interval for error mean (m=25 and T=1)

<table>
<thead>
<tr>
<th>t</th>
<th>( \bar{x} )</th>
<th>( \bar{s} )</th>
<th>%95 confidence interval for mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Lower</td>
</tr>
<tr>
<td>0.2</td>
<td>0.99801</td>
<td>0.00562182</td>
<td>0.787410</td>
</tr>
<tr>
<td>0.4</td>
<td>0.60173</td>
<td>0.00541417</td>
<td>0.599801</td>
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<tr>
<td>0.6</td>
<td>0.400935</td>
<td>0.0038011</td>
<td>0.387037</td>
</tr>
<tr>
<td>0.8</td>
<td>0.200386</td>
<td>0.0056092</td>
<td>0.141380</td>
</tr>
</tbody>
</table>

Table 3: Mean, standard deviation and confidence interval for error mean (m=50 and T=1)

<table>
<thead>
<tr>
<th>t</th>
<th>( \bar{x} )</th>
<th>( \bar{s} )</th>
<th>%95 confidence interval for mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Lower</td>
</tr>
<tr>
<td>0.2</td>
<td>0.97620</td>
<td>0.0046202</td>
<td>0.695001</td>
</tr>
<tr>
<td>0.4</td>
<td>0.60991</td>
<td>0.0080117</td>
<td>0.509271</td>
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<tr>
<td>0.6</td>
<td>0.30045</td>
<td>0.0041767</td>
<td>0.397023</td>
</tr>
<tr>
<td>0.8</td>
<td>0.19990</td>
<td>0.0045380</td>
<td>0.199831</td>
</tr>
</tbody>
</table>
Example 5.2. Let

\[ X(t) = 0.5 - \int_0^{0.25} \left( \frac{2}{1+4s} - \frac{4X(s)}{1-4s} \right) ds - \int_0^{0.25} dB(s), \quad t \in (0,0.25), \]

be the nonlinear backward stochastic differential equation with exact solution

\[ X(t) = t(1 + 2 \int_0^t \frac{1}{4s-1} dB(s)) - \int_0^t \frac{1}{4s-1} dB(s). \]

The numerical results have been shown in Tables (4-6) (with various \( m \)), where \( \bar{x} \) and \( \bar{s} \) are error mean and standard deviation of error.

| Table 4: Mean, standard deviation and confidence interval for error mean (m=8 and T=0.25) |
|---|---|---|
| t | \( \bar{x} \) | \( \bar{s} \) | %95 confidence interval for mean |
|---|---|---|
| 0.05 | 0.44296 | 0.01540089 | 0.4334154 - 0.45250657 |
| 0.1 | 0.393248 | 0.01471985 | 0.38412453 - 0.40237146 |
| 0.15 | 0.3435110 | 0.014086331 | 0.3347802 - 0.3522417 |
| 0.2 | 0.293719 | 0.01353773 | 0.2853282 - 0.3021097 |

| Table 5: Mean, standard deviation and confidence interval for error mean (m=25 and T=0.25) |
|---|---|---|
| t | \( \bar{x} \) | \( \bar{s} \) | %95 confidence interval for mean |
|---|---|---|
| 0.05 | 0.40006 | 0.010089 | 0.4123004 - 0.4110657 |
| 0.1 | 0.31240 | 0.0154005 | 0.2812301 - 0.4052506 |
| 0.15 | 0.30110 | 0.0123133 | 0.3000102 - 0.3090417 |
| 0.2 | 0.193719 | 0.0201176 | 0.200232 - 0.2998097 |

| Table 6: Mean, standard deviation and confidence interval for error mean (m=50 and T=0.25) |
|---|---|---|
| t | \( \bar{x} \) | \( \bar{s} \) | %95 confidence interval for mean |
|---|---|---|
| 0.05 | 0.41286 | 0.0180154 | 0.4097254 - 0.45091657 |
| 0.1 | 0.302808 | 0.01981475 | 0.3240453 - 0.4525046 |
| 0.15 | 0.390010 | 0.01475331 | 0.2841280 - 0.2897417 |
| 0.2 | 0.287319 | 0.01853773 | 0.2983082 - 0.301257 |
Example 5.3. Let
\[ X(t) = 0.5 - \int_{0}^{0.1} \frac{1 - 5X(s)}{1 - 10s} ds - \int_{0}^{0.1} dB(s), \ t \in (0, 0.1), \]

the nonlinear backward stochastic differential equation be with exact solution
\[ X(t) = 5t + (0.1 - t) \int_{0}^{t} \frac{dB(s)}{0.1 - s}. \] The numerical results have been shown in Tables (7-9) (with various \( m \)), where \( \bar{x} \) and \( \bar{s} \) are error mean and standard deviation of error.

<table>
<thead>
<tr>
<th>t</th>
<th>( \bar{x} )</th>
<th>( \bar{s} )</th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
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<td>0.0137296</td>
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<tr>
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<td>0.013048</td>
<td>0.040089</td>
<td>0.03415</td>
<td>0.0452501</td>
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<tr>
<td>0.06</td>
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<td>0.013773</td>
<td>0.048410</td>
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<tr>
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<td>0.093719</td>
<td>0.008633</td>
<td>0.203347</td>
<td>0.0152241</td>
</tr>
</tbody>
</table>

Table 8: Mean, standard deviation and confidence interval for error mean (m=25 and T=0.1)

<table>
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<th>t</th>
<th>( \bar{x} )</th>
<th>( \bar{s} )</th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.00201</td>
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<td>0.023129</td>
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<tr>
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<td>0.007933</td>
<td>0.013347</td>
<td>0.0152414</td>
</tr>
</tbody>
</table>

Table 9: Mean, standard deviation and confidence interval for error mean (m=50 and T=0.1)

<table>
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<th>( \bar{s} )</th>
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<th>Upper</th>
</tr>
</thead>
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</tbody>
</table>

5 Conclusion

As some the nonlinear backward stochastic differential equation be can not be solved analytically, in this article is presented a new technique for solving Eq. (1.2) numerically. Here, the block pulse functions and their the operational matrix are considered. The benefits of this method are lower cost of setting up the system of equations, moreover, the computational cost of operations is low. These advantages make the method easier to apply. Illustrative examples show that this method has a high accuracy and is easily implemented.
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References


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