Dynamics and stability of Hilfer-Hadamard type fractional differential equations with boundary conditions

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Abstract
In this paper, we investigate the existence, uniqueness and Ulam stabilities of solutions for Hilfer-Hadamard type fractional differential equations with boundary conditions in weighted spaces of continuous functions. The existence results rely on Schaefer’s fixed point theorem. The Banach contraction principle is also considered to obtain uniqueness and stability results. An example is provided to illustrate the usefulness of the obtained results.

Keywords: Boundary value problem; Hilfer-Hadamard fractional derivative; Existence; Ulam stability; Fixed point.

1 Introduction
In this paper, we prove the existence, uniqueness and Ulam stabilities analysis of solutions of Hilfer-Hadamard type fractional differential equations with boundary conditions of the form

\[
\begin{align*}
\{ \mathlr{H}^{\alpha, \beta}_{t^\gamma} x(t) &= f(t, x(t)), & t \in J := [1, T], \\
I_{1^+}^{\gamma} x(1) &= a, & I_{1^+}^{\gamma} x(T) &= b,
\end{align*}
\]

(1.1)

where \( \mathlr{H}^{\alpha, \beta}_{t^\gamma} \) is the Hilfer-Hadamard fractional derivative, \( 0 < \alpha < 1, 0 \leq \beta \leq 1 \) and let \( X \) be a Banach space, \( f : J \times X \rightarrow X \) is given continuous function.

It is seen that equation (1.1) is equivalent to the integral equation

\[
x(t) = \frac{a}{\Gamma(\gamma)} (\log t)^{\gamma-1} + \left( b - a - \frac{1 - \beta}{1 + \beta} (f(T, x(T))) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{(\log t)^{\gamma + 2\beta - 2}}{(\log T)^{2\beta - 1}} \\
+ \frac{1}{\Gamma(\alpha)} \int_1^t \left( \frac{\log s}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s},
\]

(1.2)

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. There has been a significant development in fractional differential (FDEs) and partial differential equations in recent years; see the monographs of of Hilfer [21], Kilbas [24] and Podlubny [26]. There are some works on FDEs with Hadamard fractional derivative, even if it has been studied many

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years ago (see for example [2, 6, 7]). Recently, several works reporting Hilfer type of equations have been published. See [2, 6, 7] for more examples and remarks concerning Hilfer fractional derivative.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces. Thereafter, this type of stability is called the Ulam-Hyers stability [5, 13, 14, 17]. In 1978, Rassias [17] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The stability properties of all kinds of equations have attracted the attention of many mathematicians. In particular, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability have been taken up by a number of mathematicians and the study of this area has developed to be one of the central subjects in the mathematical analysis area. For more details on the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of differential equations, see [13, 19, 28, 29].

Recently, Kassim et. al. [16] investigated well-posedness and stability for a differential equations with Hilfer-Hadamard fractional derivative. In this paper we mainly focus on developing the theory of dynamics and stability for FDEs via Hilfer-Hadamard derivative. The problem of the existence of solutions for FDEs with boundary conditions has been recently treated in the literature in [2, 4, 8, 9, 20, 18].

The rest of this paper is organized as follows. In Section 2, we give some basic definitions and results concerning the Hilfer-Hadamard fractional derivative. In Section 3, we present our main result by using Schaefer’s fixed point theorem. In section 4, we introduce four types of Ulam stability definitions for FDEs: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. We present the four types of Ulam stability results for FDEs (1.1).

2 Prerequisites

This section is devoted to basic definitions and lemmas from [1, 10, 11, 15, 21, 22, 23] the theory of Hilfer fractional derivative which are used in subsequent sections.

Definition 2.1. Let $C[J, X]$ denotes the Banach space of continuous function on $[1, T]$ with the norm

$$
\|x\|_C := \sup \{x(t) : t \in J\}.
$$

We denote $L^1 \{R_+ \}$, the space of Lebesgue integrable functions on $J$. By $C_{\gamma, \log}[J, X]$ and $C_1[\gamma, \log][J, X]$, we denote the weighted spaces of continuous functions defined by

$$
C_{\gamma, \log}[J, X] := \{f(t) : J \to X | (\log t)^{\gamma}f(t) \in C[J, X] \},
$$

with norm

$$
\|f\|_{C_{\gamma, \log}} = \| (\log t)^{\gamma}f(t) \|_C,
$$

and

$$
\|f\|_{C_1[\gamma, \log]} = \sum_{k=0}^{n-1} \| f^k \|_C + \| f^n \|_{C_{\gamma, \log}}, \quad n \in \mathbb{N}.
$$

Moreover, $C_0_{\gamma, \log}[J, X] := C_{\gamma, \log}[J, X]$.

Now, we give some results and properties of Hadamards fractional calculus.
Definition 2.2. [2, 7] The Hadamard fractional integral of order $\alpha$ for a function $h$ is defined as

$$I_{1+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{h(s)}{s} ds, \quad \alpha > 0,$$

provided the integral exists.

Notice that for all $\alpha, \alpha_1, \alpha_2 > 0$ and each $h \in C[J, X]$, we have $I_{1+}^\alpha h \in C[J, X]$, and

$$(I_{1+}^\alpha I_{1+}^\beta h)(t) = (I_{1+}^{\alpha+\beta} h)(t); \quad \text{for a.e. } t \in J.$$

Definition 2.3. [2, 7] The Hadamard derivative of fractional order $\alpha$ for a function $h : [1, \infty) \rightarrow X$ is defined as

$$H D_{1+}^\alpha h(t) = \frac{d}{dt} \left( \frac{d}{dt} \right)^{n-\alpha-1} \frac{h(s)}{s^n} ds, \quad n-1 < \alpha < n, \quad n = \lfloor \alpha \rfloor + 1,$$

where $\lfloor \alpha \rfloor$ denotes the integer part of real number $\alpha$ and $\log(\cdot) = \log_x(\cdot)$.

Let $\alpha \in (0, 1]$, $\gamma \in (0, 1)$ and $h \in C_{1-\gamma, log}[J, X]$. Then the following expression leads to the left inverse operator as follows.

$$(H D_{1+}^\alpha I_{1+}^\beta h)(t) = h(t); \quad \text{for all } t \in [1, b].$$

Moreover, if $I_{1+}^{1-\alpha} h \in C_{1-\gamma, log}[J, X]$, then the following composition

$$(I_{1+}^\alpha, H D_{1+}^\alpha) h(t) = h(t) - \frac{(I_{1+}^{1-\alpha} h)(1^+)}{1-\alpha} \left( \log t \right)^{\alpha-1}; \quad \text{for all } t \in [1, b].$$

In [21], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see also [22, 15]).

Definition 2.4. (Hilfer-Hadamard derivative)

Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $h \in L^1 \{R_+\}$, $I_{1+}^{1-\alpha}(1-\beta) \in C_{1-\gamma, log}[J, X]$. The Hilfer-Hadamard fractional derivative of order $\alpha$ and type $\beta$ of $h$ is defined as

$$(H D_{1+}^{\alpha, \beta} h)(t) = \left( I_{1+}^{1-\alpha} \frac{d}{dt} \right)^{1-\beta} I_{1+}^{\alpha} h(t); \quad \text{for a.e. } t \in J.$$  \hspace{1cm} (2.3)

Properties: Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha \beta$, and $h \in L^1 \{R_+\}$.

1. The operator $(H D_{1+}^{\alpha, \beta} h)(t)$ can be written as

$$(H D_{1+}^{\alpha, \beta} h)(t) = \left( I_{1+}^{1-\alpha} \frac{d}{dt} I_{1+}^{\gamma} h \right)(t) = \left( I_{1+}^{\beta(1-\alpha)} H D_{1+}^{\alpha} h \right)(t); \quad \text{for a.e. } t \in J.$$

Moreover, the parameter $\gamma$ satisfies

$$0 < \gamma \leq 1, \quad \gamma \geq \alpha, \quad \gamma > \beta, \quad 1 - \gamma < 1 - \beta(1 - \alpha).$$

2. The generalization (2.3) for $\beta = 0$, coincides with the Hadamard Riemann-Liouville derivative and for $\beta = 1$ with the Hadamard Caputo derivative.

$$H D_{1+}^{\alpha, 0} = H D_{1+}^{\alpha}, \quad \text{and} \quad H D_{1+}^{\alpha, 1} = \mathcal{C} D_{1+}^{\alpha}.$$

3. If $H D_{1+}^{\beta(1-\alpha)} h$ exists and in $L^1 \{R_+\}$, then

$$(H D_{1+}^{\alpha, \beta} I_{1+}^{\beta(1-\alpha)} h)(t) = \left( I_{1+}^{\beta(1-\alpha)} H D_{1+}^{\beta(1-\alpha)} h \right)(t); \quad \text{for a.e. } t \in J.$$

Furthermore, if $h \in C_{\gamma, log}[J, X]$ and $I_{1+}^{1-\beta(1-\alpha)} h \in C_{\gamma, log}[J, X]$, then

$$H D_{1+}^{\alpha, \beta} I_{1+}^{\beta(1-\alpha)} h(t) = h(t); \quad \text{for a.e. } t \in J.$$
Lemma 2.2. \textit{Let }$x \in \mathbb{R}$\textit{, then equations (2.4)-(2.5) hold.}
3 Existence and uniqueness results

In this section, we obtain the existence and uniqueness of solutions to the problem (1.1). For this, let us make the following conditions.

(C1) $f : J \times X \to X$ is continuous function.

(C2) There exists a constant $L > 0$ such that

$$|f(t, u) - f(t, \overline{u})| \leq L|u - \overline{u}|,$$

for any $u, \overline{u} \in X$, and $t \in J$.

(C3) The function $f : J \times X \to X$ is completely continuous and there exists a function $\mu(t) \in F^1 \{R_+ \}$ such that

$$|f(t, u)| \leq \mu(t), \quad t \in J, \quad u \in X.$$

(C4) There exists an increasing function $\varphi \in C_{1-\gamma, \log}[J, X]$ and there exists $\lambda \varphi > 0$ such that for any $t \in J$

$$\int_1^t \varphi(t) \leq \lambda \varphi(t).$$

Theorem 3.1. (Existence of solution) Assume that the conditions (C1), (C3) are satisfied. Then the problem (1.1) has at least one solution defined on $J$.

Proof. Consider the operator $P : C_{1-\gamma, \log}[J, X] \to C_{1-\gamma, \log}[J, X]$ defined by

$$(Px)(t) = \frac{a}{\Gamma(\gamma)} (\log t)^{\gamma - 1} + \left( b - a - I_{\alpha}^{1-\beta(1-\alpha)} f(T, x(T)) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} (\log t)^{\gamma + 2\beta - 2} + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s, x(s)) \frac{ds}{s}.$$  \hspace{1cm} (3.6)

It is obvious that the operator $P$ is well defined.

Step 1: $P$ is continuous.

Let $x_n$ be a sequence such that $x_n \to x$ in $C_{1-\gamma, \log}[J, X]$. Then for each $t \in J$,

$$\|(Px_n)(t) - (Px)(t)\| (\log t)^{1-\gamma} \leq \frac{\left( \frac{a}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} |f(s, x_n(s)) - f(s, x(s))| \frac{ds}{s} \right)}{\Gamma(2\beta)} \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \left( \frac{1}{\Gamma(\gamma + 2\beta - 1)} \int_1^T \left( \log \frac{T}{s} \right)^{1-\beta(1-\alpha)} |f(s, x_n(s)) - f(s, x(s))| \frac{ds}{s} \right) (\log t)^{2\beta - 1}.$$  \hspace{1cm} (3.6)

Since $f$ is continuous, then we have

$$\|P x_n - P x\|_{C_{1-\gamma, \log}} \to 0 \quad \text{as} \quad n \to \infty.$$  \hspace{1cm} (3.6)

Step 2: $P$ maps bounded sets into bounded sets in $C_{1-\gamma, \log}[J, X]$.

Indeed, it is enough to show that for $q > 0$, there exists a positive constant $l$ such that $x \in B_q \{ x \in C_{1-\gamma, \log}[J, X] : \|x\| \leq q \}$. 

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we have $\| (Px) \|_{C_{1-\gamma} \log} \leq l$.

\[
\begin{align*}
\| (Px)(t) (\log t)^{1-\gamma} \| &\leq \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \left( b - a - \frac{1}{\Gamma(1 - \beta(1 - \alpha))} \int_1^T \left( \log \frac{T}{s} \right)^{(1-\beta(1-\alpha))^{-1}} \| f(s, x(s)) \| ds \right) \frac{(\log t)^{2\beta - 1}}{(\log T)^{2\beta - 1}} \\
&\quad + \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} |f(s, x(s))| ds \frac{1}{s} \\
&\leq \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} (b - a) + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \left( \frac{(\log T)^{1-\gamma + \alpha}}{\Gamma(\alpha + 1)} - \frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2 - \beta(1-\alpha))} \right) \| \mu \|_{C_{1-\gamma} \log} \\
&=: l.
\end{align*}
\]

**Step 3:** $P$ maps bounded sets into equicontinuous set of $C_{1-\gamma} \log [J, X]$.

Let $t_1, t_2 \in J, t_1 < t_2, B_q$ be a bounded set of $C_{1-\gamma} \log [J, X]$ as in Step 2, and $x \in B_q$. Then,

\[
\begin{align*}
\| (Px)(t_2) - (Px)(t_1) \| &\leq \frac{a}{\Gamma(\gamma)} \left( (\log t_2)^{\gamma - 1} - (\log t_1)^{\gamma - 1} \right) \\
&\quad + \left( b - a - \int_{t_1}^{t_2} f(T, x(T)) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \left[ \frac{(\log t_2)^{\gamma + 2\beta - 2} - (\log t_1)^{\gamma + 2\beta - 2}}{(\log T)^{2\beta - 1}} \right] \\
&\quad + \left( \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left( \log \frac{t_2}{s} \right)^{\alpha - 1} f(s, x(s)) \frac{ds}{s} \right) - \left( \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left( \log \frac{t_1}{s} \right)^{\alpha - 1} f(s, x(s)) \frac{ds}{s} \right) \\
&\quad \leq \frac{a}{\Gamma(\gamma)} \left( (\log t_2)^{\gamma - 1} - (\log t_1)^{\gamma - 1} \right) \\
&\quad + \left( b - a - \frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2 - \beta(1-\alpha))} \| \mu \|_{C_{1-\gamma} \log} \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \left[ \frac{(\log t_2)^{\gamma + 2\beta - 2} - (\log t_1)^{\gamma + 2\beta - 2}}{(\log T)^{2\beta - 1}} \right] \\
&\quad + \left( \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left( \log \frac{t_2}{s} \right)^{\alpha} \| \mu \|_{C_{1-\gamma} \log} \right) - \left( \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left( \log \frac{t_1}{s} \right)^{\alpha} \| \mu \|_{C_{1-\gamma} \log} \right) \frac{ds}{s}.
\end{align*}
\]

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of step 1 to 3, together with Arzela-Ascoli theorem, we can conclude that $P : C_{1-\gamma} \log [J, X] \to C_{1-\gamma} \log [J, X]$ is continuous and completely continuous.

**Step 4:** A priori bounds.

Now it remains to show that the set

$$
\omega = \{ x \in C_{1-\gamma} \log [J, X] : x = \delta(Px), \ 0 < \delta < 1 \}
$$

is bounded set. Let $x \in \omega, x = \delta(Px)$ for some $0 < \delta < 1$. Thus for each $t \in J$. We have,

$$
x(t) = \delta \left[ \frac{a}{\Gamma(\gamma)} (\log t)^{\gamma - 1} + \left( b - a - \int_{t_1}^{t_2} f(T, x(T)) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{(\log t)^{\gamma + 2\beta - 2}}{(\log T)^{2\beta - 1}} \\
+ \left( \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left( \log \frac{t_2}{s} \right)^{\alpha} f(s, x(s)) \frac{ds}{s} \right) \right].
$$
This implies by (C3) that for each $t \in J$, we have
\[
\left| x(t) \right| (\log t)^{1-\gamma} \\
\leq \left| (Px)(t) \right| (\log t)^{1-\gamma} \\
\leq \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \left( b - a - \frac{1}{\Gamma(2 - (1-\beta)(1-\alpha))} \right) \int_{1}^{T} \left( \log \frac{T}{s} \right)^{(1-\beta(1-\alpha)) - 1} |f(s, x(s))| \frac{ds}{s} \frac{(\log t)^{2\beta - 1}}{(\log T)^{2\beta - 1}} \\
+ \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{\alpha - 1} |f(s, x(s))| \frac{ds}{s} \frac{(\log T)^{1-\gamma + \gamma}}{(\log T)^{1-\gamma + \gamma}} \\
\leq \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \left( b - a - I_{1+}^{\beta(1-\alpha)} f(T, x(T)) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta + 1)} + \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{\alpha - 1} |f(s, x(s))| \frac{ds}{s} \frac{(\log T)^{1-\gamma + \gamma}}{(\log T)^{1-\gamma + \gamma}} \\
\leq L \left( \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2 - (1-\beta)(1-\alpha))} \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta + 1)} + \frac{(\log T)^{1-\gamma + \gamma}}{\Gamma(\alpha + 1)} \right) \| x - y \|_{C_{1-\gamma, log}}.
\]

that $\| x \|_{C_{1-\gamma, log}} \leq R.$

This shows that the set $\phi$ is bounded. As a consequence of Schaefer’s fixed point theorem, we deduce that $P$ has a fixed point which is a solution of problem (1.1).

\[\square\]

**Lemma 3.1.** *(Uniqueness of solution)* Assume that the conditions (C1),(C2) are satisfied. If

\[
L \left( \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2 - (1-\beta)(1-\alpha))} + \frac{(\log T)^{1-\gamma + \gamma}}{\Gamma(\alpha + 1)} \right) < 1,
\]

then the problem (1.1) has a unique solution.

**Proof.** Consider the operator $P : C_{1-\gamma, log}[J, X] \to C_{1-\gamma, log}[J, X].$

\[
(Px)(t) = \frac{a}{\Gamma(\gamma)} (\log t)^{\gamma - 1} + \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \left( b - a - I_{1+}^{\beta(1-\alpha)} f(T, x(T)) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta + 1)} \frac{(\log t)^{\gamma + 2\beta - 2}}{(\log T)^{\gamma + 2\beta - 2}} \\
+ \frac{1}{\Gamma(\alpha)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s, x(s)) \frac{ds}{s} \frac{(\log T)^{\gamma}}{(\log T)^{\gamma}}.
\]

It is clear that the fixed points of $P$ are solutions of (1.1). Let $x, y \in C_{1-\gamma, log}[J, X]$ and $t \in J$, then we have

\[
\left| (Px)(t) - (Py)(t) \right| (\log t)^{1-\gamma} \\
\leq \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \left( b - a - I_{1+}^{\beta(1-\alpha)} f(T, x(T)) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta + 1)} \frac{(\log t)^{\gamma + 2\beta - 2}}{(\log T)^{\gamma + 2\beta - 2}} \\
+ \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{\alpha - 1} \left| f(s, x(s)) - f(s, y(s)) \right| \frac{ds}{s} \frac{(\log T)^{\gamma}}{(\log T)^{\gamma}} \\
\leq L \left( \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2 - (1-\beta)(1-\alpha))} + \frac{(\log T)^{1-\gamma + \gamma}}{\Gamma(\alpha + 1)} \right) \| x - y \|_{C_{1-\gamma, log}}.
\]

Hence,

\[
\left| (Px) - (Py) \right|_{C_{1-\gamma, log}} \leq L \left( \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2 - (1-\beta)(1-\alpha))} + \frac{(\log T)^{1-\gamma + \gamma}}{\Gamma(\alpha + 1)} \right) \| x - y \|_{C_{1-\gamma, log}}.
\]

From (3.7), it follows that $P$ has a unique fixed point which is a solution of problem (1.1).

\[\square\]
4 Stability analysis

In this section, we consider the Ulam stability of Hilfer-Hadamard type FDEs with boundary conditions (1.1). We adopt the definitions in [19, 27]. We consider the following inequalities

\[ |D^α_t z(t) - f(t, z(t))| \leq ε, \quad t \in J; \]  
(4.8)

\[ |D^α_t z(t) - f(t, z(t))| \leq ε\varphi(t), \quad t \in J; \]  
(4.9)

\[ |D^α_t z(t) - f(t, z(t))| \leq \varphi(t), \quad t \in J. \]  
(4.10)

**Definition 4.1.** The equation (1.1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^γ_1[J, X]$ of the inequality (4.8) there exists a solution $x \in C^γ_1[J, X]$ of equation (1.1) with

\[ |z(t) - x(t)| \leq C_f \varepsilon, \quad t \in J. \]

**Definition 4.2.** The equation (1.1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C([0, \infty), [0, \infty))$, $\psi_f(0) = 0$ such that for each solution $z \in C^γ_1[J, X]$ of the inequality (4.8) there exists a solution $x \in C^γ_1[J, X]$ of equation (1.1) with

\[ |z(t) - x(t)| \leq \psi_f(\varepsilon), \quad t \in J. \]

**Definition 4.3.** The equation (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C^γ_1[J, X]$ if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^γ_1[J, X]$ of the inequality (4.9) there exists a solution $x \in C^γ_1[J, X]$ of equation (1.1) with

\[ |z(t) - x(t)| \leq C_f \varphi(t), \quad t \in J. \]

**Definition 4.4.** The equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C^γ_1[J, X]$ if there exists a real number $C_f, \varphi > 0$ such that for each solution $z \in C^γ_1[J, X]$ of the inequality (4.10) there exists a solution $x \in C^γ_1[J, X]$ of equation (1.1) with

\[ |z(t) - x(t)| \leq C_f \varphi(t), \quad t \in J. \]

**Remark 4.1.** A function $z \in C^γ_1[J, X]$ is a solution of the inequality

\[ |D^α_t z(t) - f(t, z(t))| \leq ε, \quad t \in J, \]

if and only if there exists a function $g \in C^γ_1[J, X]$ (which depend on solution $x$) such that

1. $|g(t)| \leq ε, \quad t \in J$;
2. $H^{D^α_t} z(t) = f(t, z(t)) + g(t), \quad t \in J$.

**Remark 4.2.** Clearly,

1. Definition 4.1 $\Rightarrow$ Definition 4.2.
2. Definition 4.3 $\Rightarrow$ Definition 4.4.

**Lemma 4.1.** [30] Suppose $1 > α > 0$, $\overline{α} > 0$ and $\overline{β} > 0$ and suppose $u(t)$ is nonnegative and locally integral on $[1, +\infty)$ with

\[ u(t) \leq \overline{α} + \overline{β} \int_1^t \left( \log \frac{t}{s} \right)^{\overline{α}-1} u(s) \frac{ds}{s}, \quad t \in [1, +\infty). \]

Then

\[ u(t) \leq \overline{α} + \int_1^t \left( \sum_{n=1}^{\infty} \frac{(\overline{β} \Gamma(\overline{α}))^n}{\Gamma(n\overline{α})} \left( \log \frac{t}{s} \right)^{n\overline{α}-1} \right) \frac{ds}{s}, \quad t \in [1, +\infty). \]
Remark 4.3. Under the assumptions of Lemma 4.1, let $u(t)$ be a nondecreasing function on $[1, \infty)$. Then we have
$$
u(t) \leq \pi E_{\alpha, 1} (b \Gamma(\alpha) (\log t)^{\alpha}) ,$$
where $E_{\alpha, 1}$ is the Mittag-Leffler function defined by
$$E_{\alpha, 1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C}.$$
We ready to prove our stability results for problem (1.1).

Theorem 4.1. Assume that the conditions (C1), (C2) and (3.7) are satisfied, then the problem (1.1) is Ulam-Hyers stable.

Proof. Let $\varepsilon > 0$ and let $z \in C^\gamma_{1-T, \log}[J, X]$ be a function which satisfies the inequality:
$$|HD^\alpha x(t) - f(t, z(t))| \leq \varepsilon, \quad \text{for any } t \in J,$$
and let $x \in C^\gamma_{1-T, \log}[J, X]$ be the unique solution of the following Hilfer-Hadamard type BVP
$$HD^\alpha \gamma (t) = f(t, x(t)), \quad t \in J := [1, T],$$
$$I^{\gamma - \gamma} x(1) = I^{\gamma - \gamma} z(1) = a, \quad I^{\gamma - \gamma} x(T) = I^{\gamma - \gamma} z(T) = b \quad \gamma = \alpha + \beta - \alpha \beta ,$$
where $0 < \alpha < 1, 0 \leq \beta \leq 1$. Using Lemma 2.2, we obtain
$$x(t) = A_x + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s, x(s)) \frac{ds}{s}$$
where
$$A_x = \frac{a}{\Gamma(\gamma)} (\log t)^{\gamma - 1} + \left( b - a - I_1^{\gamma - \gamma} \frac{t}{s} \right) f(T, x(T)) \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} (\log T)^{2\beta - 1}.$$
On the other hand, if $I_1^{\gamma - \gamma} x(T) = I_1^{\gamma - \gamma} z(T)$ and $I_1^{\gamma - \gamma} x(1) = I_1^{\gamma - \gamma} z(1)$, then $A_x = A_z$.

Indeed,
$$|A_x - A_z| \leq \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} (\log T)^{2\beta - 1} |x(T) - z(T)|$$
$$= 0.$$
Thus, $A_x = A_z$.

Then, we have
$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s, x(s)) \frac{ds}{s}$$
By integration of the inequality (4.11), we obtain
$$z(t) - x(t) \leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)} $$
We have
$$|z(t) - x(t)| \leq \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s, z(s)) \frac{ds}{s} \right|$$
$$+ \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} |f(s, z(s)) - f(s, x(s))| \frac{ds}{s} \right|$$
$$\leq \varepsilon (\log T)^{\alpha} + \frac{L}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} |z(s) - x(s)| \frac{ds}{s} .$$
and to apply Lemma 4.1 and Remark 4.3, we obtain

\[ |z(t) - x(t)| \leq \frac{(\log T)^\alpha E_{\alpha,1}(L(\log T)^\alpha)}{\Gamma(\alpha + 1)} \cdot \varepsilon \]

\[ := C_f \varepsilon. \]

Thus, the equation (1.1) is Ulam-Hyers stable.

**Theorem 4.2.** Assume that the conditions (C1), (C2), (C4) and (3.7) are satisfied. Then, the problem (1.1) is generalized Ulam-Hyers-Rassias stable.

**Proof.** Let \( z \in C_{1,7-\log}^{\gamma}[J,X] \) be solution of the inequality

\[ |nD_{1,7-\log}^{\alpha,\beta} z(t) - f(t, z(t))| \leq \varepsilon \varphi(t), \quad t \in J, \quad \varepsilon > 0, \]

(4.12)

and let \( x \in C_{1,7-\log}^{\gamma}[J,X] \) be the unique solution of the following Hilfer-Hadamard type BVP

\[ nD_{1,7-\log}^{\alpha,\beta} x(t) = f(t, x(t)), \quad t \in J := [1, T], \]

\[ I_{1,7-\log}^{\gamma} z(1) = a, \quad I_{1,7-\log}^{\gamma} z(T) = b \]

where \( 0 < \alpha < 1, 0 \leq \beta \leq 1 \).

Using Lemma 2.2, we obtain

\[ x(t) = A_z + \frac{1}{\Gamma(\gamma)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s, x(s)) \frac{ds}{s}, \]

where

\[ A_z = \frac{\alpha}{\Gamma(\gamma)} (\log T)^{\gamma - 1} + \left( b - a - I_{1,7-\log}^{\gamma} f(T, x(T)) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} (\log T)^{\gamma + 2\beta - 2}. \]

By integration of the inequality (4.12), we get

\[ |z(t) - x(t)| \leq \left| z(t) - A_z \right| + \frac{1}{\Gamma(\gamma)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s, z(s)) \frac{ds}{s} \leq \varepsilon \lambda \varphi(t). \]

(4.13)

On the other hand, we have

\[ |z(t) - x(t)| \leq \left| z(t) - A_z \right| + \frac{1}{\Gamma(\gamma)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} f(s, z(s)) \frac{ds}{s} \]

\[ + \frac{L}{\Gamma(\gamma)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} |z(s) - x(s)| \frac{ds}{s} \]

\[ \leq \varepsilon \lambda \varphi(t) + \frac{L}{\Gamma(\gamma)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} |z(s) - x(s)| \frac{ds}{s}. \]

By applying Lemma 4.1 and Remark 4.3, we get

\[ |z(t) - x(t)| \leq \varepsilon \lambda \varphi(t) E_{\alpha,1}(L(\log T)^\alpha), \quad t \in [1, T]. \]

Thus, the equation (1.1) is generalized Ulam-Hyers-Rassias stable.

\[ \square \]

5 **Example**

In this section, we present an example to illustrate the theory results.
Example 5.1. Let us consider the following Hilfer-Hadamard type fractional BVP

\[ H_{1+}^{\alpha, \beta} x(t) = \frac{1}{10} \left( \frac{|x(t)|}{1 + |x(t)|} \right), \quad t \in J := [1, e], \quad (5.14) \]

\[ I_{1+}^{\gamma} x(1) = 1, \quad I_{1+}^{1-\gamma} x(e) = 2, \quad \gamma = \alpha + \beta - \alpha \beta. \quad (5.15) \]

Notice that this problem is a particular case of \((1.1)\), where \(\alpha = \frac{2}{3}, \beta = \frac{1}{2}\) and choose \(\gamma = \frac{5}{6}\).

Set \(f(t, u) = \frac{1}{10} \left( \frac{u}{1 + u} \right)\), for any \(u \in X\), and \(t \in J\).

Clearly, the function \(f\) satisfies condition of Theorem 3.1.

For each \(u, v \in X\) and \(t \in J\),

\[ |f(t, u) - f(t, v)| \leq \frac{1}{10} |u - v|. \]

Hence, the condition (C2) is satisfied with \(L = \frac{1}{10}\). Here \(T = e\).

Thus, condition from (3.7)

\[ L \left( \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2 - \beta(1-\alpha))} + \frac{(\log T)^{1-\gamma+\alpha}}{\Gamma(\alpha + 1)} \right) \approx 0.2050 < 1, \]

It follows from Lemma 3.1 that the problem (5.14)-(5.15) has a unique solution. Moreover, Theorem 4.1 implies that the problem (5.14)-(5.15) is Ulam-Hyers stable.

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