Journal Nonlinear Analysis and Application 2018 No.1 (2018) 34-47

Available online at www.ispacs.com/jnaa

Volume 2018, Issue 1, Year 2018 Article ID jnaa-00367, 14 Pages

doi:10.5899/2018/jnaa-00367

Research Article

Hybrid fixed point in CAT(0) spaces

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Abstract

In this paper, we introduce an ultrapower approach to prove fixed point theorems for H^+-nonexpansive multi-valued mappings in the setting of CAT(0) spaces and prove several hybrid fixed point results in CAT(0) spaces for families of single-valued nonexpansive or quasinonexpansive mappings and multi-valued upper semicontinuous, almost lower semicontinuous or H^+-nonexpansive mappings which are weakly commuting. We also establish a result about structure of the set of fixed points of H^+-quasinonexpansive mapping on a CAT(0) space.

Keywords: Fixed point; H^+-nonexpansive mapping; H^+-quasinonexpansive mapping; demiclosed; CAT(0) space; proximinal set.

Mathematics Subject Classification (2010): 47H10; 47H09; 54H25; 47H04.

1 Introduction

The study of fixed points for multi-valued contraction and nonexpansive mappings using the Hausdorff metric was initiated independently by Markin [33] and Nadler [35]. Afterward an interesting and rich fixed point theory for such mappings has been developed. Inspired from the work of Nadler [35], the fixed point theory of multi-valued contraction was further developed in different directions by many authors, in particular, by Beg et al. [4, 5], Reich [39, 40], Mizoguchi and Takahashi [34], Kaneko [21], Lim [31], Lami Dozo [30], Feng and Liu [18], Klim and Wardowski and [28], Suzuki [45], Pathak and Shahzad [36] and many others. For further details and more references, see [38, 44]. On the other hand,the study of metric spaces without linear structure has played a vital role in various branches of pure and applied sciences. One of such space is a CAT(0) space. A useful example of it is an $\mathbb{R}$-tree (see, e.g., [13, 46]), whose study found applications in convex optimization, control theory, economics, biology/medicine and computer science (see, e.g., [6, 26, 42]). Fixed point theory in a CAT(0) space was first studied by Kirk (see [23, 25]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory in CAT(0) spaces has been rapidly developing and recently significant research work is done in this area (see, e.g., [1, 2, 3, 14, 15, 16, 17]). The aim of this article is to introduce an ultrapower approach to prove fixed point theorems for H^+-nonexpansive multi-valued mappings in the setting of CAT(0) spaces. We obtained several hybrid fixed point results in CAT(0) spaces for families of single-valued nonexpansive or quasinonexpansive mappings and multi-valued upper semicontinuous, almost lower semicontinuous or H^+-nonexpansive mappings which are weakly commuting. Rest of the paper is arranged as follows: Section 2 recalls the basic ideas and notations of the Hausdorff metric, multi-vaued contraction mappings, ultraproduct

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and CAT(0) spaces. Section 3 is devoted to the study of necessary conditions for the existence of fixed points of H^+-nonexpansive and H^+-quasinonexpansive mappings in CAT(0) spaces. In Section 4, we obtained hybrid fixed point results for certain single-valued noncommuting class of mappings and multi-valued nonexpansive mappings.

2 Preliminaries

Let K be a nonempty subset of a metric space (X,d) and 2^K, $\mathscr{CB}(K)$ and $\mathscr{K}(K)$ denote the collection of all nonempty subsets of K, the collection of all nonempty closed and bounded subsets of K and the collection of all compact subsets of K, respectively.
For $E,F \in \mathscr{CB}(X)$, let

$$H(E,F) = \max\Big\{\rho(E,F),\rho(F,E)\Big\},\ \ H^+(E,F) = \frac{1}{2}\Big\{\rho(E,F)+\rho(F,E)\Big\},$$

where $\rho(E,F) = \sup_{x\in E} dist(x,F)$ and $dist(x,F) = \inf_{y\in F} d(x,y)$.
It is well known that H is a metric on $\mathscr{CB}(X)$. Such a map H is called *Hausdorff metric* induced by the metric d of X.
A multi-valued mapping $T : X \to \mathscr{CB}(X)$ is said to be a

(i) *multi-valued k-contraction mapping* if there exists a fixed real number $k, 0 < k < 1$ such that

$$H(Tx,Ty) \le k\,d(x,y)$$

(ii) *multi-valued nonexpansive mapping* if

$$H(Tx,Ty) \le d(x,y)$$

for all $x,y \in X$.

An element $x \in X$ is called a fixed point of a multi-valued map $T : K \subset X \to 2^X$ if $x \in T(x)$. We denote by $Fix(T)$ the set of fixed points of T. Nadler [35] in his seminal paper proved the following result;

Theorem 2.1. *[35] Every multi-valued contraction mapping T on a complete metric space has a fixed point.*

Proposition 2.1. *[37]. H^+ is a metric on $CB(X)$*

Notice that the two metrics H and H^+ are equivalent [29] since

$$\frac{1}{2}H(E,F) \le H^+(F,E) \le H(E,F).$$

In the light of this equivalence and referring to Kuratowski [29], we conclude that $(\mathscr{CB}(X),H^+)$ is complete whenever (X,d) is complete. Indeed, it is a simple consequence of the completeness of the Hausdorff metric H. Also $H^+ : \mathscr{CB}(X)\times\mathscr{CB}(X) \to \mathbb{R}$ is a continuous function.

Theorem 2.2. *[37]. If the metric space (X,d) is complete, then so is $(\mathscr{CB}(X),H^+)$ and also $\mathscr{K}(X)$ is a closed subspace of $(\mathscr{CB}(X),H^+)$.*

We recall the following definitions.

Definition 2.1. *[15] Let (X,d) be a metric space. A multi-valued map $T : X \to \mathscr{CB}(X)$ is called*

1. *H^+-contraction*

   (i) if there exists k in $(0,1)$ such that

$$H^+(Tx,Ty) \le kd(x,y)\,\textit{for all}\,x,y \in X,$$

   (ii) for every x in X, y in $T(x)$ and $\varepsilon > 0$, there exists z in $T(y)$ such that

$$d(y,z) \le H^+(T(y),T(x))+\varepsilon.$$

2. *H^+-nonexpansive*

   (i) if

$$H^+(Tx,Ty) \leq d(x,y) \textit{ for all } x,y \in X,$$

   (ii) if for every x in X, y in $T(x)$ and $\varepsilon > 0$, there exists z in $T(y)$ such that

$$d(y,z) \leq H^+(T(y),T(x)) + \varepsilon.$$

Recently, Pathak and Shahzad [37] generalize the above cited result of Nadler by weakening the multi-valued contraction. Indeed, they proved the following result.

Theorem 2.3. *Every H^+-multi-valued contraction mapping $T : X \to \mathscr{CB}(X)$ with Lipschitz constant $k < 1$ has a fixed point.*

Let (X,d) be a metric space. For any pair of points x,y in X, a geodesic path joining these points is a map c from a closed interval $[0,r] \subset \mathbb{R}$ to X such that $c(0) = x, c(r) = y$ and $d(c(t),c(s)) = |t-s|$ for all $s,t \in [0,r]$. The mapping c is an isometry and $d(x,y) = r$. The image of c is called a geodesic segment joining x and y which when unique is denoted by $[x,y]$. For any $x,y \in X$, denote the point $z \in [x,y]$ such that $d(x,z) = \alpha d(x,y)$ by $z = (1-\alpha)x \bigoplus \alpha y$, where $0 \leq \alpha \leq 1$. The space (X,d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset K of X is called convex if K includes every geodesic segment joining any two of its points.
In what follows, let (X,d) be a geodesic metric space in which each two points $x,y \in X$ are joined by a unique geodesic (metric) segment denoted by $[x,y]$. Denote by M^2_κ the following classical metric spaces:

(1) if $\kappa = 0$ then M^2_0 is the Euclidean plane $\mathbb{E}^2$;

(2) if $\kappa < 0$ then M^2_κ is obtained from the classical hyperbolic plane $\mathbb{H}^2$ by multiplying the hyperbolic distance by $1/\sqrt{-\kappa}$.

Recall that a *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$) and a *comparison triangle* for a geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) = \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in M^2_κ such that $d_{M^2_\kappa}(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.
A metric space X is a CAT(0) space (the term is due to M. Gromov – see, e.g., [7, Page 159]) if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane $\mathbb{E}^2$. For a precise definition and a detailed discussion of the properties of such spaces, see Bridson and Haefliger [7] or Burago, et al. [8]. It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include the classical hyperbolic spaces, the complex Hilbert ball with a hyperbolic metric (see [20]; also [41, inequality (4.3)] and subsequent comments), and many others.
Let (x_n) be a bounded sequence in a complete CAT(0) space X and for $x \in X$ set

$$r(x,(x_n)) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x,(x_n)) : x \in X\}.$$

The asymptotic center $A((x_n))$ of $(x_n)$is the set

$$A((x_n)) = \{x \in X : r(x,(x_n)) = r((x_n))\}.$$

Recall that a bounded sequence (x_n) is regular if $r(x_n) = r(u_n)$ for every subsequence (u_n) of (x_n). It is easy to see that every bounded sequence in X has a regular subsequence (see, e.g., [20, page 166]).
The following lemma was proved by Goebel [19] and Lim [32] in the framework of Banach spaces. Since the proof has a metric nature the lemma holds true in complete geodesic metric spaces with convex metric.

Lemma 2.1. *[19] Let X be a complete geodesic metric space with convex metric, $\{x\}$ be a bounded sequence in X and let K be a nonempty closed convex subset of X. Then there exists a subsequence of $\{x_n\}$ which is regular relative to K.*

It is known [11, Proposition 7] that in a CAT(0) space, $A((x_n))$ consists of exactly one point.
We will also need the following important fact about asymptotic centers.

Proposition 2.2. *[10]. If K is a closed convex subset of X and if (x_n) is a bounded sequence in K, then the asymptotic center of (x_n) is in K.*

Let C be a subset of a complete CAT(0) space X. For convenience and brevity we work in an ultrapower setting (see [27]). Let $\mathscr{U}$ be a nontrivial ultrafilter on the natural numbers $\mathbb{N}$. Fix $p \in X$, and let $\tilde{X}$ denote the metric space ultrapower of X over $\mathscr{U}$ relative to p. Thus the elements of $\tilde{X}$ consist of equivalence classes $[(x_i)]_{i\in\mathbb{N}}$ for which

$$\lim_{\mathscr{U}} d(x_i, p) < \infty,$$

with $(u_i) \in [(x_i)] \iff \lim_{\mathscr{U}} d(x_i, u_i) = 0$. Note that $\tilde{X}$ is also a CAT(0) space ([2, page 187]).
For $C \subseteq X$, let

$$\tilde{C} = \{x = [(x_n)] : x_n \in C \text{ for each } n\},$$

and for $x \in X$, let $\dot{x} = [(x_n)]$ where $x_n = x$ for each $n \in \mathbb{N}$. Finally, let $\dot{X}$ and $\dot{C}$ denote the respective canonical isometric copy of X and C in $\tilde{X}$.
Now let $\mathscr{U}$ be a nontrivial ultrafilter over the natural numbers $\mathbb{N}$ and let $\tilde{X}$ denote the Banach space ultrapower of X over $\mathscr{U}$. We will use the standard notation for this setting, see for example [22].
Notice that a H^+-type nonexpansive multi-valued mapping $T : C \to \mathscr{CB}(X)$ induces a H^+-nonexpansive multi-valued mapping $\tilde{T}$ defined on $\tilde{C}$ as follows:

$$\tilde{T}(\tilde{x}) = \{\tilde{u} \in \tilde{X} : \exists \text{ a representative } (u_n) \text{ of } \tilde{u} \text{ with } u_n \in T(x_n) \text{ for each } n\}.$$

To see that $\tilde{T}$ is H^+-nonexpansive (and hence well-defined), let $\tilde{x}, \tilde{y} \in \tilde{C}$, with $\tilde{x} = [(x_n)]$ and $\tilde{y} = [(y_n)]$. Then

$$\begin{aligned} H^+(\tilde{T}(\tilde{x}), \tilde{T}(\tilde{y}) &\leq \lim_{\mathscr{U}} H^+(T(x_n); T(y_n)) \\ &\leq \lim_{\mathscr{U}} d(x_n, y_n) \\ &= d_{\mathscr{U}}(\tilde{x}, \tilde{y}). \end{aligned}$$

Since for each $n \in \mathbb{N}$, $x_n \in C$, $y_n \in T(x_n)$ and $\varepsilon > 0$, there exists $z_n \in T(y_n)$ such that $d(y_n, z_n) \leq H^+(T(x_n), T(y_n)) + \varepsilon$, it follows that

$$\lim_{\mathscr{U}} d(y_n, z_n) \leq \lim_{\mathscr{U}} H^+(T(x_n), T(y_n)) + \varepsilon.$$

Thus, for any $\tilde{x} = [(x_n)] \in \tilde{C}$, $\tilde{y} = [(y_n)] \in \tilde{T}(\tilde{x})$ and $\varepsilon > 0$, there exists $\tilde{z} = [(z_n)] \in \tilde{T}(\tilde{y})$ such that

$$d_{\mathscr{U}}(\tilde{y}, \tilde{z}) \leq H^+(\tilde{T}(\tilde{x}), \tilde{T}(\tilde{y}) + \varepsilon.$$

For our further discussion, the following fact will be needed.

$$\text{If } S \subseteq C \text{ is compact, then } \dot{S} = \tilde{S}. \tag{2.1}$$

Proposition 2.3. *[10]. Element x is the asymptotic center of a regular sequence $(x_n) \subset X \iff \dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$ in the ultrapower $\tilde{X}$.*

Dhompongsa, Kaewkhao, and Panyanak [10] also proved the following fixed point result for commuting mappings.

Theorem 2.4. *[10] Let K be a nonempty closed bounded convex subset of a complete CAT(0) space X, f a nonexpansive self-mapping of K and $T : K \to 2^K$ is nonexpansive, where for any $x \in K, Tx$ is nonempty compact convex. Assume that for some $p \in Fix(f)$, $\alpha p \bigoplus (1-\alpha)Tx$ is convex for all $x \in X$ and $\alpha \in [0,1]$. If f and T commute, then there exists an element $z \in X$ such that $z = f(z) \in T(z)$.*

Lemma 2.2. *[20] Let X be a geodesic metric space with convex metric, $\{z_n\}$ and $\{w_n\}$ be two bounded sequences in X, and let $0 < \lambda < 1$. If for every natural number n we have $z_{n+1} = \lambda w_n \bigoplus (1-\lambda) z_n$ and $d(w_{n+1}, w_n) \leq d(z_{n+1}, z_n)$, then $\lim_{n\to\infty} d(w_n, z_n) = 0$.*

Lemma 2.3. *[12] Let X be a CAT(0) space. Then for all $x, y, z \in X$ and all $t \in [0,1]$ we have*
$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2.$$

3 Fixed points of H^+-nonexpansive and H^+-quasinonexpansive mappings in CAT(0) spaces

Theorem 3.1. *Let K be a closed convex subset of a complete CAT(0) space X, and let $T : K \to \mathscr{K}(X)$ be H^+-nonexpansive mapping. Suppose $dist(x_n, T(x_n)) \to 0$ for a unique bounded sequence $(x_n) \subset K$. Then T has a fixed point.*

Proof. By passing to a subsequence we may suppose (x_n) is regular. Let x be the asymptotic center of (x_n) and by convexity of K the asymptotic center of (x_n) lies in K. By Proposition 2.3 $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$. By Proposition 2.2, $x \in K$ and also $\dot{x} \in \dot{K}$. Since $\tilde{T}(\tilde{x}) = \{\tilde{x}\}$, $\tilde{x}$ must lie in a ρ_1-neighborhood of $\tilde{T}(\dot{x})$ for $\rho_1 \leq \rho_0 = H^+(\tilde{T}(\tilde{x}), \tilde{T}(\dot{x}))$. Since $\tilde{T}(\dot{x})$ is compact, $dist(\tilde{x}, \tilde{T}(\dot{x})) = d_{\mathscr{U}}(\tilde{x}, \dot{u}) \leq \rho_1$ for some $\dot{u} \in \tilde{T}(\dot{x})$. But since $\tilde{T}(\dot{x}) \subset \dot{X}$, if $\dot{u} \neq \dot{x}$ we have the contradiction

$$d_{\mathscr{U}}(\tilde{x}, \dot{u}) > d_{\mathscr{U}}(\tilde{x}, \dot{x}) \geq H^+(\tilde{T}(\tilde{x}), \tilde{T}(\dot{x})) = \rho_0 \geq \rho_1.$$

Therefore $\dot{x} = \dot{u} \in \tilde{T}(\dot{x})$. However $\tilde{T}(\dot{x}) = \widetilde{T(x)}$, so (2.1) further implies $x \in T(x)$.

□

Remark 3.1. *It may be observed that Theorem 3.1 holds under the weaker assumption that K is closed and contains the asymptotic centers of all of its regular sequences.*

Let (X,d) be a geodesic metric space. A subset $K \subset X$ is called proximal if for each $x \in X$, there exists an element $y \in K$ such that

$$d(x,y) = dist(x,K) = \inf\{d(x,z) : z \in K\}.$$

We denote by $P(K), KC(K)$ and $CBP(K)$ the collection of all nonempty proximal subsets, nonempty compact convex subsets, and nonempty closed bounded proximal subsets of K, respectively.

Definition 3.1. *Let (X,d) be a geodesic metric space, $K \subset X$ and $T : K \to \mathscr{CB}(K)$ a multi-valued map. Then T is said to be*

1°. *H^+-quasinonexpansive if $Fix(T) \neq \emptyset$,*

(i) *$H^+(T(x), T(p)) \leq d(x,p)$ for all $x \in K$ and all $p \in Fix(T)$, and*

(ii) *for $x, y \in Fix(T)$ and $z \in [x,y]$, unique point $w \in Tz$ closest to both x and y.*

2°. *condition (C_λ) for some $\lambda \in (0,1)$ provided that*

$$\lambda\ dist(x, T(x)) \leq d(x,y) \Rightarrow H(T(x), T(y)) \leq d(x,y), x, y \in K.$$

3°. *satisfy condition $(E_{\mu,\eta})$ provided that*

$$dist(x, T(y)) \leq \mu\ dist(x, T(x)) + \eta d(x,y), x, y \in K.$$

We say that T satisfies condition (E) whenever T satisfies $(E_{\mu,\eta})$ for some $\mu, \eta \geq 1$.

Lemma 3.1. *Let K be a closed convex subset of a CAT(0) space X. Let $T : K \to P(K)$ be a multi-valued mapping such that P_T is H^+-quasinonexpansive, where $P_T(x) = \{y \in T(x) : d(x,y) = dist(x,T(x))\}$. Then $F(T)$ is a closed and convex set.*

Proof. Let $\{p_n\}$ be a sequence in $Fix(T)$ such that $p_n \to z$ as $n \to \infty$. Then $P_T(p_n) = \{p_n\}$. By H^+-quasinonexpansivness of P_T we have

$$\begin{aligned}
dist(z,T(z)) \leq dist(z,P_T(z)) &\leq d(z,p_n) + dist(p_n,P_T(z)) \\
&\leq d(z,p_n) + \frac{1}{2}\Big[\rho(\{p_n\},P_T(z)) + \rho(P_T(z),\{p_n\})\Big] \\
&= d(z,p_n) + \frac{1}{2}\Big[\rho(P_T(p_n),P_T(z)) + \rho(P_T(z),P_T(p_n))\Big] \\
&= d(z,p_n) + H^+(P_T(p_n),P_T(z)) \\
&\leq 2d(z,p_n) \to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

It implies that $z \in T(z)$, hence $z \in Fix(T)$.
We now show that $Fix(T)$ is convex. For $x,y \in Fix(T)$ we have $P_T(x) = \{x\}$ and $P_T(y) = \{y\}$. For $\alpha \in [0,1]$, put $z = \alpha x \bigoplus (1-\alpha)y$. Let $w \in P_T(z)$ be the unique closest point to both x and y, then by Lemma 2.3 we have

$$\begin{aligned}
d(w,z)^2 &= d(\alpha x \bigoplus (1-\alpha)y, w)^2 \\
&\leq \alpha d(w,x)^2 + (1-\alpha)d(w,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\
&= \alpha dist(w,P_T(x))^2 + (1-\alpha)dist(w,P_T(y))^2 - \alpha(1-\alpha)d(x,y)^2 \\
&\leq \alpha\Big\{\frac{1}{2}[\rho(P_T(z),P_T(x)) + \rho(P_T(x),P_T(z))]\Big\}^2 \\
&\quad + (1-\alpha)\Big\{\frac{1}{2}[\rho(P_T(z),P_T(y)) + \rho(P_T(y),P_T(z))]\Big\}^2 - \alpha(1-\alpha)d(x,y)^2 \\
&= \alpha H^+(P_T(z),P_T(x))^2 + (1-\alpha)H^+(P_T(z),P_T(y))^2 - \alpha(1-\alpha)d(x,y)^2 \\
&= \alpha d(x,z)^2 + (1-\alpha)d(z,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\
&= \alpha(1-\alpha)^2 d(x,y)^2 + (1-\alpha)\alpha^2 d(x,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\
&= \alpha(1-\alpha)(1-\alpha+\alpha-1)d(x,y)^2 = 0.
\end{aligned}$$

Thus $z = w \in P_T(z) \subset T(z)$ and finally $z \in Fix(T)$. □

Definition 3.2. *Let A be a nonempty subset of $Fix(T)$. Then A is said to be T-closed if for each sequence$\{p_n\} \subset A$ converges to some $z \in Fix(T)$ implies that $T(z) = \{z\}$.*

Example 3.1. *Let $T, T_1, T_2 : R^+ \to CB(R^+)$ be defined by*

$$Tx := \begin{cases} \{x\}, & \text{if } 0 \leq x \leq 1, \\ [1,2], & \text{if } x > 1, \end{cases} \quad T_1 x := \begin{cases} \{x\}, & \text{if } 0 \leq x < 1, \\ [1,2], & \text{if } x \geq 1, \end{cases} \quad T_2 x := \begin{cases} [0,x], & \text{if } 0 \leq x < 1, \\ [1,2], & \text{if } x \geq 1. \end{cases}$$

Obviously $Fix(T) = Fix(T_1) = Fix(T_2) = [0,2]$. Take $A = [0,1]$, then we notice that A is T-closed but it is neither T_1-closed nor T_2-closed.

Lemma 3.2. *Let K be a closed convex subset of a CAT(0) space X. Let $T : K \to CB(K)$ be a H^+-quasinonexpansive multi-valued mapping. Let A be a nonempty subset of $Fix(T)$ such that $T(p) = \{p\}$ for all $p \in A$ and that A is T-closed. Then A is closed and convex.*

Proof. Let $\{p_n\}$ be a sequence in A such that $p_n \to z$ as $n \to \infty$. Then by assumption $T(p_n) = \{p_n\}$. It follows that

$$\begin{aligned} dist(z,T(z)) &\leq d(z,p_n) + dist(p_n,T(z)) \\ &\leq d(z,p_n) + \frac{1}{2}\Big[\rho(\{p_n\},T(z)) + \rho(T(z),\{p_n\})\Big] \\ &= d(z,p_n) + \frac{1}{2}\Big[\rho(T(p_n),T(z)) + \rho(T(z),T(p_n))\Big] \\ &= d(z,p_n) + H^+(T(p_n),T(z)) \\ &\leq 2d(z,p_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

It implies that $z \in T(z)$. By T-closedness of A, we have $z \in A$. We now show that A is convex. For $x,y \in A$ we have $T(x) = \{x\}$ and $T(y) = \{y\}$. For $\alpha \in [0,1]$, put $z = \alpha x \bigoplus (1-\alpha)y$. Let $w \in T(z)$ be the unique closest point to both x and y, then we have

$$\begin{aligned} d(w,z)^2 &= d(\alpha x \bigoplus (1-\alpha)y, w)^2 \\ &\leq \alpha d(w,x)^2 + (1-\alpha)d(w,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\ &= \alpha dist(w,T(x))^2 + (1-\alpha)dist(w,T(y))^2 - \alpha(1-\alpha)d(x,y)^2 \\ &\leq \alpha\Big\{\frac{1}{2}[\rho(T(z),T(x)) + \rho(T(x),T(z))]\Big\}^2 \\ &\quad + (1-\alpha)\Big\{\frac{1}{2}[\rho(T(z),T(y)) + \rho(T(y),T(z))]\Big\}^2 - \alpha(1-\alpha)d(x,y)^2 \\ &= \alpha H^+(T(z),T(x))^2 + (1-\alpha)H^+(T(z),T(y))^2 - \alpha(1-\alpha)d(x,y)^2 \\ &= \alpha d(x,z)^2 + (1-\alpha)d(z,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\ &= \alpha(1-\alpha)^2 d(x,y)^2 + (1-\alpha)\alpha^2 d(x,y)^2 - \alpha(1-\alpha)d(x,y)^2 \\ &= \alpha(1-\alpha)(1-\alpha+\alpha-1)d(x,y)^2 = 0 \end{aligned}$$

so that $z = w \in T(z)$ and finally $z \in A$. □

As a corollary, we obtain the following result of Chaoha and Phonon [9].

Corollary 3.1. *Let K be a closed convex subset of a CAT(0) space X and $T : K \to K$ a quasinonexpansive single-valued mapping. Then $Fix(T)$ is closed and convex.*

Corollary 3.2. *Let A be a nonempty subset of a CAT(0) space X, and $S,T : X \to 2^X$ be two multi-valued mappings. We say that the pair (S,T) weakly commutes in A if for all $x \in A$*

$$S(T(x)) \subset T(S(x),$$

where $S(T(x)) = \bigcup_{y \in T(x)} S(y)$.

Notice that the order of mappings in the pair is crucial in this definition. Indeed, if the pair (S,T) weakly commutes then it does not necessarily implies that the pair (T,S) weakly commutes.
Next we state and prove main result of this section.

Theorem 3.2. *Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $S : K \to CB(K)$ be H^+-quasinonexpansive multi-valued mapping, and let $T : K \to KC(K)$ be a multi-valued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If the pair (S,T) weakly commutes in a nonempty subset A of $Fix(S)$, $S(p) = \{p\}$ for all $p \in A$, and that A is S-closed, then S and T have a common fixed point.*

Proof. By Lemma 3.2, it follows that A is a nonempty closed convex subset of X. We show that for $x \in A, T(x) \cap A \neq \emptyset$. To see this, let $x \in A$, and let $y \in T(x)$ be the unique closest point to x. Since S and T weakly commute at $x \in A$ and $S(x) = \{x\}$ we have

$$S(y) \subset ST(x) \subset TS(x) = T(x),$$

and since S is H^+-quasinonexpansive, we have

$$dist(S(y),x) \leq H^+(S(y),S(x)) \leq d(y,x).$$

Therefore there exists $z \in S(y) \subset T(x)$ such that

$$d(z,x) = dist(S(y),x) \leq d(y,x).$$

Now by the uniqueness of y as the closest point to x, we get $y = z \in S(y)$ and therefore $T(x) \cap A \neq \emptyset$, for all $x \in A$. It follows that $T(x) \cap A \neq \emptyset$, for all $x \in A$.. Now we find an approximate fixed point sequence in A for T. Take $x_0 \in A$, since $T(x_0) \cap A \neq \emptyset$, we can find $y_0 \in T(x_0) \cap A$. Define

$$x_1 = (1-\lambda)x_0 \bigoplus \lambda y_0.$$

Since A is convex, we have $x_1 \in A$. Let $y_1 \in T(x_1)$ be taken in such a way that

$$d(y_0,y_1) = dist(y_0,T(x_1)).$$

By the method described above we can prove that $y_1 \in A$. Similarly, we put $x_2 = (1-\lambda)x_1 \bigoplus \lambda y_1$, and again we choose $y_2 \in T(x_2)$ in such a way that

$$d(y_1,y_2) = dist(y_1,T(x_2)).$$

By the same argument, we get $y_2 \in A$. In this way we will find a sequence $\{x_n\}$ in A such that $x_{n+1} = (1-\lambda)x_n \bigoplus \lambda y_n$ where $y_n \in T(x_n) \cap A$ and

$$d(y_{n-1},y_n) = dist(y_{n-1},T(x_n)).$$

Therefore for every natural number $n \geq 1$ we have

$$\lambda d(x_n,y_n) = d(x_n,x_{n+1}).$$

It follows that

$$\lambda dist(x_n,T(x_n)) \leq \lambda d(x_n,y_n) = d(x_n,x_{n+1}), n \geq 1.$$

Since T satisfies the condition (C_λ) we have

$$H^+(T(x_n),T(x_{n+1})) \leq d(x_n,x_{n+1}), n \geq 1.$$

Thus for each $n \geq 1$

$$d(y_n,y_{n+1}) = dist(y_n,T(x_{n+1})) \leq H^+(T(x_n),T(x_{n+1})) \leq d(x_n,x_{n+1}).$$

We now apply Lemma 2.2 to conclude that $\lim_{n\to\infty} d(x_n,y_n) = 0$ where $y_n \in T(x_n)$. From Lemma 2.1 by passing to a subsequence we may assume that $\{x_n\}$ is regular. Put $A(Fix(S),\{x_n\}) = \{z\}$. If

$$r = r(Fix(S),\{x_n\}) = 0,$$

then $x_n \to z$ as $n \to \infty$. Since T satisfies condition (E), there exist $\eta, \mu \geq 1$ such that

$$\begin{aligned} dist(z,T(z)) &\leq d(z,x_n) + dist(x_n,T(z)) \\ &\leq (\eta+1)d(z,x_n) + dist(x_n,T(x_n)) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies that $z \in T(z)$. In the other case, if $r > 0$, there exists a natural number n_0 such that for every $n \geq n_0$,

$$\lambda dist(x_n,T(x_n)) \leq d(x_n,z)$$

and therefore from our assumption we have

$$H^+(T(x_n),T(z)) \leq d(x_n,z),\ \forall n \geq n_0.$$

The compactness of $T(z)$ implies that for each $n \geq 1$ we can take $z_n \in T(z)$ such that

$$d(y_n,z_n) = dist(y_n,T(z)).$$

Since $y_n \in A$, by a similar argument, we obtain $z_n \in A$. We also have

$$d(y_n,z_n) = dist(y_n,T(z)) \leq H(T(x_n),T(z)) \leq d(x_n,z),\ \forall n \geq n_0.$$

Since $T(z)$ is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = w \in T(z)$. Now, the closedness of A implies that $w \in A$. Note that

$$\begin{aligned} d(x_{n_k},w) &\leq d(x_{n_k},y_{n_k}) + d(y_{n_k},z_{n_k}) + d(z_{n_k},w)d(x_{n_k},y_{n_k}) \\ &\leq d(x_{n_k},z) + d(z_{n_k},w), \quad \text{for } n_k \geq n0. \end{aligned}$$

It implies that

$$\limsup_{k\to\infty} d(x_{n_k},w) \leq \limsup_{k\to\infty} d(x_{n_k},z) \leq r.$$

It further implies, by the regularity of $\{x_n\}$ and by the uniqueness of asymptotic center, that $z = w \in T(z)$. Hence $z \in A \cap Fix(T) \subset Fix(S) \cap Fix(T)$. □

Theorem 3.3. *Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $S: K \to P(K)$ be a multi-valued mapping such that P_S is H^+-quasinonexpansive, and let $T: K \to KC(K)$ be a multi-valued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If the pair (P_S,T) weakly commutes in Fix(S), then they have a common fixed point.*

Proof. From Lemma 3.1, it follows that $Fix(S)$ is a nonempty closed convex subset of X. We now show that for $x \in Fix(S), T(x) \cap Fix(S) \neq \emptyset$. To see this, let $x \in Fix(S)$ then we have $P_S(x) = \{x\}$. Let $y \in T(x)$ be the unique closest point to x, since the pair (P_S,T) commutes in x and we have

$$P_S(y)P_ST(x) \subset TP_S(x) = T(x)$$

and since P_S is H^+-quasinonexpansive, we have

$$dist(P_S(y),x) \leq H^+(P_S(y),P_S(x)) \leq d(y,x).$$

Thus there exists $z \in P_S(y) \subset T(x)$ such that

$$d(z,x) = dist(P_S(y),x) \leq d(y,x).$$

Now by the uniqueness of y as the closest point to x, we get $y = z \in P_S(y) \subset S(y)$ and therefore $T(x) \cap Fix(S) \neq \emptyset$, for $x \in Fix(S)$. The rest of proof is exactly similar to the proof of Theorem 3.2. □

Corollary 3.3. *Let K be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $ft: K \to K$ be a quasinonexpansive single-valued mapping, and let $T: K \to KC(K)$ be a multi-valued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. If f and T weakly commute, then they have a common fixed point, i.e. there exists a point $z \in K$ such that $z = f(z) \in T(z)$.*

We now give an example to illustrate Theorem 3.2.

Example 3.2. *Take $K = [0,1]$ and define S and T by*

$$S(x) = \begin{cases} [0,2x], & \text{if } 0 \leq x < \frac{1}{2} \\ [0,1], & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad \text{and} \quad T(x) = \begin{cases} [0,\frac{x}{2}], & \text{if } x \neq 1 \\ \{\frac{1}{2}\}, & \text{if } x = 1. \end{cases}$$

Then S is H^+-quasinonexpansive, and $[0,1]$ is the set of fixed points of S. Also T satisfies conditions (C_λ) and E. It is easy to see that S and T weakly commutes in $\{0\} \subset Fix(S) = [0,1]$ and that 0 is a common fixed point of T and S.

Notice that S is neither nonexpansive nor quasinonexpansive.

4 Hybrid fixed point in CAT(0) spaces

The following observation of Shahzad [43] gives impetus to the study of hybrid fixed point results for certain single-valued noncommuting class of mappings and multi-valued nonexpansive mappings.

Theorem 4.1. *[43] Let K be a nonempty closed convex subset of a complete CAT(0) space X, f a mapping of K into X such that $C = \{x \in K : d(f(x),x) = d(f(x),K)\}$ is nonempty and f is nonexpansive with respect to C. Then C is a closed set on which f is continuous. In addition, suppose that f is isometric on C. If for $u,v \in C, x \in [u,v]$, then $f(x) \in [f(u), f(v)]$.*
If $f(K) \subset K$, then $Fix(f) = C$ and so we have the following result of Chaoha and Phonon [9] as a corollary.

Corollary 4.1. *[9] Let K be a closed convex subset of a complete CAT(0) space X, f a quasi-nonexpansive self-mapping of K. Then $C = Fix(f)$ is a nonempty closed convex set on which f is continuous.*
The following result Theorem 4.2 extends and improves Theorem 2.4. due to [10]. It basically shows that the following assumptions in Theorem 2.4 can be further relaxed

- *the mapping f is nonexpansive can be replaced by the assumption that f is only quasinonexpansive;*
- *the mapping $T : K \to 2^K$ is nonexpansive can be replaced by the assumption that T is only H^+-nonexpansive.*

Theorem 4.2. *Let K be a nonempty closed bounded convex subset of a complete CAT(0) space X, f a quasinonexpansive self-mapping of K, and let $T : K \to 2^K$ be H^+-nonexpansive mapping, where for any $x \in K, Tx$ is nonempty compact convex. If f and T commute weakly, then there exists an element $z \in K$ such that $z = f(z) \in T(z)$.*

Proof. By Corollary 4.1, Fix(f) is nonempty closed convex. Let $x \in Fix(f)$. Then $f(\partial_K T(x)) \subset T(f(x)) = T(x)$. Let $u \in \partial_K T(x)$ be a unique closest point to x. Since f is nonexpansive with respect to Fix(f), we have $d(f(u),x) \le d(u,x)$ and so $f(u) = u$, by uniqueness of the closest point u. Thus

$$T(x) \cap Fix(f) \neq \emptyset.$$

Let $F(x) = T(x) \cap Fix(f)$. Then F is a mapping of Fix(f) into $2^{Fix(f)}$. Notice that for any $x,y \in Fix(f)$,

$$\begin{aligned} H^+(F(x),F(y)) &= \frac{1}{2}\Big[\sup_{u\in F(x)} d(u,F(y)) + \sup_{v\in F(y)} d(v,F(x))\Big] \\ &\le \frac{1}{2}\Big[\sup_{u\in F(x)} d(u,T(y)) + \sup_{v\in F(y)} d(v,T(x))\Big] \\ &\le \frac{1}{2}\Big[\sup_{u\in T(x)} d(u,T(y)) + \sup_{v\in T(y)} d(v,T(x))\Big] \\ &= H^+(T(x),T(y)) \\ &\le d(x,y). \end{aligned}$$

Now Dhompongsa, Kaewkhao and Panyanak [10, Corollary 3.5] guarantees the existence of $z \in Fix(f)$ such that $z \in F(z)$. As a result, we have $f(z) = z \in T(z)$. □

Corollary 4.2. *Let K be a nonempty closed bounded convex subset of a complete CAT(0) space X, f a nonexpansive self-mapping of K, and let $T : K \to 2^K$ be H^+-type nonexpansive mapping, where for any $x \in K$, Tx is nonempty compact convex. If f and T commute weakly, then there exists an element $z \in K$ such that $z = f(z) \in T(z)$.*

Proof. By [24, Theorem 12] Fix(f) is nonempty closed bounded convex. So the result follows from Theorem 4.2. □

Theorem 4.3. *Let K be a nonempty closed bounded convex subset of a complete CAT(0) space X and f a nonexpansive self-mapping of K. Then for any closed convex subset Y of K such that $f(\partial_K Y) \subset Y$, we have $P_{Fix(f)}(Y) \subset Y$.*

Proof. Fix $u \in Y$, and define the mapping $f_t : Y \to K$ by taking $f_t(x)$ to be the point of $[u, f(x)]$ at distance $td(u, f(x))$ from u. Then by convexity of the metric

$$d(f_t(x), f_t(y)) \leq td(x, y)$$

for all $x, y \in Y$. This shows that $f_t : Y \to K$ is a contraction. Let P_Y be the proximinal nonexpansive retraction of K into Y. Then $P_Y f_t$ is a contraction self-mapping of Y. By the Banach Contraction Principle, there exists a unique fixed point $y_t \in Y$ of $P_Y f_t$. Thus $d(f_t y_t, y_t) = \inf d(f_t y_t, z) : z \in Y$. Since $f(\partial_K Y) \subset Y$, we have $f_t(\partial_K Y) \subset Y$ and so we have $f_t(y_t) = y_t \in [u, f(y_t)]$. Note that $C = \text{Fix}(f)$ is nonempty closed bounded convex by [24, Theorem 12]**.** Now [23, Theorem 26] guarantees that $\lim_{t \to 1^-} y_t$ converges to the unique fixed point of f which is nearest u. As a result, $\lim_{t \to 1^-} y_t = P_C(u) \in Y$. Since X is a CAT(0) space, P_C is nonexpansive and $P_C(Y) \subset Y$. □

Remark 4.1. *Let K be a nonempty closed bounded convex subset of a complete CAT(0) space $X, f : K \to X$ a nonexpansive mapping. Then there exists an element $z \in K$ such that*

$$d(f(z), z) = d(f(z), K).$$

To see this, let P_K be the proximinal nonexpansive retraction of X into K. Then P_K. f is a nonexpansive self-mapping of K and so has a fixed point z. Hence

$$d(f(z), z) = d(f(z), K).$$

The following result also follows from Theorem 4.2 but the proof given here is constructive one.

Theorem 4.4. *Let K be a nonempty closed bounded convex subset of a complete CAT(0) space $X, f : K \to X$ a nonexpansive mapping, and $T : K \to 2^K$ an H^+-nonexpansive mapping, where for any $x \in K, Tx$ is nonempty compact convex. If for each $x \in X, P_K f(\partial_K T(x)) \subset T(P_K f(x))$, where P_K is the proximinal nonexpansive retraction of X into K, then there exists an element $z \in K$ with $z \in T(z)$ such that*

$$d(f(z), z) = d(f(z), K).$$

Proof. By Remark 4.1, we observe that

$$C = \{x \in K : d(f(x), x) = d(f(x), K)$$

is nonempty and $C = \text{Fix}(P_K f)$. Define $F : K \to 2^K$ by $F(x) = T(P_C(x))$. Then F is nonexpansive and has a fixed point $v \in K$ by [10, Corollary 3.9]. Notice that

$$P_K f(\partial_K F(v)) = P_K f(\partial_K T(P_C(v)) \subset T(P_K f(P_C v)) = T(P_C v) = F(v).$$

Also $C = \text{Fix}(P_K f)$. Now Theorem 4.1 guarantees that $P_C(F(v)) \subset F(v)$. In particular, $P_C v \in F(v)$. Let $z = P_C v$. Then $P_K f(z) = z \in T(z)$ and $d(f(z), z) = d(f(z), K)$. □

Acknowledgment

The research of Hemant Kumar Pathak was supported by University Grants Commission, New Delhi, F. No.-43-422/2014 (SR) (MRP-MAJOR-MATH-2013- 18394) .

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