A Stage-Structured Prey-Predator Fishery Model In The Presence Of Toxicity With Taxation As A Control Parameter of Harvesting Effort

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Abstract
In this paper we have considered stage-structured fishery model in the presence of toxicity, which is diminishing due to the current excessive use of fishing efforts resulting in devastating consequences. The purpose of this study is to propose a bio-economic mathematical model by introducing taxes to the profit per unit biomass of the harvested fish of each species with the intention of controlling fishing efforts in the presence of toxicity. We obtained both boundary and interior equilibrium points along with the conditions ensuring their validity. Local stability for the interior equilibrium point has been found by the trace-determinant criterion and global stability has been analyzed through a suitable Lyapunov function. We have also obtained the optimal harvesting policy with the help of Pontryagin's maximum principle. Lastly, numerical simulation with the help of MATLAB have been done and thus, the results of the formulated model have been established.

Keywords: Stage Structure; Toxicity; Equilibrium points; Local Stability; Optimal Harvesting.

AMS Subject Classification: 34A34, 65L05.

1 Introduction

Overfishing of commercial fish species is a grave problem due to the rapid growth of industrialization and population. Various surveys have indicated that there has been a rapid decline of fish species. Given the economic importance of the fishery, management measures aiming at controlling fishing efforts are needed for sustainability of the species. Possible control instruments for regulating harvesting efforts as were pointed out by [3] could be taxation, license fees, lease of property rights, seasonal harvesting, fishing period control, creating reserve zones and many more depending on the nature of the fishery. Open access is the condition where access to the fishery (for the purpose of harvesting fish) is unrestricted; i.e., the right to catch fish is free and open to all. So the taxation method could apply as the efficiency method. Authors [1]-[8] have suggested that governments or fishing regulatory authorities can use taxation as an effective control instrument to regulate the extent of fishing efforts. [10] proposed a mathematical model to study the growth and exploitation of a schooling fish species by imposing a tax on the catch to control the overexploitation of fish species. [2] discussed a dynamical model for a single species fishery, which depends partially on logistically growing resource with functional response and taxation as a control instrument to protect fish population.

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from overexploitation. [9] studied a fishery model containing predator fish and prey fish in which the predator was the commercial fish by including spawning periods and taxation. [5] studied a dynamic model for fishery resource with reserve area and taxation. [4] further analyzed a non-linear mathematical model to study the dynamics of an inshore-offshore fishery under variable harvesting by considering taxation as the control instrument [9]-[19], [22]. Moreover, the effect of toxic substance on ecological communities have posed huge problems on environment. Such type of problems in mathematical modeling were studied by scholars. In the real world, almost all animals have the stage-structure of immature and mature. However, no attempt has been made to study the optimal taxation policy of a stage-structured three species fishery in the presence of toxicity in which they interact in a prey-predator manner and all species being subjected to harvesting [20]-[21].

2 Mathematical Model

In this section, we discuss the mathematical model describing the stage-structured predator-prey model where each fish species- prey, immature predator and mature predator is infected by toxic materials released by some external sources like factories, industries, etc. and also subjected to harvesting. The model is developed as under:

1. In the Lotka Volterra prey-predator model, the prey population is growing logistically at the rate $r_1$ with carrying capacity $L$ in the absence of predator species. So we consider the first term of prey species as $r_1x_1(1 - \frac{x_1}{L})$.

The mature predator consumes the prey species at the rate $\beta$. We suppose that the mature predator attack the prey at the rate of $\beta x_1$. In this model, we have also considered different harvesting rates of each species rather than the same harvesting rate.

Also, the effect of toxicity on the prey population is measured by $\gamma_1 x_1^2$. Since, $\frac{d(\gamma_1 x_1^2)}{dx_1} = 2\gamma_1 x_1 > 0$ and $\frac{d^2(\gamma_1 x_1^2)}{dx_1^2} = 2\gamma_1 > 0$, it shows that there is a growth in the production of toxic materials of the prey population and more of the prey species consume the toxic substance. Thus, the dynamics of prey population is governed by:

$$\frac{dx_1}{dt} = r_1x_1(1 - \frac{x_1}{L}) - \gamma_1 x_1^2 - \beta x_1 x_3 - q_1 E_1 x_1 \tag{2.1}$$

where,

- $x_1 = x_1(t)$ is the stock biomass of prey species at time $t$,
- $x_3 = x_3(t)$ is the stock biomass of mature predator species at time $t$,
- $r_1$ is the intrinsic growth rate of prey species,
- $\beta$ is the rate of interaction of prey with mature predator,
- $\gamma_1$ is the coefficient of toxicity to the prey species,
- $q_1$ is the catchability coefficient of prey species,
- $E_1 = E_1(t)$ is the fishing effort of prey population.

2. The dynamics of immature predator is governed by:

$$\frac{dx_2}{dt} = \alpha x_3 - \mu x_2 - \gamma_2 x_2 - \delta x_2 - q_2 E_2 x_2 \tag{2.2}$$

where,

- $x_2 = x_2(t)$ is the stock biomass of immature predator population at time $t$,
- $x_3 = x_3(t)$ is the stock biomass of mature predator population at time $t$,
- $\alpha$ is the growth rate of immature predator because of mature predator,
- $\mu$ is the death rate of predator species,
We impose taxation on the fishing efforts so as to sustain fishing of the species. Thus, using (2.1), (2.2), (2.3) and above equations, the system of equations becomes:

The dynamics of mature predator is governed by:

\[
\frac{dx_3}{dt} = \theta \beta x_1 x_3 - \mu x_3 - \gamma_3 x_3 + \delta x_2 - q_3 E_3 x_3
\]  

where,

\( x_1 = x_1(t) \) is the stock biomass of prey population at time \( t \),

\( x_2 = x_2(t) \) is the stock biomass of immature predator population at time \( t \),

\( x_3 = x_3(t) \) is the stock biomass of mature predator population at time \( t \),

\( \theta \) is the conversion rate from prey to predator,

\( \gamma_3 \) is the conversion rate of immature predator population to mature predator population,

\( \delta \) is the conversion rate of immature predator population to mature predator population,

\( q_3 \) is the catchability coefficient of mature predator species,

\( E_3 = E_3(t) \) is the fishing effort of mature predator species.

3. The dynamics of mature predator is governed by:

\[
\frac{dx_3}{dt} = \theta \beta x_1 x_3 - \mu x_3 - \gamma_3 x_3 + \delta x_2 - q_3 E_3 x_3
\]  

4. We impose taxation on the fishing efforts so as to sustain fishing of the species. Thus, \( E_1 E_2 \) and \( E_3 \) are dynamic variables i.e. time dependent. Let \( p_1, p_2 \) and \( p_3 \) be the fixed selling price per unit population of prey, immature predator and mature predator species respectively and let \( c_1, c_2 \) and \( c_3 \) be the fixed cost of harvesting per unit effort for the prey, immature predator and mature predator species respectively.

Thus, the economic revenue for the species will be:

\[
R_1 = p_1 q_1 E_1 x_1 - c_1 E_1
\]  

\[
R_2 = p_2 q_2 E_2 x_2 - c_2 E_2
\]  

\[
R_3 = p_3 q_3 E_3 x_3 - c_3 E_3
\]

Let \( \tau_1 > 0, \tau_2 > 0 \) and \( \tau_3 > 0 \) be the taxes imposed per unit prey population, immature predator population and mature predator population harvested respectively. The net economic revenue is obtained by the introduction of taxes to the fixed selling price per unit population of fish species. Hence, the above equations become:

\[
R_{1\text{net}} = (p_1 - \tau_1) q_1 E_1 x_1 - c_1 E_1
\]  

\[
R_{2\text{net}} = (p_2 - \tau_2) q_2 E_2 x_2 - c_2 E_2
\]  

\[
R_{3\text{net}} = (p_3 - \tau_3) q_3 E_3 x_3 - c_3 E_3
\]

5. Thus, using (2.1), (2.2), (2.3) and above equations, the system of equations become:

\[
\frac{dx_1}{dt} = r_1 x_1 (1 - \frac{x_1}{L}) - \gamma_1 x_1^2 - \beta x_1 x_3 - q_1 E_1 x_1
\]  

\[
\frac{dx_2}{dt} = \alpha x_3 - \mu x_2 - \gamma_2 x_2 - \delta x_2 - q_2 E_2 x_2
\]  

\[
\frac{dx_3}{dt} = \theta \beta x_1 x_3 - \mu x_3 - \gamma_3 x_3 + \delta x_2 - q_3 E_3 x_3
\]  

\[
\frac{dE_1}{dt} = \phi_1 [(p_1 - \tau_1) q_1 E_1 x_1 - c_1 E_1]
\]  

\[
\frac{dE_2}{dt} = \phi_2 [(p_2 - \tau_2) q_2 E_2 x_2 - c_2 E_2]
\]
The steady state solutions are obtained from the following system of equations:

\[
\begin{align*}
    r_1 x_1 (1 - \frac{x_1}{L}) - \gamma_1 x_1^2 - \beta x_1 x_3 - q_1 E_1 x_1 &= 0 \\
    \alpha x_3 - \mu x_2 - \gamma_2 x_2 - \delta x_2 - q_2 E_2 x_2 &= 0 \\
    \theta \beta x_1 x_3 - \mu x_3 - \gamma_3 x_3 + \delta x_2 - q_3 E_3 x_3 &= 0
\end{align*}
\]

(3.16)

Also \(x_1(0) > 0, x_2(0) > 0, x_3(0) > 0, E_1(0) > 0, E_2(0) > 0\) and \(E_3(0) > 0\).

\(\phi_j\) for \(j=1,2,3\) are adjustment coefficients.

### 3 Dynamical Behaviour

#### 3.1 Equilibrium points

The steady state solutions are obtained from the following system of equations:

\[
\begin{align*}
    r_1 x_1 (1 - \frac{x_1}{L}) - \gamma_1 x_1^2 - \beta x_1 x_3 - q_1 E_1 x_1 &= 0 \\
    \alpha x_3 - \mu x_2 - \gamma_2 x_2 - \delta x_2 - q_2 E_2 x_2 &= 0 \\
    \theta \beta x_1 x_3 - \mu x_3 - \gamma_3 x_3 + \delta x_2 - q_3 E_3 x_3 &= 0
\end{align*}
\]

(3.16)

From (3.16) we get,

\[
\begin{align*}
    x_1 [r_1 (1 - \frac{x_1}{L}) - \gamma_1 x_1 - \beta x_3 - q_1 E_1] &= 0 \\
    r_1 (1 - \frac{x_1}{L}) - \gamma_1 x_1 - \beta x_3 - q_1 E_1 &= 0 \\
    r_1 - x_1 (a_1 + \gamma_1) - \beta x_3 - q_1 E_1 &= 0
\end{align*}
\]

where \(a_1 = \frac{r_1}{L}\)

Thus,

\[
E_1 = \frac{1}{q_1} (r_1 - \beta x_3 - x_1 (a_1 + \gamma_1))
\]

From (3.17) we get,

\[
\begin{align*}
    \alpha x_3 - \mu x_2 - \gamma_2 + \delta + q_2 E_2 &= 0 \\
    x_2 &= \frac{\alpha x_3}{\mu + \gamma_2 + \delta + q_2 E_2} \\
    E_2 &= \frac{1}{q_2} \left( \frac{\alpha x_3}{x_2} - (\mu + \gamma_2 + \delta) \right)
\end{align*}
\]

From (3.18) we get,

\[
\begin{align*}
    -x_3 (-\theta \beta x_1 + \mu + \gamma_3 + q_3 E_3) + \delta x_2 &= 0 \\
    x_3 &= \frac{\delta x_2}{(-\theta \beta x_1 + \mu + \gamma_3 + q_3 E_3)}
\end{align*}
\]

Thus,

\[
E_3 = -\left( \frac{-\theta \beta x_1 + \mu + \gamma_3}{q_3} \right) + \frac{\delta x_2}{q_3 x_3}
\]

i.e.,

\[
E_3 = \frac{1}{q_3} \left( \frac{\delta x_2}{x_3} - (-\theta \beta x_1 + \mu + \gamma_3) \right)
\]
From (3.19) we get,
\[ \dot{x}_1 = \frac{c_1}{q_1(p_1 - \tau_1)}, p_1 > \tau_1 \]
From (3.20) we get,
\[ \dot{x}_2 = \frac{c_2}{q_2(p_2 - \tau_2)}, p_2 > \tau_2 \]
From (3.21) we get,
\[ \dot{x}_3 = \frac{c_3}{q_3(p_3 - \tau_3)}, p_3 > \tau_3 \]
Thus, the system has the following 18 possible equilibrium points:
\[ P_1 = (0, 0, 0, \frac{r_1}{q_1}, 0, 0) \]
where, \( p_1 > \tau_1 \).
\[ P_2 = (0, \frac{c_1}{q_1(p_1 - \tau_1)}, 0, 0, 0, 0) \]
where, \( p_2 > \tau_2 \).
\[ P_3 = (0, 0, \frac{c_2}{q_2(p_2 - \tau_2)}, 0, 0, 0) \]
where, \( p_3 > \tau_3 \).
\[ P_4 = (0, 0, 0, \frac{r_1 - \beta \dot{x}_3}{q_1}, 0, 0) \]
where \( r_1 > \beta \dot{x}_3, p_3 > \tau_3 \).
\[ P_5 = (0, 0, 0, \frac{c_1}{q_1(p_1 - \tau_1)}, \frac{r_1 - \dot{x}_1(a_1 + \gamma_1)}{q_1}, 0, 0) \]
where \( r_1 > \dot{x}_1(a_1 + \gamma_1), p_1 > \tau_1 \).
\[ P_6 = (0, 0, 0, \frac{c_2}{q_2(p_2 - \tau_2)}, 0, \frac{r_1}{q_1}, 0, 0) \]
where \( p_2 > \tau_2 \).
\[ P_8 = (0, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{c_3}{q_3(p_3 - \tau_3)}, \frac{r_1 - \beta \dot{x}_3}{q_1}, \frac{1}{q_2}(\alpha \dot{x}_3 - (\mu + \gamma_2)), \frac{1}{q_3}(\delta \dot{x}_3 - (\mu + \gamma_3))) \]
where \( r_1 > \beta \dot{x}_3, \alpha \dot{x}_3 > (\mu + \gamma_2 + \delta) \) and \( \delta \dot{x}_3 > (\mu + \gamma_3) \), \( p_2 > \tau_2, p_3 > \tau_3 \).
\[ P_9 = (0, \frac{c_3}{q_3(p_3 - \tau_3)}, 0, \frac{r_1 - \beta \dot{x}_3 - \dot{x}_1(a_1 + \gamma_1)}{q_1}, 0, \frac{\theta \beta \xi_1 - \mu - \gamma_3}{q_3}) \]
where \( r_1 > \beta \dot{x}_3 + \dot{x}_1(a_1 + \gamma_1) \) and \( \theta \beta \xi_1 > \mu + \gamma_3, p_1 > \tau_1, p_3 > \tau_3 \).
\[ P_{10} = (0, \frac{c_1}{q_1(p_1 - \tau_1)}, 0, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{1}{q_1}(r_1 - \dot{x}_1(a_1 + \gamma_1)), 0, 0) \]
where \( r_1 > \dot{x}_1(a_1 + \gamma_1), p_1 > \tau_1, p_2 > \tau_2 \).
where $p_1 > \tau_1, p_2 > \tau_2, p_3 > \tau_3$.

\[ P_{12} = \left( \frac{c_1}{q_1(p_1 - \tau_1)}, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{c_3}{q_3(p_3 - \tau_3)}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right) \]

where $\frac{\delta \xi}{\xi} > (\theta \beta \xi_1 + \mu + \gamma_1), p_1 > \tau_1, p_2 > \tau_2, p_3 > \tau_3$.

\[ P_{13} = \left( \frac{c_1}{q_1(p_1 - \tau_1)}, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{c_3}{q_3(p_3 - \tau_3)}, 0, 0, \frac{1}{q_3} \left( \frac{\alpha \xi_3}{\xi_2} - (\mu + \gamma_2 + \delta) \right), 0, 0 \right) \]

where $\frac{\alpha \xi}{\xi} > (\mu + \gamma_2 + \delta), p_2 > \tau_2, p_3 > \tau_3$.

\[ P_{14} = \left( \frac{c_1}{q_1(p_1 - \tau_1)}, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{c_3}{q_3(p_3 - \tau_3)}, 0, 0, \frac{1}{q_1} (r_1 - \beta \xi_3 - \xi_1 (a_1 + \gamma_1)), 0, 0 \right) \]

where $r_1 > \beta \xi_3 + \xi_1 (a_1 + \gamma_1), p_1 > \tau_1, p_2 > \tau_2, p_3 > \tau_3$.

\[ P_{15} = \left( \frac{c_1}{q_1(p_1 - \tau_1)}, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{c_3}{q_3(p_3 - \tau_3)}, 0, 0, \frac{1}{q_2} (\frac{\alpha \xi_3}{\xi_2} - (\mu + \gamma_2 + \delta)), 0, 0 \right) \]

where $\frac{\alpha \xi}{\xi} > (\mu + \gamma_2 + \delta)$ and $\frac{\delta \xi}{\xi} > (\theta \beta \xi_1 + \mu + \gamma_1), p_1 > \tau_1, p_2 > \tau_2, p_3 > \tau_3$.

\[ P_{16} = \left( \frac{c_1}{q_1(p_1 - \tau_1)}, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{c_3}{q_3(p_3 - \tau_3)}, 0, 0, \frac{1}{q_3} (\frac{\delta \xi_3}{\xi_3} + (\theta \beta \xi_1 - \mu - \gamma_3)) \right) \]

where $r_1 > \beta \xi_3 + \xi_1 (a_1 + \gamma_1)$ and $\frac{\delta \xi}{\xi} > (\theta \beta \xi_1 + \mu + \gamma_1), p_1 > \tau_1, p_2 > \tau_2, p_3 > \tau_3$.

\[ P_{17} = \left( \frac{c_1}{q_1(p_1 - \tau_1)}, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{c_3}{q_3(p_3 - \tau_3)}, 0, 0, \frac{1}{q_1} (r_1 - \beta \xi_3 - \xi_1 (a_1 + \gamma_1)), 0, 0 \right) \]

where $r_1 > \beta \xi_3 + \xi_1 (a_1 + \gamma_1)$ and $\frac{\alpha \xi}{\xi} > (\mu + \gamma_2 + \delta), p_1 > \tau_1, p_2 > \tau_2, p_3 > \tau_3$.

\[ P_{18} = \left( \frac{c_1}{q_1(p_1 - \tau_1)}, \frac{c_2}{q_2(p_2 - \tau_2)}, \frac{c_3}{q_3(p_3 - \tau_3)}, 0, 0, \frac{1}{q_1} (r_1 - \beta \xi_3 - \xi_1 (a_1 + \gamma_1)), \frac{1}{q_3} (\frac{\alpha \xi_3}{\xi_2} - (\mu + \gamma_2 + \delta)), \frac{1}{q_3} (\frac{\delta \xi_3}{\xi_3} + (\theta \beta \xi_1 - \mu - \gamma_3)) \right) \]

where $r_1 > \beta \xi_3 + \xi_1 (a_1 + \gamma_1), \frac{\alpha \xi}{\xi} > (\mu + \gamma_2 + \delta)$ and $\frac{\delta \xi}{\xi} > (\theta \beta \xi_1 + \mu + \gamma_3), p_1 > \tau_1, p_2 > \tau_2, p_3 > \tau_3$. 

\[ 1 \]
3.2 Local stability

We investigate the local stability of the equilibrium points. Here, the trace-determinant criteria is used. An equilibrium point is locally stable if the Jacobian matrix evaluated at that point has a positive determinant and negative trace. The Jacobian matrix of the system is a 6X6 matrix given by:

$$\begin{bmatrix}
\frac{\partial A}{\partial x_1} & \frac{\partial A}{\partial x_2} & \frac{\partial A}{\partial x_3} & \frac{\partial A}{\partial E_1} & \frac{\partial A}{\partial E_2} & \frac{\partial A}{\partial E_3} \\
\frac{\partial B}{\partial x_1} & \frac{\partial B}{\partial x_2} & \frac{\partial B}{\partial x_3} & \frac{\partial B}{\partial E_1} & \frac{\partial B}{\partial E_2} & \frac{\partial B}{\partial E_3} \\
\frac{\partial C}{\partial x_1} & \frac{\partial C}{\partial x_2} & \frac{\partial C}{\partial x_3} & \frac{\partial C}{\partial E_1} & \frac{\partial C}{\partial E_2} & \frac{\partial C}{\partial E_3} \\
\frac{\partial D}{\partial x_1} & \frac{\partial D}{\partial x_2} & \frac{\partial D}{\partial x_3} & \frac{\partial D}{\partial E_1} & \frac{\partial D}{\partial E_2} & \frac{\partial D}{\partial E_3} \\
\frac{\partial E}{\partial x_1} & \frac{\partial E}{\partial x_2} & \frac{\partial E}{\partial x_3} & \frac{\partial E}{\partial E_1} & \frac{\partial E}{\partial E_2} & \frac{\partial E}{\partial E_3} \\
\frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \frac{\partial F}{\partial x_3} & \frac{\partial F}{\partial E_1} & \frac{\partial F}{\partial E_2} & \frac{\partial F}{\partial E_3}
\end{bmatrix}$$

where,

$$A = x_1[r_1(1 - \frac{x_1}{L}) - r_1x_1 - \beta x_3 - q_1E_1] = 0$$

$$B = \alpha x_3 - x_2(\mu + \gamma_2 + \delta + q_2E_2) = 0$$

$$C = \delta x_2 - x_3(-\theta \beta x_1 + \mu + \gamma + q_3E_3) = 0$$

$$D = \frac{dE_1}{dt} = \phi_1[(p_1 - \tau_1)q_1E_1x_1 - c_1E_1]$$

$$E = \frac{dE_2}{dt} = \phi_2[(p_2 - \tau_2)q_2E_2x_2 - c_2E_2]$$

$$F = \frac{dE_3}{dt} = \phi_3[(p_3 - \tau_3)q_3E_3x_3 - c_3E_3]$$

Therefore,

$$\frac{\partial A}{\partial x_1} = -2a_1x_1 - 2\gamma x_1 + r_1 - \beta x_3 - q_1E_1, \frac{\partial A}{\partial x_2} = 0$$

$$\frac{\partial A}{\partial x_3} = -\beta x_1, \frac{\partial A}{\partial E_1} = -q_1x_1, \frac{\partial A}{\partial E_2} = 0, \frac{\partial A}{\partial E_3} = 0$$

$$\frac{\partial B}{\partial x_1} = 0, \frac{\partial B}{\partial x_2} = -(\mu + \gamma_2 + \delta + q_2E_2), \frac{\partial B}{\partial x_3} = \alpha$$

$$\frac{\partial B}{\partial E_1} = 0, \frac{\partial B}{\partial E_2} = -q_2x_2, \frac{\partial B}{\partial E_3} = 0$$

$$\frac{\partial C}{\partial x_1} = \theta \beta x_3, \frac{\partial C}{\partial x_2} = \delta, \frac{\partial C}{\partial x_3} = -(\mu + \gamma_3 + q_3E_3 - \theta \beta x_1)$$
\[
\frac{\partial C}{\partial E_1} = 0, \frac{\partial C}{\partial E_2} = 0, \frac{\partial C}{\partial E_3} = -q_3 x_3
\]
\[
\frac{\partial D}{\partial x_1} = \phi_1 [(p_1 - \tau_1) q_1 E_1], \frac{\partial D}{\partial x_2} = 0, \frac{\partial D}{\partial x_3} = 0
\]
\[
\frac{\partial E}{\partial x_1} = 0, \frac{\partial E}{\partial x_2} = \phi_2 [(p_2 - \tau_2) q_2 E_2], \frac{\partial E}{\partial x_3} = 0
\]
\[
\frac{\partial F}{\partial x_1} = 0, \frac{\partial F}{\partial x_2} = \phi_3 [(p_3 - \tau_3) q_3 E_3]
\]

Thus, the Jacobian at the equilibrium point \( P_{18} \) is given by:

\[
J[P_{18}] = \begin{bmatrix}
n_{11} & n_{13} & n_{14} & 0 & 0 \\
0 & n_{22} & n_{23} & 0 & n_{25} \\
n_{31} & n_{32} & n_{33} & 0 & 0 & n_{36} \\
n_{41} & 0 & 0 & 0 & 0 \\
0 & n_{52} & 0 & 0 & 0 & 0 \\
0 & 0 & n_{63} & 0 & 0 & 0 \\
\end{bmatrix}
\]

(3.22)

where,

\[
n_{11} = \frac{\partial A}{\partial x_1 | P_{18}} = -(a_1 + \gamma_1) \frac{c_1}{\eta_1 (p_1 - \tau_1)}
\]
\[
n_{13} = \frac{\partial A}{\partial x_3 | P_{18}} = -\beta\frac{c_1}{\eta_1 (p_1 - \tau_1)}
\]
\[
n_{14} = \frac{\partial A}{\partial E_1 | P_{18}} = -\frac{c_1}{(p_1 - \tau_1)}
\]
\[
n_{22} = \frac{\partial B}{\partial x_2 | P_{18}} = -\alpha\frac{c_3 q_2 (p_2 - \tau_2)}{c_2 q_3 (p_3 - \tau_3)}
\]
\[
n_{23} = \frac{\partial B}{\partial x_3 | P_{18}} = \alpha
\]
\[
n_{25} = \frac{\partial B}{\partial E_2 | P_{18}} = -\frac{c_2}{p_2 - \tau_2}
\]
Thus, the time derivative is given by:

\[ \frac{\partial C}{\partial x_1} |_{x_1} = \frac{\partial C}{\partial x_2} |_{x_2} = \frac{\partial C}{\partial x_3} |_{x_3} = 0 \]

Thus, the equilibrium point is given by:

\[ n_{31} = \frac{\partial C}{\partial x_1} |_{x_1} = \frac{\theta \beta e_3}{q_3 (p_3 - \tau_1)} \]

\[ n_{32} = \frac{\partial C}{\partial x_2} |_{x_2} = \delta \]

\[ n_{33} = \frac{\partial C}{\partial x_3} |_{x_3} = -\delta \frac{c_2 q_3 (p_3 - \tau_1)}{c_3 q_2 (p_2 - \tau_2)} \]

\[ n_{36} = \frac{\partial C}{\partial E_3} |_{E_3} = \frac{c_3}{p_3 - \tau_3} \]

\[ n_{41} = \frac{\partial D}{\partial x_1} |_{x_1} = \phi_1 [(p_1 - \tau_1) q_1 \tilde{E}_1] \]

\[ n_{52} = \frac{\partial E}{\partial x_2} |_{x_2} = \phi_2 [(p_2 - \tau_2) q_2 \tilde{E}_2] \]

\[ n_{63} = \frac{\partial F}{\partial x_3} |_{x_3} = \phi_3 [(p_3 - \tau_3) q_3 \tilde{E}_3] \]

Thus, trace is given by:

\[ \text{trace}[J(P_{18})] = n_{11} + n_{22} + n_{33} \]

\[ = -(a_1 + \gamma_1) \frac{c_1}{q_1 (p_1 - \tau_1)} - \alpha \frac{c_2 q_2 (p_2 - \tau_2)}{c_2 q_3 (p_3 - \tau_3)} - \delta \frac{c_2 q_3 (p_3 - \tau_3)}{c_3 q_2 (p_2 - \tau_2)} \]

\[ = -[(a_1 + \gamma_1) \frac{c_1}{q_1 (p_1 - \tau_1)} + \alpha \frac{c_2 q_2 (p_2 - \tau_2)}{c_2 q_3 (p_3 - \tau_3)} + \delta \frac{c_2 q_3 (p_3 - \tau_3)}{c_3 q_2 (p_2 - \tau_2)}] < 0 \]

and determinant is given by:

\[ \text{det}[J(P_{18})] = -n_{14} * n_{41} * -n_{32} * n_{63} * n_{25} * -n_{36} > 0 \]

Thus, the equilibrium point \( P_{18} \) is locally stable.

3.3 Global stability

Global stability is investigated through a suitable Lyapunov function.

Consider the Lyapunov function,

\[ V(x_1, x_2, x_3, E_1, E_2, E_3) = m_1 \left[ x_1 - \dot{x}_1 - x_1 \ln \left( \frac{x_1}{\beta \chi_1} \right) \right] + m_2 \left[ x_2 - \dot{x}_2 - x_2 \ln \left( \frac{x_2}{\beta \chi_2} \right) \right] \]

\[ + m_3 \left[ x_3 - \dot{x}_3 - x_3 \ln \left( \frac{x_3}{\beta \chi_3} \right) \right] + m_4 \left[ E_1 - \dot{E}_1 - E_1 \ln \left( \frac{E_1}{\beta \chi_1} \right) \right] \]

\[ + m_5 \left[ E_2 - \dot{E}_2 - E_2 \ln \left( \frac{E_2}{\beta \chi_2} \right) \right] + m_6 \left[ E_3 - \dot{E}_3 - E_3 \ln \left( \frac{E_3}{\beta \chi_3} \right) \right] \]

where \( m_i > 0 \) for \( i = 1, 2, 3, 4, 5, 6 \).

Thus, the time derivative is given by:
\[
\frac{dV}{dt} = m_1 \left(1 - \frac{\dot{x}_1}{x_1}\right) \frac{dx_1}{dt} + m_2 \left(1 - \frac{\dot{x}_2}{x_2}\right) \frac{dx_2}{dt} + m_3 \left(1 - \frac{\dot{x}_3}{x_3}\right) \frac{dx_3}{dt} + m_4 \left(1 - \frac{\tilde{E}_1}{E_1}\right) \frac{d\tilde{E}_1}{dt} \\
+ m_5 \left(1 - \frac{\tilde{E}_2}{E_2}\right) \frac{d\tilde{E}_2}{dt} + m_6 \left(1 - \frac{\tilde{E}_3}{E_3}\right) \frac{d\tilde{E}_3}{dt}
\]

Let
\[
\frac{dV}{dt} = m_1 G + m_2 H + m_3 I + m_4 K + m_5 L + m_6 R
\]
where,
\[
G = (1 - \frac{\dot{x}_1}{x_1}) \frac{dx_1}{dt} = (x_1 - \dot{x}_1)\left[-(x_1 - \dot{x}_1)(a_1 + \gamma_1) - \beta(x_3 - \dot{x}_3) - q_1(E_1 - \tilde{E}_1)\right]
\]
\[
H = (1 - \frac{\dot{x}_2}{x_2}) \frac{dx_2}{dt} = \frac{x_2 - \dot{x}_2}{x_2} \left[\alpha(x_3 - \dot{x}_3) - (x_2 - \dot{x}_2)(\mu + \gamma_2 + \delta) - x_2q_2E_2 + \tilde{x}_q\tilde{E}_2\right]
\]
\[
I = (1 - \frac{\dot{x}_3}{x_3}) \frac{dx_3}{dt} = \frac{x_3 - \dot{x}_3}{x_3} \left[\delta(x_2 - \dot{x}_2) - (x_3 - \dot{x}_3)(\mu + \gamma_3) + \theta \beta x_1x_3 - q_3E_3x_3 - \theta \beta \tilde{x}_1 \tilde{x}_3 + q_3\tilde{E}_3\right]
\]
\[
K = (1 - \frac{\tilde{E}_1}{E_1}) \frac{d\tilde{E}_1}{dt} = \phi_1(E_1 - \tilde{E}_1)[q_1(p_1 - \tau_1)(x_1 - \dot{x}_1)]
\]
\[
L = (1 - \frac{\tilde{E}_2}{E_2}) \frac{d\tilde{E}_2}{dt} = \phi_2(E_2 - \tilde{E}_2)[q_2(p_2 - \tau_2)(x_2 - \dot{x}_2)]
\]
\[
R = (1 - \frac{\tilde{E}_3}{E_3}) \frac{d\tilde{E}_3}{dt} = \phi_3(E_3 - \tilde{E}_3)[q_3(p_3 - \tau_3)(x_3 - \dot{x}_3)]
\]

Now, substituting the values of G, H, I, K, L, R in \(\frac{dV}{dt}\), we get:
\[
\frac{dV}{dt} = -m_1(x_1 - \dot{x}_1)(a_1 + \gamma_1) - m_1\beta(x_1 - \dot{x}_1)(x_3 - \dot{x}_3) - m_1q_1(x_1 - \dot{x}_1)(E_1 - \tilde{E}_1) \\
+ m_2\alpha(x_2 - \dot{x}_2)(x_3 - \dot{x}_3) - m_2(x_2 - \dot{x}_2)^2(\mu + \gamma_2 + \delta) - m_2q_2E_2(x_2 - \dot{x}_2) \\
+ m_2\tilde{x}_q\tilde{E}_2(x_2 - \dot{x}_2) + m_3\delta(x_2 - \dot{x}_2)(x_3 - \dot{x}_3) - m_3(x_3 - \dot{x}_3)^2(\mu + \gamma_3) + m_3\theta \beta x_1x_3 - q_3E_3x_3 - \theta \beta \tilde{x}_1 \tilde{x}_3 + q_3\tilde{E}_3 \\
- m_3q_3E_3(x_3 - x_3) - m_3\theta \beta \tilde{x}_1 \tilde{x}_3(x_3 - \dot{x}_3) + m_3q_3\tilde{E}_3 \tilde{x}_3(x_3 - \dot{x}_3) \\
+ m_4\phi_1(E_1 - \tilde{E}_1)[q_1(p_1 - \tau_1)(x_1 - \dot{x}_1)] + m_5\phi_2(E_2 - \tilde{E}_2)[q_2(p_2 - \tau_2)(x_2 - \dot{x}_2)] \\
+ m_6\phi_3(E_3 - \tilde{E}_3)[q_3(p_3 - \tau_3)(x_3 - \dot{x}_3)] \\
= -[m_1x_1(x_1 - \dot{x}_1)(a_1 + \gamma_1) + m_1\beta(x_1 - \dot{x}_1)(x_3 - \dot{x}_3) + m_1q_1(x_1 - \dot{x}_1)(E_1 - \tilde{E}_1) \\
+ m_2(x_2 - \dot{x}_2)^2(\mu + \gamma_2 + \delta) + m_2q_2E_2(x_2 - \dot{x}_2) + m_3(x_3 - \dot{x}_3)^2(\mu + \gamma_3) \\
+ m_3q_3E_3(x_3 - \dot{x}_3) + m_3\theta \beta \tilde{x}_1 \tilde{x}_3(x_3 - \dot{x}_3)]
\]
Optimal Harvesting Policy

The present value \( J \) of a continuous time-stream of revenues is:

\[
J = \int_0^\infty e^{-\delta t} [(p_1 q_1 E_1 x_1 - c_1 E_1) + (p_2 q_2 E_2 x_2 - c_2 E_2) + (p_3 q_3 E_3 x_3 - c_3 E_3)] dt,
\]

(4.23)

where \( \delta \) is the instantaneous rate of annual discount. Thus, our objective is to maximize \( J \) subject to the equations in (2.10-2.15) and to the control parameters:

\[
\begin{align*}
&\frac{dV}{dt} = \alpha (x_2 - \bar{x}_2)(x_3 - \bar{x}_3) + \frac{x_2 q_2 E_2 (x_2 - \bar{x}_2)}{x_2} + \frac{x_3 \delta (x_2 - \bar{x}_2)(x_3 - \bar{x}_3)}{x_3} \nonumber \\
&+ m_3 \theta \beta (x_3 - \bar{x}_3) + m_3 q_3 \bar{E}_3 \frac{x_3 (x_3 - \bar{x}_3)}{x_3} + m_4 \phi (E_1 - \bar{E}_1) [q_1 (p_1 - \tau_1) (x_1 - \bar{x}_1)] \\
&+ m_5 \phi_2 (E_2 - \bar{E}_2) [q_2 (p_2 - \tau_2) (x_2 - \bar{x}_2)] + m_6 \phi_3 (E_3 - \bar{E}_3) [q_3 (p_3 - \tau_3) (x_3 - \bar{x}_3)] \\
\end{align*}
\]

which is:

\[
= -S_1 + S_2,
\]

where,

\[
S_1 = m_1 (x_1 - \bar{x}_1) (a_1 + \gamma_1) + m_1 \beta (x_1 - \bar{x}_1) (x_3 - \bar{x}_3) + m_1 q_1 (x_1 - \bar{x}_1) (E_1 - \bar{E}_1) \\
+ \frac{m_2 (x_2 - \bar{x}_2)^2 (\mu + \gamma_2 + \delta)}{x_2} + m_2 q_2 E_2 (x_2 - \bar{x}_2) + m_3 (x_3 - \bar{x}_3)^2 (\mu + \gamma_3) \\
+ m_3 q_3 E_3 (x_3 - \bar{x}_3) + m_3 \theta \beta \frac{x_1 \bar{x}_3 (x_3 - \bar{x}_3)}{x_3}
\]

and

\[
S_2 = m_2 \alpha (x_2 - \bar{x}_2) (x_3 - \bar{x}_3) + \frac{x_2 q_2 E_2 (x_2 - \bar{x}_2)}{x_2} + \frac{x_3 \delta (x_2 - \bar{x}_2)(x_3 - \bar{x}_3)}{x_3} \\
+ m_3 \theta \beta (x_3 - \bar{x}_3) + m_3 q_3 \bar{E}_3 \frac{x_3 (x_3 - \bar{x}_3)}{x_3} + m_4 \phi (E_1 - \bar{E}_1) [q_1 (p_1 - \tau_1) (x_1 - \bar{x}_1)] \\
+ m_5 \phi_2 (E_2 - \bar{E}_2) [q_2 (p_2 - \tau_2) (x_2 - \bar{x}_2)] + m_6 \phi_3 (E_3 - \bar{E}_3) [q_3 (p_3 - \tau_3) (x_3 - \bar{x}_3)].
\]

Thus, we get,

(i) \( \frac{dV}{dt} = 0 \) for all \( (x_1, x_2, x_3, E_1, E_2, E_3) = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{E}_1, \bar{E}_2, \bar{E}_3) \).

(ii) If \( m_1 = m_2 = m_3 \) and \( S_1 > S_2 \),

then \( \frac{dV}{dt} < 0 \) for all \( (x_1, x_2, x_3, E_1, E_2, E_3) \neq (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{E}_1, \bar{E}_2, \bar{E}_3) \).

Therefore, the equilibrium point \( P_{eq} \) is globally stable.

4 Optimal Harvesting Policy

In this section, we analyze the optimal harvesting policy for the system in (2.10-2.15) so as to maximize the total discounted net revenue using taxation as a control parameter. The present value \( J \) of a continuous time-stream of revenues is:

\[
J = \int_0^\infty e^{-\delta t} [(p_1 q_1 E_1 x_1 - c_1 E_1) + (p_2 q_2 E_2 x_2 - c_2 E_2) + (p_3 q_3 E_3 x_3 - c_3 E_3)] dt,
\]

(4.23)
\[ \tau_{minj} < \tau < \tau_{maxj} \text{ for } j=1,2,3. \]

To find the optimal equilibrium, Pontryagin’s maximum principle is used. The associated Hamiltonian function is given by:

\[
H = e^{-\hat{\mu}t}[(p_1q_1x_1 - c_1E_1) + (p_2q_2x_2 - c_2E_2) + (p_3q_3x_3 - c_3E_3)] \\
+ \lambda_1[x_1(r_1(1 - \frac{x_1}{L}) - y_1x_1 - \beta x_3 - q_1E_1)] \\
+ \lambda_2[\alpha x_3 - x_2(\mu + y_2 + \delta + q_2E_2)] \\
+ \lambda_3[\delta x_2 - x_3(\mu + y_3 + q_3E_3 - \theta \beta x_1)] \\
+ \lambda_4[\phi_1(p_1 - \tau_1)q_1E_1x_1 - c_1E_1] \\
+ \lambda_5[\phi_2(p_2 - \tau_2)q_2E_2x_2 - c_2E_2] \\
+ \lambda_6[\phi_3(p_3 - \tau_3)q_3E_3x_3 - c_3E_3], 
\]

(4.24)

where, \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) and \( \lambda_6 \) are adjoint variables in terms of time.

Hamiltonian 'H' should be maximized for \( \tau(t) \in [\tau_{minj}, \tau_{maxj}] \) where \( j=1,2,3 \).

We assume that the control constraints are not binding (that is, the optimal solution does not occur at \( \tau(t) = \tau_{minj} \) or \( \tau_{maxj} \) for \( j=1,2,3 \)). Thus, we have singular control given by:

\[
\frac{\partial H}{\partial \tau_1} = 0, \quad \frac{\partial H}{\partial \tau_2} = 0, \quad \frac{\partial H}{\partial \tau_3} = 0. 
\]

(4.25)

Applying (4.25), we obtain,

\[ \lambda_4 = \lambda_5 = \lambda_6 = 0. \]  

(4.27)

Applying (4.26), we obtain,

\[ \frac{\partial H}{\partial \tau_1} = e^{-\hat{\mu}t}(p_1q_1x_1 - c_1) - \lambda_1 q_1x_1 = 0 \]

\[ \Rightarrow \lambda_1(t) = e^{-\hat{\mu}t}(p_1 - \frac{c_1}{q_1x_1}) \]  

(4.28a)

\[ \frac{\partial H}{\partial \tau_2} = e^{-\hat{\mu}t}(p_2q_2x_2 - c_2) - \lambda_2 x_2q_2 = 0 \]

\[ \Rightarrow \lambda_2(t) = e^{-\hat{\mu}t}(p_2 - \frac{c_2}{q_2x_2}) \]  

(4.28b)

\[ \frac{\partial H}{\partial \tau_3} = e^{-\hat{\mu}t}(p_3q_3x_3 - c_3) - \lambda_3 x_3q_3 = 0 \]

\[ \Rightarrow \lambda_3(t) = e^{-\hat{\mu}t}(p_3 - \frac{c_3}{q_3x_3}) \]  

(4.28c)

By Pontryagin’s maximum principle, again we have:

\[ \frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i} \]

(4.29a)
\[
\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2}
\]  
(4.29b)

\[
\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial x_3}
\]  
(4.29c)

Considering (4.29a) we get,

\[
\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1} + \lambda_1 \left( r_1 - \frac{2r_1x_1}{L} - 2\gamma x_1 - \beta x_3 - q_1E_1 \right)
\]

Substituting the value of \(\lambda_3\) from above, we get:

\[
\frac{d\lambda_1}{dt} - A_1\lambda_1 = -A_2 e^{-\delta t},
\]  
(4.30)

where,

\[
A_1 = -(r_1 - \frac{2r_1x_1}{L} - 2\gamma x_1 - \beta x_3 - q_1E_1)
\]  
(4.31a)

\[
A_2 = p_1q_1E_1 - (p_3 - \frac{c_1}{q_3x_3})\theta\beta x_3
\]  
(4.31b)

Now, employing an I.F. \(e^{-A_1t}\) to solve (4.30) resulted into:

\[
\lambda_1(t) \ast I.F. = \int \left( \text{RHS} \ast I.F. \right) + T_1
\]

(4.32)

where \(T_1\) is a constant of integration.

\[
\Rightarrow \lambda_1(t) = \frac{A_2}{A_1 + \delta} e^{-\delta t} + T_1 e^{A_1t}.
\]  
(4.33)

Let \(\mu_1(t) = \lambda_1 e^{\delta t} = (p_1 - \frac{c_1}{q_1x_1})\), the shadow price per unit population of prey harvested. When \(t \to \infty\), then \(\mu_1(t)\) is bounded iff \(T_1 = 0\).

Hence, (4.33) can be rewritten as:

\[
\Rightarrow \lambda_1(t) = \frac{A_2}{A_1 + \delta} e^{-\delta t}.
\]  
(4.34)

Now, considering (4.29b),we get,

\[
\frac{d\lambda_2}{dt} = -\left[ e^{-\delta t} \frac{p_2q_2E_2}{q_3x_3} - \lambda_2(\mu + \gamma_2 + \delta + q_2E_2) + \lambda_3\delta \right]
\]

Substituting the value of \(\lambda_3\), we get,

\[
\frac{d\lambda_2}{dt} - B_1\lambda_2 = -B_2 e^{-\delta t},
\]  
(4.35)

where,

\[
B_1 = \mu + \gamma_2 + \delta + q_2E_2
\]  
(4.36a)
\[ B_2 = p_2 q_2 E_2 + (p_3 - \frac{c_1}{q_3 x_3}) \delta \]  
\[(4.36b)\]

Now, employing an I.F. \( e^{-B_1 t} \) to solve (4.35) resulted into the following,
\[
\Rightarrow \lambda_2(t) = \frac{B_2}{B_1 + \delta} e^{-\delta t} + T_2 e^{B_1 t}.
\]
\[(4.37)\]

where \( T_2 \) is a constant of integration.

Let \( \mu_2(t) = \lambda_2 e^{\delta t} = (p_2 - \frac{c_1}{q_3 x_3}) \), the shadow price per unit population of immature predator harvested.
When \( t \to \infty \), then \( \mu_2(t) \) is bounded iff \( T_2 = 0 \).

Hence, (4.37) can be rewritten as,
\[
\Rightarrow \lambda_2(t) = \frac{B_2}{B_1 + \delta} e^{-\delta t}.
\]
\[(4.38)\]

Now, considering 4.29c) we get,
\[
\frac{d\lambda_3}{dt} = -[e^{-\delta t} p_3 q_3 E_3 - \lambda_1 \beta + \lambda_2 \alpha - \lambda_3 (\mu + \gamma_3 + q_3 E_3 - \theta \beta x_1)]
\]

Substituting the values of \( \lambda_1 \) and \( \lambda_2 \), we get,
\[
\frac{d\lambda_3}{dt} - D_1 \lambda_3 = -D_2 e^{-\delta t}
\]
\[(4.39)\]

where,
\[
D_1 = \mu + \gamma_3 + q_3 E_3 - \theta \beta x_1
\]
\[(4.40a)\]

\[
D_2 = p_3 q_3 E_3 - (p_1 - \frac{c_1}{q_1 x_1}) \beta + (p_2 - \frac{c_2}{q_2 x_2}) \alpha
\]
\[(4.40b)\]

Now, employing an I.F. \( e^{-D_1 t} \) to solve (4.39) resulted into the following,
\[
\Rightarrow \lambda_3(t) = \frac{D_2}{D_1 + \delta} e^{-\delta t} + T_3 e^{D_1 t}.
\]
\[(4.41)\]

where \( T_3 \) is a constant of integration.

Let \( \mu_3(t) = \lambda_3 e^{\delta t} = (p_3 - \frac{c_1}{q_3 x_3}) \), the shadow price per unit biomass of harvested mature predator.
When \( t \to \infty \), then \( \mu_3(t) \) is bounded iff \( T_3 = 0 \).

Hence, (4.41) can be rewritten as,
\[
\Rightarrow \lambda_3(t) = \frac{D_2}{D_1 + \delta} e^{-\delta t}.
\]
\[(4.42)\]

Now, equating eqn. (4.28a) with (4.34), we get,
\[
\lambda_1(t) = e^{-\delta t} (p_1 - \frac{c_1}{q_1 x_1}) = \frac{A_2}{A_1 + \delta} e^{-\delta t}
\]
\[
\Rightarrow (p_1 - \frac{c_1}{q_1 x_1}) = \frac{A_2}{A_1 + \delta}
\]
\[(4.43)\]
Now, substituting the values of $A_1$ and $A_2$ in above, we get,

$$\left(p_1 - \frac{c_1}{q_1 x_1}\right) = \frac{p_1 q_1 E_1 - (p_3 - \frac{c_3}{q_3 x_3}) \theta \beta x_3}{- (r_1 - \frac{2r_1 x_1}{L} - 2\gamma x_3 - \beta x_3 - q_1 E_1) + \delta}$$

Making further simplifications, we get,

$$E_{1\delta} = \frac{1}{q_1 c_1} \left[ (c_1 - p_1 q_1 x_{1\delta})(r_1 - \frac{2r_1 x_{1\delta}}{L} - 2\gamma x_{1\delta} - \beta x_{1\delta}) + \delta (p_1 q_1 x_{1\delta} - c_1) + q_1 x_{1\delta} (p_3 - \frac{c_3}{q_3 x_{3\delta}}) \theta \beta x_{3\delta}\right] \quad (4.44)$$

Now, equating equation (4.28b) with (4.38), we get,

$$\lambda_2(t) = e^{-\delta t} (p_2 - \frac{c_2}{q_2 x_2}) = \frac{B_2}{B_1 + \delta} e^{-\delta t}$$

$$\implies \left(p_2 - \frac{c_2}{q_2 x_2}\right) = \frac{B_2}{B_1 + \delta} \quad (4.45)$$

Now substituting the values of $B_1$ and $B_2$ in above, we get,

$$\left(p_2 - \frac{c_2}{q_2 x_2}\right) = \frac{p_2 q_2 E_2 + (p_3 - \frac{c_3}{q_3 x_3}) \delta}{\mu + \gamma_2 + \delta + q_2 E_2 + \delta}$$

Making further simplifications, we get,

$$E_{2\delta} = \frac{1}{q_2 c_2} \left[ (p_2 q_2 x_{2\delta} - c_2) (\mu + \gamma_2 + \delta) + \delta (p_2 q_2 x_{2\delta} - c_2) - q_2 x_{2\delta} (p_3 - \frac{c_3}{q_3 x_{3\delta}}) \delta\right] \quad (4.46)$$

Now, equating equation (4.28c) with (4.42), we get,

$$\lambda_3(t) = e^{-\delta t} (p_3 - \frac{c_3}{q_3 x_3}) = \frac{D_2}{D_1 + \delta} e^{-\delta t}$$

$$\implies \left(p_3 - \frac{c_3}{q_3 x_3}\right) = \frac{D_2}{D_1 + \delta} \quad (4.47)$$

Now substituting the values of $D_1$ and $D_2$ in above, we get,

$$\left(p_3 - \frac{c_3}{q_3 x_3}\right) = \frac{p_3 q_3 E_3 - (p_1 - \frac{c_1}{q_1 x_1}) \beta + (p_2 - \frac{c_2}{q_2 x_2}) \alpha}{\mu + \gamma_3 + q_3 E_3 - \theta \beta x_1 + \delta}$$

Making further simplifications, we get,

$$E_{3\delta} = \frac{1}{q_3 c_3} \left[ (p_3 q_3 x_{3\delta} - c_3) (\mu + \gamma_3 - \theta \beta x_{1\delta}) + \delta (p_3 q_3 x_{3\delta} - c_3) + q_3 x_{3\delta} (p_1 - \frac{c_1}{q_1 x_{1\delta}}) \beta - q_3 x_{3\delta} (p_2 - \frac{c_2}{q_2 x_{2\delta}}) \alpha\right] \quad (4.48)$$
At the optimal level, equations (2.10), (2.11) and (2.12) becomes,

\[ r_1 \left(1 - \frac{x_{1\delta}}{L}\right) - \gamma_1 x_{1\delta} - \beta x_{3\delta} - q_1 E_{1\delta} = 0 \]  
\[ -x_{2\delta} (\mu + \gamma_2 + \delta + q_2 E_{2\delta}) + \alpha x_{3\delta} = 0 \]  
\[ \delta x_{2\delta} - x_{3\delta} (-\theta \beta x_{1\delta} + \mu + \gamma_3 + q_3 E_{3\delta}) = 0 \]

Therefore, the optimal values \(x_{1\delta}, x_{2\delta}, x_{3\delta}, E_{1\delta}, E_{2\delta}\) and \(E_{3\delta}\) are computed using equations (4.44), (4.46), (4.48) and (4.49) where as the optimal taxations \(\tau_{1\delta}, \tau_{2\delta}\) and \(\tau_{3\delta}\) are computed using equations (4.50) below:

\[ \tau_{1\delta} = \left(p_1 - \frac{c_1}{q_1 x_{1\delta}}\right), \]  
\[ \tau_{2\delta} = \left(p_2 - \frac{c_2}{q_2 x_{2\delta}}\right), \]  
\[ \tau_{3\delta} = \left(p_3 - \frac{c_3}{q_3 x_{3\delta}}\right). \]

5 Numerical Simulation

We present numerical simulation of some theoretical results which are discussed in the previous sections. We have studied the behaviour of system for the following set of parameters:

\[ r = 0.99, q_1 = 0.002, q_2 = 0.01, q_3 = 0.03, c_1 = 100, c_2 = 800, c_3 = 780, \phi_1 = 0.10, \phi_2 = 0.10, \phi_3 = 0.1, p_1 = 2800, p_2 = 4500, p_3 = 1500, \gamma_1 = 0.0125, L = 100, \beta = 0.04, \alpha = 0.99, \mu = 0.05, \gamma_2 = 0.01, \delta = 0.01, \theta = 0.99, \gamma_3 = 0.01. \]

Different values of \(\tau_1, \tau_2\) and \(\tau_3\) are considered.

Thus, the following results are obtained:

1. The behavior of the fish population \(x_1, x_2, x_3\) and harvesting efforts \(E_1, E_2, E_3\) with respect to time for different values of tax rates \(\tau_1, \tau_2\) and \(\tau_3\) are shown in Fig 1, 2, 3 and 4. From figure 4, we see that the system is locally asymptotically stable and from figure 1, 2, 3 and 4, we observe that the harvesting efforts decreases with the increase in tax rates.

2. We have also plotted the fish population \(x_1, x_2, x_3\) with respect to time for different values of harvesting efforts \(E_1, E_2, E_3\) which are shown in Fig 5, 6, 7 and 8 and observed that as the harvesting efforts increases, the fish population decreases and gradually move towards extinction.
Figure 1: Effect of taxation on the system

Figure 2: Effect of taxation on the system
Figure 3: Effect of taxation on the system

Figure 4: Effect of taxation on the system
Figure 5: Effect of harvesting efforts on the system

Figure 6: Effect of harvesting efforts on the system
6 Conclusion

We have developed the mathematical model involving stage structure, taxation policy and effect of toxicity on fish species. We have found out eighteen equilibrium points out of which one is interior and the others are boundary equilibrium points. For the interior equilibrium point, we have proved that it is locally stable using trace-determinant criteria. Also by using Lyapunov function, we have proved that the interior equilibrium point is globally stable when \( S_1 > S_2 \). We have also found out the optimal harvesting policy by using Pontryagin’s Maximum Principle. Finally, we have verified our results with the help of numerical simulations by using MATLAB software to solve our system of equations.

We conclude that \( \tau_1, \tau_2 \) and \( \tau_3 \) are important parameters which governs the dynamics of the system of equations. Suitable tax policies are proper measures to manage fishery. However, implementations of these policies has to be done with great care in order to attain bioeconomic equilibrium. Low tax rates will provide higher net revenue to fishers and hence encouraging higher harvesting efforts which may lead to extinction of fish species, where as higher tax rates will result into lower net revenue to fishers which may lead to extreme reduction of fishing efforts and hence...
abundance of fish species which does not favour the ecosystem. Hence, the bioeconomic equilibrium is attained at the optimal tax rates and at the optimal harvesting efforts.

References


