Existence and uniqueness results for pantograph equations with generalized fractional derivative

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Abstract
In this paper, we study the existence and uniqueness results for nonlinear pantograph equations with generalized fractional derivative (Katugampola Caputo fractional derivative). We use the Krasnoselkii’s fixed point theorem to show the existence results. An example is provided to illustrate the results.

Keywords: Generalized fractional derivative; Pantograph equation; Nonlocal condition; Existence; Fixed point.

1 Introduction

The importance of fractional differential equations (FDEs) has emerged as a new branch of applied mathematics, which has been used for constructing many mathematical models in science and engineering. In fact FDEs are considered as models alternative to nonlinear differential equations [6] and other kinds of equations [12, 16, 19]. The theory of FDEs has been widely studied by many authors [2, 1, 7, 17]. Many problems can be modelled with the help of the FDEs in many areas such as seismic analysis, viscous damping, viscoelastic materials and polymer physics in [10, 20].

It is well known that in the deterministic situation there is very special delay differential equation known as pantograph equation. It arises in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, quantum mechanics, and electrodynamics. Pantograph equation has been studied by many researchers and solved by several numerical methods; see [3, 8, 11, 18] and references therein. K. Balachandran et. al. introduced abstract fractional pantograph equations and investigated the existence and uniqueness results for considered equation in [3].

These days generalization of the derivatives of both Riemann-Liouville and Caputo types are introduced and shown the effect of utilizing it in equations of mathematical physics or related to probability. This was done using the definition of generalized fractional derivatives given by Katugampola [13]. The author initiated a new fractional integral, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form. Later, Katugampola [14] introduced a new fractional derivative, which generalizes the two derivatives in question. At this writing, only one paper in the literature has been devoted to pantograph equations with generalized fractional derivative. Motivated by the papers [3, 15], we consider the following nonlocal pantograph equation with generalized fractional...
derivative
\[
\begin{aligned}
(\mathcal{D}_a^\alpha f)(t) &= f(t, x(t), x(\lambda t)), \quad t \in J := [0, T], \\
x(0) + g(x) &= x_0
\end{aligned}
\] (1.1)

where \(\mathcal{D}_a^\alpha\) is the Caputo generalized fractional derivative. Let \(\alpha \in \mathbb{R}, 0 < \lambda < 1, \rho > 0\) and \(f : J \times X \times X \rightarrow X, g : C([0, T], X) \rightarrow X\) are given continuous functions.

It is seen that system (1.1) is equivalent to the following nonlinear integral equation (see [4, 5] for more details).

In passing, we note that the application of nonlinear condition \(x(0) + g(x) = x_0\) in physical problems yields better effect than the initial condition \(x(0) = x_0\) [5].

Let \(C(J, X)\) be the Banach space of continuous function \(x(t)\) with \(x(t) \in X\) for \(t \in J\) and \(\|x\|_{C(J, X)} = \sup_{t \in J} \|x(t)\|\). The rest of this paper is organized as follows. In Section 2, preliminaries and notations are given. In Section 3, we present our main results by Krasnoselkii’s fixed point theorems. An example is given in Section 4 to demonstrate the application of our main results.

2 Preliminaries

This section deals with some preliminaries and notations which are used throughout this paper.

**Definition 2.1.** The Riemann-Liouville fractional integral and derivative of order \(\alpha \in \mathbb{C}, \Re(\alpha) \geq 0\) are given by

\[
(I^\alpha_{a+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,
\]

and

\[
(D^\alpha_{a+} f)(t) = \left(\frac{d}{dt}\right)^n (I^\alpha_{a+} f)(t), \quad t > a,
\]

respectively, where \(n = \lfloor\Re(\alpha)\rfloor\) and \(\Gamma(\alpha)\) is the Gamma function.

**Definition 2.2.** The Hadamard fractional integral and derivative are given by

\[
(I^\alpha_{a+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^n f(s) \frac{ds}{s},
\]

and

\[
(D^\alpha_{a+} f)(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha+1} f(s) \frac{ds}{s},
\]

respectively, for \(t > a \geq 0\) and \(\Re(\alpha) > 0\).

Now we give the definitions of the generalized fractional operators introduced in [14, 15].

**Definition 2.3.** The generalized left-sided fractional integral \(\rho I^\alpha_{a+} f\) of order \(\alpha \in \mathbb{C}, \Re(\alpha)\) is defined by

\[
(\rho I^\alpha_{a+} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (\rho^s - s^\alpha)^{\alpha-1}\rho^{\alpha-1} f(s) ds,
\] (2.3)

for \(t > a\), if the integral exists.

The generalized fractional derivative, corresponding to the generalized fractional integral (2.3), is defined for \(0 \leq \alpha < t\), by

\[
(\rho D^\alpha_{a+} f)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-1)} \left(1-\rho \frac{d}{dt}\right)^n (\rho^s - s^\alpha)^{n-\alpha+1}\rho^{\alpha-1} ds,
\] (2.4)

if the integral exists.
Definition 2.4. The Caputo-type generalized fractional derivative, $\rho D_{a+}^\alpha f$, is defined via the above generalized fractional derivative (2.4) as follows

$$\rho D_{a+}^\alpha f(t) = \left(\rho D_{a+}^\alpha \left[f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (s-a)^k\right]\right)(t)$$

where $n = \lceil \text{Re}(\alpha) \rceil$.

Lemma 2.1. (Krasnoselkii’s fixed point theorem) Let $K$ be a closed convex and nonempty subset of a Banach space $X$. Let $T$ and $S$, be two operators such that

- $T x + S y \in K$ for any $x, y \in K$;
- $T$ is compact and continuous;
- $S$ is contraction mapping.

Then there exists $z_1 \in K$ such that $z_1 = T z_1 + S z_1$.

We are ready to present our results. We adopt some ideas from [9, 15].

3 Main results

This section is devoted to study the existence and uniqueness of solutions of the system (1.1). Let us list some assumptions to prove the main results.

(A1) $f : J \times X \times X \rightarrow X$ is continuous function.

(A2) there exists a positive constant $L > 0$ such that

$$\|f(t,x,u) - f(t,y,v)\| \leq L(\|x-y\| + \|u-v\|), \quad t \in J, \quad u,v,x,y \in C(J,X).$$

(A3) $g : C(J,X) \rightarrow X$ is continuous and there exists $b > 0$, such that

$$\|g(x) - g(y)\| \leq b \|x - y\|, \quad \text{for all} \quad x, y \in C(J,X).$$

(A4) there exists a function $\mu \in L^1(J,\mathbb{R}^+)$ such that $\|f(t,x,y)\| \leq \mu(t)$, for all $t \in J, x,y \in X$.

The uniqueness results are based on applications of the Banach contraction principle.

Theorem 3.1. Assume the assumptions (A1)-(A3) are fulfilled. If

$$b < \frac{1}{2} \quad \text{and} \quad L \leq \frac{\rho^\alpha \Gamma(\alpha + 1)}{2^{\rho^\alpha}},$$

then the system (1.1) has a unique solution.

Proof. Define the operator $P : C(J,X) \rightarrow C(J,X)$ by

$$(P x)(t) := x_0 - g(x) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (\rho^\alpha - s^\alpha)^{\alpha-1} s^\alpha f(s,x(s),x(\lambda s))ds.$$ 

Choose $r \geq 2 \left(\|x_0\| + G + \frac{MT^\alpha}{\rho^\alpha \Gamma(\alpha + 1)}\right)$ and let $\sup_{t \in J} \|f(t,0,0)\| = M$.

Then we can show that $P B_r \subset B_r$, where $B_r := \{x \in C(J,X) : \|x\| \leq r\}$.
So let $x \in B_r$ and set $G = \sup_{x \in C(J,X)} \|g(x)\|$. Then we get

$$\|Px(t)\| \leq \|x_0\| + G + \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} \|f(s,x(s),x(\lambda s))\| \, ds$$

$$\leq \|x_0\| + G + \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} (\|f(s,x(s),x(\lambda s)) - f(s,0,0)\| + \|f(s,0,0)\|) \, ds$$

$$\leq \|x_0\| + G + (2Lt + M) \frac{T^{\rho \alpha}}{\rho^{\alpha} \Gamma(\alpha + 1)} \leq r.$$ 

By the choice of $L$ and $r$. Now take $x, y \in C(J,X)$. Then we get

$$\|(Px)(t) - (Py)(t)\| \leq \|g(x) - g(y)\| + \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} \|f(s,x(s),x(\lambda s)) - f(s,y(s),y(\lambda s))\| \, ds$$

$$\leq b \|x - y\| + \frac{2Lp^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} \|x(s) - y(s)\| \, ds$$

$$\leq \left( b + \frac{2LT^{\rho \alpha}}{\rho^{\alpha} \Gamma(\alpha + 1)} \right) \|x - y\|_{C(J,X)}.$$ 

Thus

$$\|(Px)(t) - (Py)(t)\|_{C(J,X)} \leq \Omega_{b,L,T,\alpha,\rho} \|x - y\|_{C(J,X)},$$

where $\Omega_{b,L,T,\alpha,\rho} := \left( b + \frac{2LT^{\rho \alpha}}{\rho^{\alpha} \Gamma(\alpha + 1)} \right)$ depends only on the parameters of the problem. And since $\Omega_{b,L,T,\alpha,\rho} < 1$, the results follows in view of the contraction mapping principle. 

The main results are based on Krasnoselkii’s fixed point theorem. Now, we are in the position to state and prove the following:

**Theorem 3.2.** Assume that the assumptions (A1), (A3) with $b < 1$ and (A4) hold. Then, system (1.1) has at least one fixed point on $J$.

**Proof.** Let $A$ and $B$ the two operators defined on $B_r$ by

$$(Ax)(t) := \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} f(s,x(s),x(\lambda s)) \, ds,$$

and

$$(Bx)(t) := x_0 - g(x),$$

respectively. Note that $x, y \in B_r$, then $Ax + By \in B_r$.

Indeed it is easy to check the inequality

$$\|Ax + By\| = \left\| x_0 - g(y) + \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} f(s,x(s),x(\lambda s)) \, ds \right\|$$

$$\leq \|x_0\| + \|g(y)\| + \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} \|f(s,x(s),x(\lambda s))\| \, ds$$

$$\leq \|x_0\| + G + \frac{\|g\|_1 T^{\rho \alpha}}{\rho^{\alpha} \Gamma(\alpha + 1)} \leq r.$$ 

Thus,

$$Ax + By \in B_r.$$
By (A3), it is also clear that \( B \) is a contraction mapping. Produced from continuity of \( x \), the operator \( (Ax)(t) \) is continuous in accordance with (A1). Also we observe that

\[
\| (Ax)(t) \| \leq \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \| f(s,x(s),x(\lambda s)) \| \, ds \\
\leq \| \mu \|_1 T^{\rho \alpha} \rho^\alpha \Gamma(\alpha + 1).
\]

Then \( A \) is uniformly bounded on \( B_r \).

Now let’s prove that \( (Ax)(t) \) is equicontinuous. Let \( t_1, t_2 \in J, t_2 \leq t_1 \) and \( x \in B_r \). Using the fact \( f \) is bounded on the compact set \( J \times B_r \), (these \( \sup_{(t,x,y)\in J\times B_r} \| f(t,x,y) \| := C_0 < \infty \)).

We will get

\[
\| (Ax)(t_1) - (Ax)(t_2) \| = \left| \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s,x(s),x(\lambda s)) \, ds \right| \\
\leq \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \left[ (t^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right] s^{\rho-1} f(s,x(s),x(\lambda s)) \, ds \\
+ \int_{t_1}^{t_2} (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s,x(s),x(\lambda s)) \, ds \\
\leq C_0 \frac{p^{1-\alpha}}{\Gamma(\alpha)} \left( \int_0^{t_1} \left[ (t^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right] s^{\rho-1} \, ds \\
+ \int_{t_1}^{t_2} (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \, ds \right).
\]

The second integral in the right-hand side of the last inequality has the value \( \frac{p^{1-\alpha}}{\rho \alpha} (t_2^\rho - t_1^\rho)^\alpha \).

For the first integral, consider the three cases \( \alpha < 0, \alpha = 0 \) and \( \alpha > 1 \) separately. In the case \( \alpha = 1 \), the integral has the value zero.

For \( \alpha < 1 \), we have \( (t_1^\rho - s^\rho)^{\alpha-1} \geq (t_2^\rho - s^\rho)^{\alpha-1} \). Thus,

\[
\int_0^{t_1} \left[ (t_1^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right] s^{\rho-1} \, ds = \int_0^{t_1} \left[ (t_1^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right] s^{\rho-1} \, ds \\
= \frac{1}{\rho \alpha} (t_2^\rho \alpha - t_1^\rho \alpha) + \frac{1}{\rho \alpha} (t_1^\rho - t_2^\rho)^\alpha \\
= \frac{1}{\rho \alpha} (t_2^\rho - t_1^\rho)^\alpha.
\]

Finally, if \( \alpha > 1 \) then \( (t_1^\rho - s^\rho)^{\alpha-1} \leq (t_2^\rho - s^\rho)^{\alpha-1} \) and hence

\[
\int_0^{t_1} \left[ (t_1^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right] s^{\rho-1} \, ds = \int_0^{t_1} \left[ (t_1^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right] s^{\rho-1} \, ds \\
= \frac{1}{\rho \alpha} (t_2^\rho \alpha - t_1^\rho \alpha) - \frac{1}{\rho \alpha} (t_2^\rho - t_1^\rho)^\alpha \\
= \frac{1}{\rho \alpha} (t_2^\rho - t_1^\rho \alpha).
\]

Combining these results, we have

\[
\| (Ax)(t_1) - (Ax)(t_2) \| \leq \frac{2C_0}{\rho \alpha \Gamma(\alpha + 1)} (t_2^\rho - t_1^\rho)^\alpha, \quad \text{if } \alpha \leq 1, \\
\frac{2C_0}{\rho \alpha \Gamma(\alpha + 1)} [(t_2^\rho - t_1^\rho)^\alpha + (t_2^\rho - t_1^\rho)^\alpha], \quad \text{if } \alpha > 1,
\]

which is autonomous of \( x \) and head for zero as \( t_1 - t_2 \to 0 \) consequently. \( A \) is equicontinuous. Thus, \( A \) is relatively compact on \( B_r \). By the Arzelà-Ascoli theorem, \( A \) is compact.

We now conclude the results of the theorem based on the Krasnoselkii’s fixed point. Thus, the problem (1.1) has at least one fixed point on \( J \).
Remark 3.1. Theorem 3.1 and 3.2 can easily be extended to the generalized multi-pantograph equation with generalized fractional derivative of the form
\[ \rho D_0^\alpha x(t) = f(t, x(t), x(\lambda_1 t) \cdots x(\lambda_m t)), \quad t \in [0, T], \]
where \( \alpha \in \mathbb{R}, \rho > 0 \) and \( \lambda \in (0, 1). \)

4 Illustrative example

In this section we give an example to illustrate the usefulness of our main results.

Example 4.1. Consider the nonlinear pantograph equation with generalized fractional derivative
\[ \rho D_0^\alpha x(t) = \frac{1}{5} + \frac{1}{10} x(t) + \frac{1}{10} x \left( \frac{t}{2} \right), \quad t \in [0, 1], \]
where \( \alpha \in (0, 1), \rho = 0.4, \) and \( a_i > 0, i = 0, 1, 2, \cdots, m. \)

Set \( f(t, u, v) = \frac{1}{5} + \frac{1}{10} u + \frac{1}{10} v, \) \( t \in [0, 1], u, v \in X, \)

and \( g(x) = \sum_{i=1}^{m} a_i x(t_i). \)

Let \( u, \bar{u}, v, \bar{v} \in X \) and \( t \in [0, 1], \) then we have
\[ |f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq \frac{1}{10} (|u - \bar{u}| + |v - \bar{v}|), \]
and
\[ |g(u) - g(\bar{u})| \leq \sum_{i=1}^{m} a_i |u(t_i) - \bar{u}(t_i)| \leq \sum_{i=1}^{m} a_i \{|u(t_i) - \bar{u}(t_i)|\}. \]

Denote \( \alpha = \frac{1}{2}, \) \( L = \frac{1}{10}, \rho = 0.4, \) \( T = 1 \) and \( \sum a_i < \frac{1}{2}. \)

Thus
\[ L \leq \frac{\rho^\alpha \Gamma(\alpha + 1)}{2T^\rho\alpha} \Leftrightarrow \rho^\alpha \Gamma(\alpha + 1) \geq 2L = 0.2, \]
where \( \frac{\rho^\alpha \Gamma(\alpha + 1)}{2T^\rho\alpha} = 0.1982. \) Now all assumptions in Theorem 3.1 and 3.2 are satisfied, the problem (4.5) has a unique solution.

Acknowledgements

This work was financially supported by the Tamilnadu State Council for Science and Technology, Dept. of Higher Education, Government of Tamilnadu. The authors are grateful to the referees for their careful reading of the manuscript and valuable comments. The authors thank the help from editor too.

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