Journal Nonlinear Analysis and Application 2017 No.2 (2017) 113-121

Available online at www.ispacs.com/jnaa

Volume 2017, Issue 2, Year 2017 Article ID jnaa-00358, 9 Pages

doi:10.5899/2017/jnaa-00358

Research Article

Computing fixed points of nonexpansive mappings by α-dense curves

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Abstract

Given a multivalued nonexpansive mapping defined on a convex and compact set of a Banach space, with values in the class of convex and compact subsets of its domain, we present an iteration scheme which (under suitable conditions) converges to a fixed point of such mapping. This new iteration provides us another method to approximate the fixed points of a singlevalued nonexpansive mapping, defined on a compact and convex set into itself. Moreover, the conditions for the singlevalued case are less restrictive than for the multivalued case. Our main tool will be the so called α-*dense curves*, which will allow us to construct such iterations. Some numerical examples are provided to illustrate our results.

Keywords: Fixed point, Nonexpansive mappings, α-dense curves, Iterative methods.

1 Introduction

Firstly, we need to fix the notation. Throughout the paper, $(X,\|\cdot\|)$ will be a Banach space and given $C \subseteq X$, $\mathscr{P}(C)$ denotes the class of non-empty subsets of C. Following [9], given $C \in \mathscr{P}(X)$ we put

$$\mathscr{P}_{\mathrm{b}}(C) := \{D \in \mathscr{P}(C) : D \text{ is bounded}\}, \quad \mathscr{P}_{\mathrm{cl}}(C) := \{D \in \mathscr{P}(C) : D \text{ is closed}\},$$

$$\mathscr{P}_{\mathrm{cv}}(C) := \{D \in \mathscr{P}(C) : D \text{ is convex}\}, \quad \mathscr{P}_{\mathrm{cp}}(C) := \{D \in \mathscr{P}(C) : D \text{ is compact}\},$$

$$\mathscr{P}_{\mathrm{b,cl}}(C) := \mathscr{P}_{\mathrm{b}}(C) \bigcap \mathscr{P}_{\mathrm{cl}}(C), \quad \mathscr{P}_{\mathrm{cl,cv}}(C) := \mathscr{P}_{\mathrm{cl}}(C) \bigcap \mathscr{P}_{\mathrm{cv}}(C),$$

$$\mathscr{P}_{\mathrm{cp,cv}}(C) := \mathscr{P}_{\mathrm{cp}}(C) \bigcap \mathscr{P}_{\mathrm{cv}}(C).$$

We recall that the Hausdorff distance $d_H : \mathscr{P}_{\mathrm{b,cl}}(X) \times \mathscr{P}_{\mathrm{b,cl}}(X) \longrightarrow [0,+\infty)$ is given by

$$d_H(A,B) := \max\Big\{\sup\{\delta(a,B) : a \in A\}, \sup\{\delta(b,A) : b \in B\}\Big\}, \tag{1.1}$$

for each $A,B \in \mathscr{P}_{\mathrm{b,cl}}(X)$, where $\delta(x,A)$ is the usual distance from the vector x to the set A. In this context, given $C \in \mathscr{P}(X)$ and a multivalued mapping $T : C \longrightarrow \mathscr{P}(C)$, a vector $x \in C$ is said to be a *fixed point* of T if $x \in T(x)$.

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Also, recall that if there is $k \geq 0$ such that $d_H(T(x), T(y)) \leq k\|x-y\|$ for all $x, y \in C$ the mapping T is said to be *k-Lipschitzian*. If $0 \leq k < 1$ then T is said to be a *k-contraction*, while for $k = 1$ the mapping is said to be *nonexpansive*. These concepts are raised from the singlevalued mappings, where (of course) the distance generated by the norm is used instead the Hausdorff distance d_H.
The interest in the study of the existence of fixed points for multivalued mappings is not purely theoretical, but also for its many applications in Applied Sciences, particularly, in Game Theory and Mathematical Economics. Then, it is natural try to extend the known fixed points results for singlevalued mappings to the multivalued case. In fact, there are many works about this issue, see for instance [1, 4, 9, 13, 14, 16, 22, 28, 29, 30] and references therein. As expected, some issues related with the Theory of Fixed Point for multivalued mappings still remain open, see for instance [6].
So, the well known Banach contraction principle (see, for instance, [10]) for singlevalued mappings was extended in 1969 by Nadler [23] for multivalued contractions mappings. Specifically, for the case of Banach spaces, such result can be stated as:

Theorem 1.1 (Nadler fixed point theorem). *Let $C \in \mathscr{P}(X)$ and $T : C \longrightarrow \mathscr{P}_{b,cl}(C)$ a multivalued k-contraction for some $0 \leq k < 1$. Then, T has at least one fixed point.*

Moreover, the above theorem has been generalized, see for instance [7, 24] and references therein. Other well known results, as Schauder fixed point theorem (see [10]), which requires that $C \in \mathscr{P}_{\mathrm{cp,cv}}(C)$ and T only be continuous, also has been extended to the multivalued case (see, for instance, [3, 11]). The Banach contraction principle and the Schauder fixed point theorem, are just some examples which highlight the interest in extending the known fixed points results for singlevalued mappings to the multivalued case. As expected, usually, such extensions are harder to prove than for the case of singlevalued mappings.
On the other hand, in this paper (see Theorem 3.1) we propose a sequence, often called *iteration* or *iteration scheme*, to approximate under suitable conditions the fixed points of a given nonexpansive multivalued mapping $T : C \longrightarrow \mathscr{P}_{\mathrm{cp,cv}}(C)$, with $C \in \mathscr{P}_{\mathrm{cp,cv}}(X)$. Our main tool will be the so called *α-dense curves*, explained in detail in Section 2. Such curves are, from the point of view of the Hausdorff distance, a generalization of the so called *space-filling curves* (see Remark 2.1). As noted in Section 3 and Section 4, the proposed sequence presents certain improvements over other well known iterations, as the Mann or Ishikawa, for multivalued mappings.
Also, the introduced iteration for multivalued nonexpansive mappings allows us to show a new iteration for singlevalued nonexpansive mappings, $T : C \longrightarrow C$ with $C \in \mathscr{P}_{\mathrm{cp,cv}}(X)$, as we will prove in Theorem 3.2. In this case, less restrictive conditions are required. To conclude, we show some numerical examples in Section 4 to illustrate our results.

2 Densifiable sets and α-dense curves

The concept of α-dense curve was introduced in 1997 by Mora and Cherruault [18]:

Definition 2.1. *Given $\alpha > 0$ and $B \in \mathscr{P}(X)$, a continuous mapping $\gamma : I := [0,1] \longrightarrow (X, \|\cdot\|)$ is said to be an α-dense curve in B if the following conditions hold:*

1. *$\gamma(I) \subset B$.*
2. *For any $x \in B$, there is $y \in \gamma(I)$ such that $\|x-y\| \leq \alpha$.*

If for any $\alpha > 0$ there is an α-dense curve in B, then B is said to be densifiable.

Actually, in [18], the above definition is given in a metric space (E, d) but in this paper we adopt the above one.

Remark 2.1. *The positive parameter α in Definition 2.1 coincides with the Hausdorff distance from the set B to $\gamma(I)$, γ being an α-dense curve in B. So, we can say that the α-dense curves are a generalization of the so called space-filling curves (see [26]). In fact, if B is compact with non-void interior, and γ is an α-dense curve in B for all $\alpha > 0$, then γ is, precisely, a space-filling curve in B, i.e. $\gamma(I) = B$.*

Example 2.1. *([5, Prop. 9.5.4, p. 144]) The cosines curve. Let $\gamma_n : I \longrightarrow \mathbb{R}^d$ given by*

$$t \in I \longmapsto \gamma_n(t) := \Big(t, \frac{1}{2}(1-\cos(n\pi t)), \ldots, \frac{1}{2}(1-\cos(n^{d-1}\pi t))\Big),$$

for each positive integer n. Then γ_n is a $\frac{\sqrt{d-1}}{n}$-dense curve in I^d.

Other examples of α-dense curves can be found in [5, 19]. We have to note that any hypercube $K := \prod_{i=1}^{d}[a_i, b_i] \subset \mathbb{R}^d$ is densifiable. Indeed, for instance, the mapping

$$t \in I \longmapsto \gamma_n(t) := \Big(a_1 + (b_1 - a_1)t, \ldots, a_d + \frac{b_d - a_d}{2}(1-\cos(n^{d-1}\pi t))\Big),$$

is a $\max\{(b_i - a_i)\frac{\sqrt{d-1}}{n} : i = 1, \ldots, d\}$-dense curve in K, for each integer $n \geq 1$.
As expected, not every bounded set of a Banach space (even compact and connected) is densifiable:

Example 2.2. *In the Euclidean plane consider the set*

$$B := \Big\{(x, \sin(x^{-1})) : x \in [-1,0) \cup (0,1]\Big\} \cup \Big\{(0,y) : y \in [-1,1]\Big\}.$$

Then, given $\alpha > 0$ and any continuous mapping $\gamma : I \longrightarrow \mathbb{R}^2$, if $\gamma(I) \subset B$ then it is contained in some pathwise-connected component of B. So, taking $0 < \alpha < 1$, it is clear that there is no α-dense curve in B, and therefore B is not densifiable.

Therefore, the class of densifiable sets is strictly between the class of Peano Continua (i.e. those sets which are the continuous image of I) and the class of connected and precompacts sets. However, we have the following result (see [21]):

Proposition 2.1. *Let $B \in \mathscr{P}_b(X)$ be pathwise-connected. Then, B is densifiable if, and only if, it is precompact.*

The main application of the α-dense curves is the numerical computing of solutions for optimization problems; for details see [5, 17, 20] and references therein. However, given $C \in \mathscr{P}_{\text{cp,cv}}(X)$ and $T : C \times C \longrightarrow C$ nonexpansive, in [8] these curves have been used to construct an iteration that converges to a *coupled fixed point* of T, that is, a point $(x,y) \in C \times C$ with $T(x,y) = x$ and $T(y,x) = y$.
Thus, following the above research line we propose in Theorem 3.1, under suitable conditions, a new iteration for a multivalued nonexpansive mapping $T : C \longrightarrow \mathscr{P}_{\text{cp,cv}}(C)$, with $C \in \mathscr{P}_{\text{cp,cv}}(X)$. Moreover, as we will show in Theorem 3.2, our iteration can be *adapted* to the single valued case under less restrictive conditions.

3 Main results

Let $C \in \mathscr{P}_{\text{cp,cv}}(X)$ and $T : C \longrightarrow \mathscr{P}_{\text{cp,cv}}(C)$ a multivalued mapping. Some well known iterations for single valued mappings have been generalized to the multivalued case; see [25, 27] and references therein. For instance, fixed $x_1 \in C$ and a sequence $(\lambda_n)_{n\geq 1} \subset (0,1)$ the Mann iteration for multivalued mappings is given by

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)y_n,$$

where $y_n \in T(x_n)$ is such that $\|y_n - x^*\| = \delta(x^*, T(x_n))$, $x^* \in C$ being a fixed point of T. For the so called Ishikawa iteration, also is required to know some fixed point of T. Thus, we need to know some fixed point of T, or even the whole set of fixed points of T in some cases (as in [25, Th. 3.8]), to define these sequences. In the following lines we introduce a new iteration for which we do not have the *restriction* on the knowledge of fixed points of T, although some additional conditions will be required.
For each integer $n \geq 1$ take γ_n an α_n-dense curve in C, with $\alpha_n \to 0$. Note that the existence of such curves is guaranteed by Proposition 2.1. We assume that $\sum_{n\geq 1}\alpha_n < \infty$. Now, fixed $0 < \lambda < 1$ and $x_1 \in C$ define

$$x_{n+1} := (1-\lambda)\eta_n + \lambda\xi_n \in C, \tag{3.2}$$

for each $n \geq 1$, where $\xi_n \in T(x_n)$ is such that $\|\xi_n - x_n\| = \delta(x_n, T(x_n))$ and $\eta_n \in \gamma_n(I)$ satisfies $\|\xi_n - \eta_n\| \leq \alpha_n$. Note that the existence of each ξ_n follows by the compactness of $T(x_n)$. For the particular case that X is reflexive or finite dimensional, the class $\mathscr{P}_{\text{cp,cv}}(C)$ can be replaced by the class $\mathscr{P}_{\text{cl,cv}}(C)$ because in this case the closedness is enough for the existence of such ξ_n.
The following result is an immediate consequence of the definition of d_H given in (1.1), see also [24, Lem. 2.10].

Lemma 3.1. *Let $A, B \in \mathscr{P}_{b,cl}(X)$. Then, for each $x \in X$, $\delta(x,A) \leq \delta(x,B) + d_H(B,A)$.*

Also, we will need the following well known technical result (see, for instance, [2]):

Lemma 3.2. *Let $(a_n)_{n\leq 1}$ and $(b_n)_{n\leq 1}$ be two sequences of nonnegative numbers with $\sum_{n\geq 1} b_n < +\infty$, such that $a_{n+1} \leq a_n + b_n$ for each n. Thus, the limit $\lim_n a_n$ exists.*

Now, we can state and prove our main result:

Theorem 3.1. *Let $C \in \mathscr{P}_{cp,cv}(X)$, $T : C \longrightarrow \mathscr{P}_{cp,cv}(C)$ nonexpansive and assume that $\sum_{n\geq 1} \delta(x_n, T(x_n)) < \infty$, where $(x_n)_{n\geq 1}$ is defined in (3.2). Then, the sequence $(x_n)_{n\geq 1}$ converges to a fixed point of T.*

Proof. Firstly, note that as $C \in \mathscr{P}_{\text{cp,cv}}(X)$ there is a subsequence of $(x_n)_{n\geq 1}$, put $(x_{n_k})_{k\geq 1}$ such that $x_{n_k} \to x^*$, for some $x^* \in C$. We claim that x^* is a fixed point of T, i.e. $x^* \in T(x^*)$. We have

$$\|x_{n_{k+1}} - \xi_{n_k}\| \leq \|x_{n_{k+1}} - x_{n_k}\| + \|x_{n_k} - \xi_{n_k}\| = \|x_{n_{k+1}} - x_{n_k}\| + \delta(x_n, T(x_n)) \longrightarrow 0, \tag{3.3}$$

as $\sum_{n\geq 1} \delta(x_n, T(x_n)) < \infty$ and therefore $\delta(x_n, T(x_n)) \longrightarrow 0$, and $(x_{n_k})_{k\geq 1}$ is convergent.
Now, by Lemma 3.1 and noticing the nonexpansiveness of T, we have

$$\delta(x^*, T(x^*)) \leq \|x^* - x_{n_{k+1}}\| + \delta(x_{n_{k+1}}, T(x^*)) \leq \|x^* - x_{n_{k+1}}\| + \delta(x_{n_{k+1}}, T(x_{n_k})) + d_H(T(x_{n_k}), T(x^*)) \leq$$

$$\leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - x_{n_k}\| + \|x_{n_k} - \xi_{n_k}\| + d_H(T(x_{n_k}), T(x^*)) \leq$$

$$\leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - x_{n_k}\| + \|x_{n_{k+1}} - \xi_{n_k}\| + \|x_{n_k} - x^*\| \leq 2(\|x^* - x_{n_k}\| + \|x_{n_{k+1}} - x_{n_k}\|) + \|x_{n_{k+1}} - \xi_{n_k}\| \longrightarrow 0,$$

where the above convergence to zero holds by (3.3) and the fact that $x_{n_k} \longrightarrow x^*$. So, $\delta(x^*, T(x^*)) = 0$ and in view of the closedness of $T(x^*)$, must be $x^* \in T(x^*)$ as claimed.
On the other hand, we find that

$$\|x_{n+1} - x^*\| \leq \lambda\|\xi_n - x^*\| + (1-\lambda)\|\eta_n - x^*\| \leq \|\xi_n - x^*\| + (1-\lambda)\|\eta_n - \xi_n\| \leq \|\xi_n - x^*\| + (1-\lambda)\alpha_n \leq$$

$$\leq \delta(x_n, T(x_n)) + \|x_n - x^*\| + (1-\lambda)\alpha_n.$$

So, by Lemma 3.2, the sequence $\|x_{n+1} - x^*\|$ is convergent. But, as $x_{n_k} \to x^*$ we conclude that $\lim_n \|x_{n+1} - x^*\| = 0$ and this completes the proof. □

On the other hand, for the particular case that $T : C \longrightarrow C$ with $C \in \mathscr{P}_{\text{cp,cv}}(X)$, be a singlevalued nonexpansive mapping, the sequence (3.2) is actually given by

$$x_{n+1} := (1-\lambda)\eta_n + \lambda T(x_n) \in C, \tag{3.4}$$

that is, $\xi_n := T(x_n)$ for each $n \geq 1$. As above, we assume that $\sum_{n\geq 1} \alpha_n < \infty$ but as we will show in our next result, any condition is required for the series $\sum_{n\geq 1} \|x_n - T(x_n)\|$.
Then, in view of Theorem 3.1, one might expect that the sequence given in (3.4) converges to a fixed point of T. And indeed it is:

Theorem 3.2. *Let $C \in \mathscr{P}_{cp,cv}(X)$ and $T : C \longrightarrow C$ nonexpansive. Then, the sequence $(x_n)_{n\geq 1}$ defined in (3.4) converges to a fixed point of T.*

Proof. Firstly, note that for each $n \geq 1$

$$\|x_{n+1} - T(x_n)\| \leq (1-\lambda)\|\eta_n - T(x_n)\| \leq (1-\lambda)\alpha_n \longrightarrow 0, \tag{3.5}$$

because $\sum_{n\geq 1} \alpha_n < \infty$ and therefore $\alpha_n \longrightarrow 0$. As $C \in \mathscr{P}_{\text{cp,cv}}(X)$ there is a subsequence of $(x_n)_{n\geq 1}$, put $(x_{n_k})_{k\geq 1}$, with $x_{n_k} \longrightarrow x^*$ for some $x^* \in C$.
Now, we have

$$\|x^* - T(x^*)\| \leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T(x_{n_k})\| + \|T(x_{n_k}) - T(x^*)\| \longrightarrow 0,$$

by the continuity of T and the inequality (3.5). Then, as

$$\|x_{n+1} - x^*\| = \|(1-\lambda)\eta_n + \lambda T(x_n) - \lambda T(x^*) + (1-\lambda)T(x^*)\| \leq (1-\lambda)\|\eta_n - T(x^*)\| + \lambda\|T(x_n) - T(x^*)\| \leq$$

$$\leq (1-\lambda)\|\eta_n - T(x_n)\| + \|T(x_n) - T(x^*)\| \leq (1-\lambda)\alpha_n + \|x_n - x^*\|,$$

holds for each n, Lemma 3.2 can be applied and therefore the sequence $\|x_n - x^*\|$ is convergent. But, as $\|x_{n_k} - x^*\| \to 0$ we conclude that $\|x_n - x^*\| \to 0$ as needed. □

4 Numerical experiences

This section is devoted to illustrate our results. We will work in the Euclidean plane $(\mathbb{R}^2, \|\cdot\|)$.

Example 4.1. *Let $T : I^2 \longrightarrow I^2$ be the isometry $T(x,y) := (1-x, 1-y)$. The unique fixed point of T is $(0.5, 0.5)$ and can not be approximated by the Picard iteration $(T^n(x_1))_{n\geq 1}$, for an arbitrary $x_1 \in I^2$, of the Banach contraction principle. In fact, for $x_1 := (0,0)$ such sequence does not converge. However, the iteration of Theorem 3.2 allows us to approximate the fixed point of T. In fact, taking $\lambda := 0.5, x_1 := (0,0)$ and γ_{n^4} the cosines curve of Example 2.1 the obtained results are shown in Tab. 1. We denote by $\varepsilon_n := \|x_n - (0.5, 0.5)\|$ the approximation error, for each integer $n \geq 1$.*

Table 1: Some approximate values of the fixed point of T determined by the sequence $(x_n)_{n\geq 1}$ in Theorem 3.2.

n	x_n	ε_n
2	(0, 0)	0.5
4	(0.5, 0.5)	0
6	(0.49969, 0.5)	0.00031
8	(0.50002, 0.5)	0.00002
10	(0.49997, 0.5)	0.00003
12	(0.50001, 0.5)	0.00001

Example 4.2. *Let $T : I^2 \longrightarrow \mathscr{P}_{cp,cv}(I^2)$ given by $T(x,y) := \overline{Conv}\big(\{(0,0),(x,0),(0,y)\}\big)$, for each $(x,y) \in I^2$, where $\overline{Conv}$ denotes the closed convex hull. Then, we have (see [14, Example 3]) that T is a nonexpansive multivalued mapping and the set of fixed points of T is $Fix(T) := \big\{(x,y) \in I^2 : xy = 0\big\}$. Also, one can check straightforwardly that*

$$\xi_n = \begin{cases} (x_n^{(1)}, x_n^{(2)}), & \text{for} \quad x_n^{(1)} = 0 \quad \text{or} \quad x_n^{(2)} = 0 \\ \left(\dfrac{[x_n^{(1)}]^3}{[x_n^{(1)}]^2 + [x_n^{(2)}]^2}, \dfrac{[x_n^{(2)}]^3}{[x_n^{(1)}]^2 + [x_n^{(2)}]^2}\right), & \text{in other case} \end{cases},$$

for each n, where the super-index denotes the coordinate components of the point x_n.
Taking γ_{n^4} the cosines curve of Example 2.1, for the indicated values of λ and x_1, the obtained results are shown in Tab. 2. As we can see in Tab. 2, it seems that the sequence $(x_n)_{n\geq 1}$ converges to some $x^* \in Fix(T)$ (see also Fig. 1).

Table 2: Some approximate values of a fixed point of T determined by the sequence $(x_n)_{n\geq 1}$ in Theorem 3.1.

n	x_n for $\lambda := 0.6, x_1 := (1,1)$	x_n for $\lambda := 0.5, x_1 := (1,0.5)$
5	(0.15904, 0.17880)	(0.48926, 0.013144)
10	(0.0090633, 0.068989)	(0.49476,0.00041113)
15	(8.5E-5, 0.068144)	(0.49883,1.2E-5)
20	(5.1E-7, 0.068144)	(0.49899, 4E-7)
25	(2.02E-7, 0.068144)	(0.49966,1.2E-8)
30	(9.5E-8, 0.068144)	(0.49995, 3.9E-13)

Example 4.3. *Let $S : I^2 \longrightarrow \mathscr{P}_{cp,cv}(I^2)$ given by $S(x,y) :=$ the segment $\{(s,y) : 0 \leq s \leq x/2\}$, for each $(x,y) \in I^2$. So, S is nonexpansive and the set of fixed points of S is $Fix(S) := \{(0,y) : 0 \leq y \leq 1\}$ (see [14, Example 5]). Following the notation of the above example, in this case we have*

$$\xi_n = \begin{cases} (x_n^{(1)}, x_n^{(2)}), & \text{for} \quad x_n^{(1)} = 0 \\ (x_n^{(1)}/2, x_n^{(2)}), & \text{in other case} \end{cases},$$

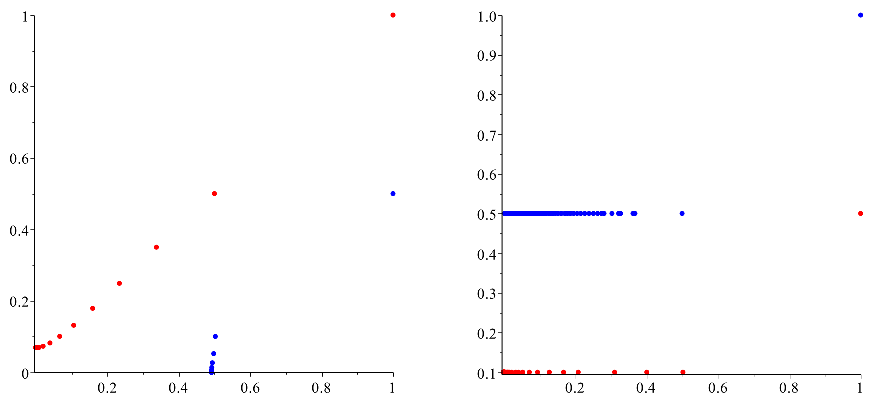
for each n. Taking γ_{n^2} the cosines curve of Example 2.1, the obtained results are shown in Tab. 3, for the indicated values of λ and x_1.

Table 3: Some approximate values of a fixed point of S determined by the sequence $(x_n)_{n\geq 1}$ in Theorem 3.1.

n	x_n for $\lambda := 0.5, x_1 := (1,0.5)$	x_n for $\lambda := 0.1, x_1 := (1,1)$
5	(0.20919, 0.10000)	(0.32778, 0.5)
10	(0.053852, 0.10000)	(0.26371, 0.5)
15	(0.013941, 0.10000)	(0.20597, 0.5)
20	(0.0044772,0.10000)	(0.16190, 0.5)
25	(0.0021433,0.10000)	(0.12613, 0.5)
30	(0.0014589,0.10000)	(0.09768, 0.5)

As we can see in the Tab. 3, it seems that the sequence $(x_n)_{n\geq 1}$ converges to some $x^* \in Fix(S)$ (see also Fig. 1).

In Fig. 1 we show the graph of first hundred points of the sequence $(x_n)_{n\geq 1}$ of Theorem 3.1, for the mappings T and S of the above Examples 4.2 and 4.3, for the values of λ and x_1 indicated in such examples (drawings in red and blue).



Graph of $\{x_1,\ldots,x_{100}\}$ for the mapping T and the values of λ and x_1 indicated in Tab. 2

Graph of $\{x_1,\ldots,x_{100}\}$ for the mapping S. and the values of λ and x_1 indicated in Tab. 3

Figure 1: Some points of the sequence $(x_n)_{n\geq 1}$, for different values of λ and x_1, in Theorem 3.1 for the mappings T (left) and S (right) of Examples 4.2 and 4.3, respectively.

Some comments are necessary before finishing. As we have pointed out above, some iterations (as that Mann or Ishikawa) need to know some fixed point of T, or even the whole set of fixed points of T (see, for instance, [25, Th. 3.8]). Moreover, some geometric properties on the Banach space X are required, for instance, uniform convexity. Then, if we consider in the above examples $\mathbb{R}^2$ endowed, for instance, the norm $\|(x,y)\|_1 := |x|+|y|$, for which $\mathbb{R}^2$ is not uniformly convex, we have no guarantee that the mentioned iterations converge to a fixed point of T. But, as all norms are equivalent in $\mathbb{R}^2$, both Theorem 3.1 and Theorem 3.2 remain true under these assumptions.

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