Chaos in dynamics of a family of transcendental meromorphic functions

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Abstract
In this paper, the chaotic behaviours in the real and complex dynamics of $\zeta_\lambda(z) = \lambda \frac{z}{z+1} e^{-z}, \lambda > 0, z \in \mathbb{C}$ are investigated. The bifurcation in the dynamics of $\zeta_\lambda(x), x \in \mathbb{R} \setminus \{-1\}$, occurs at several parameter values and the dynamics becomes chaotic when the parameter $\lambda$ crosses certain values. The Lyapunov exponent of $\zeta_\lambda(x)$ is computed for quantifying the chaos. The characterization of the Julia set of $\zeta_\lambda(z)$ as complement of the basin of attraction is found and is applied to computationally simulate the images of the Julia sets. Finally, the results on the dynamics of $\zeta_\lambda(z)$ are compared with the known results.

Keywords: Bifurcation, Chaos, Fixed points, Iteration, Julia set, Fatou set.

1 Introduction

After many investigations on the dynamical properties of polynomials and rational functions, this study is extended to the family of entire functions in [4, 8, 12, 14, 25] that provides powerful mathematical techniques and beautiful computer graphics. The initial study of iterations of transcendental meromorphic functions is mainly found in [1, 2, 3, 5]. Further, researches in this direction are pursued in [7, 13, 18, 26]. Since the Julia sets (chaotic sets) occur as one of the crucial component in these investigations, their various characterizations and identification of their intrinsic properties are primarily developed in these studies. This work has also applications in a number of diverse science and engineering areas wherein simulations of objects of fractal nature are needed [9, 10, 15, 17, 19, 20, 22, 23].

Let $\mathbb{C}$ and $\hat{\mathbb{C}}$ denote the complex plane and the extended complex plane respectively. A point $z^*$ is said to be a critical point of $f$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point $z^*$ is called a critical value of $f$. A point $\alpha \in \hat{\mathbb{C}}$ is said to be an asymptotic value for $f(z)$, if there is a continuous curve $\gamma(t)$ satisfying $\lim_{t \to \infty} \gamma(t) = \infty$ and $\lim_{t \to \infty} f(\gamma(t)) = \alpha$. A singular value of $f$ is either a critical value or an asymptotic value of $f$. A function is said to be critically finite if it has only finitely many singular values.

Let $\mathcal{S}$ be the class of critically finite transcendental meromorphic function $f(z)$. Baker [2] proved the following:

Theorem 1.1. Let $f \in \mathcal{S}$. Then, $f(z)$ has no wandering domains.

Bergweiler [3] showed the following result:

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Theorem 1.2. Let $f \in S$. Then, $f(z)$ has no Baker domains.

The basic results concerning singular values and the dynamics of functions can be seen in [16]. The Schwarzian derivative of $f(z)$ is given as

$$SD(f) = \left( \frac{f''(z)}{f'(z)} \right)^2 - \frac{1}{2} \left( \frac{f'''(z)}{f'(z)} \right)^2.$$

The Lyapunov exponent is an important tool to measure the chaos. The Lyapunov exponent of the function $f(x)$, for a given trajectory $\{x_k : k = 0, 1, 2, \ldots \}$ starting at $x_0$, is defined as

$$L = \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \ln |f'(x_j)|$$

It is well known that the behavior of dynamical system is chaotic if Lyapunov exponent of the function $f(x)$ is a positive number [11].

The purpose of this paper is to investigate the dynamics of a one parameter family of transcendental meromorphic functions. It is easily seen that the functions $f$ are one parameter family of transcendental meromorphic functions. It is well known that the behavior of dynamical system is chaotic if Lyapunov exponent of the function $f(x)$ is a positive number [11].

Let $\mathcal{M} = \left\{ \zeta_\lambda(z) = \frac{x}{z+1} e^{-z} : \lambda > 0, z \in \mathbb{C} \right\}$ be one parameter family of transcendental meromorphic functions. It is easily seen that the functions $\zeta_\lambda \in \mathcal{M}$ have two critical values $\frac{1-\sqrt{5}}{2} \lambda e^{ \frac{1-\sqrt{5}}{2} }$, $\frac{1+\sqrt{5}}{2} \lambda e^{ \frac{1+\sqrt{5}}{2} }$, one finite asymptotic value 0 and have rational Schwarzian derivative

$$SD(\zeta_\lambda) = -\frac{e^z + 2e^z - 3e^z - 4z + 18}{2(z^2 + z - 1)^2}.$$

The structure of this paper is as follows: In Section 2, the existence and nature of fixed points as well as periodic points of $\zeta_\lambda(x)$ are described. The dynamics of $\zeta_\lambda(x)$, $x \in \mathbb{R}\setminus\{-1\}$, is investigated in Section 3. In this section, it is found that $\lambda = 1$ and $\lambda = (\sqrt{2} + 1)e^{\sqrt{2}}$ are the bifurcation points of $\zeta_\lambda(x)$. The Lyapunov exponent of $\zeta_\lambda(x)$ is computed and the values of the parameter $\lambda$ are determined for which its Lyapunov exponent is positive, thereby ascertaining chaos in the dynamics of the function for these values of the parameter. In Section 4, the characterization of Julia set $J(\zeta_\lambda)$, $0 < \lambda < 1$ and $1 < \lambda \leq \lambda^*$, as the complement of the basin of attraction of an attracting real fixed point of $\zeta_\lambda(z)$ is established. Further, it is proved that Fatou set of the function $\zeta_\lambda(z)$ for $\lambda = 1$ and $\lambda = \lambda^*$ contains certain intervals of real line. Furthermore, in this section, it is shown that Julia set of $\zeta_\lambda(z)$ for $\lambda > \lambda^*$ contains the subset of real line consisting of the complement of points attracted to the attracting periodic orbits. In Section 5, the characterization of Julia set of $\zeta_\lambda(z)$ for different ranges of parameter value $\lambda$, obtained in Section 4, are used to computationally generate the images of its Julia sets. The results obtained in the present paper are finally compared with some of the results found in [5, 6, 12, 21, 24].

2 Real fixed points, periodic points of $\zeta_\lambda \in \mathcal{M}$ and their nature

The nature of fixed points (i.e. the point $x$ satisfying $\zeta_\lambda(x) = x$) and periodic points (i.e. the point $x$ satisfying $\zeta_\lambda^n(x) = x$, $n = 2, 3, \ldots$) of the functions $\zeta_\lambda \in \mathcal{M}$ are found in the present section. The number and locations of fixed points of the function $\zeta_\lambda \in \mathcal{M}$ on real axis are described now. Besides the fixed point $x = 0$, the real fixed points of the function $\zeta_\lambda(x) = \frac{x}{x^2+1}$ are given as: (a) For $0 < \lambda < 1$, $\zeta_\lambda(x)$ has exactly one fixed point and that is in the interval $(-1, 0)$. (b) For $\lambda = 1$, $\zeta_\lambda(x)$ has no non-zero fixed point. (c) For $1 < \lambda < \lambda^*$, $\zeta_\lambda(x)$ has exactly one fixed point and that is in the interval $(-1, 0)$, where $\lambda^* = (\sqrt{2} + 1)e^{\sqrt{2}}$. (d) For $\lambda \geq \lambda^*$, $\zeta_\lambda(x)$ has exactly one fixed point.
For $\lambda > \lambda^*$, the function $\zeta_\lambda(x)$ has periodic points of period greater than or equal to 2 in $(0, \infty)$ in addition to having one fixed point. These periodic points are roots of $\zeta_\lambda^n(x) = \frac{z_\lambda e^{z_\lambda (x - 1)}}{z_\lambda + 1} = x$. 

In the case $n = 2$, the periodic points $p_{11}$ and $p_{12}$ of $\zeta_{12}(x)$ are computationally obtained as $p_{11} \approx 0.748218, p_{12} \approx 2.43034$ (Fig. 1) for $\lambda > \lambda^*$. The computed value of the fixed point of $\zeta_{12}(x)$ is $-0.314923$ if $\lambda = 0.5$ ($0 < \lambda < 1$) and this value is $0.374823$ if $\lambda = 2$ ($1 < \lambda < \lambda^*$).

The nature of fixed points of $\zeta_\lambda \in \mathbb{M}$ are explored in the following theorem:

**Theorem 2.1.** Let $\zeta_\lambda \in \mathbb{M}$. (i) If $0 < \lambda < 1$, then the fixed point $r_2 \in (-1, 0)$ is repelling and the fixed point 0 is attracting. (ii) If $\lambda = 1$, then the fixed point 0 is rationally indifferent. (iii) If $1 < \lambda < \lambda^*$, then the fixed point $a_2 \in (0, \sqrt{2})$ is attracting and the fixed point 0 is repelling. (iv) If $\lambda = \lambda^*$, then the fixed point $\sqrt{2}$ is rationally indifferent and the fixed point 0 is repelling. (v) If $\lambda > \lambda^*$, then the fixed point 0 is repelling.

**Proof.** Since the non-zero fixed points of $\zeta_\lambda(x)$ are solutions of the equation $(x + 1)e^x = \lambda$, then the multiplier $\zeta_\lambda'(x_f) = -\lambda(\sqrt{x_f^2 + x_f - 1} - e^{-x})$ of the fixed point $x_f$ can be written as

$$|\zeta_\lambda'(x_f)| = \frac{|x_f^2 + x_f - 1|}{(x_f + 1)} \quad (2.1)$$

Let $p(x) = |x^2 + x - 1| - (x + 1)$. Then, $p(x) < 0$ for $x \in (0, \sqrt{2})$, $p(x) > 0$ for $x \in (-1, 0) \cup (\sqrt{2}, \infty)$. Therefore, from Equation (2.1), it follows that

$$|\zeta_\lambda'(x_f)| < 1 \quad (2.2)$$

$$|\zeta_\lambda'(x_f)| = 1 \quad (2.3)$$

$$|\zeta_\lambda'(x_f)| > 1 \quad (2.4)$$

The behaviour of the nonzero fixed points of $\zeta_\lambda(x)$ are described as follows:

(i) For fixed point $r_2 \in (-1, 0)$, by Inequality (2.4), $|\zeta_\lambda'(r_2)| > 1$. Therefore, $r_2$ is a repelling fixed point of $\zeta_\lambda(x)$.

(ii) Since, $0 < \zeta_\lambda'(0) = \lambda < 1$, the point 0 is an attracting fixed point of $\zeta_\lambda(x)$ for $0 < \lambda < 1$.

(iii) For fixed point $a_2 \in (0, \sqrt{2})$, by Inequality (2.2), $|\zeta_\lambda'(a_2)| < 1$ so that $a_2$ is an attracting fixed point of $\zeta_\lambda(x)$.

Further, since $\zeta_\lambda'(0) = \lambda > 1$, the point 0 is a repelling fixed point of $\zeta_\lambda(x)$ for $1 < \lambda < \lambda^*$.
Theorem 3.1. \( z = l \) of period greater than or equal to 2 may be attracting, repelling or indifferent. For \( \lambda > \lambda^* \), it gives that 0 is a repelling fixed point of \( \zeta_l(x) \) for \( \lambda > \lambda^* \).

For \( \lambda > \lambda^* \), the periodic cycle of \( \zeta_l(x) \) of period greater than or equal to 2 may be attracting, repelling or indifferent. For \( \lambda = 12 > \lambda^* \), it is found that \( \zeta_l(x) \) has 2-cycle periodic points \( p_{11} \approx 0.748218 \) and \( p_{12} \approx 2.430344 \), so that \( \zeta_l(p_{11}) = -0.57235, \zeta_l(p_{12}) = -0.65847 \). It follows that \( \zeta_l(p_{11}) \zeta_l(p_{12}) = 0.376878 < 1 \) for \( \lambda = 12 \). Consequently, the periodic 2-cycle of \( \zeta_l(x) \) is attracting. Similarly, for \( \lambda = 18.5 > \lambda^* \), \( \zeta_l(x) \) has 2-cycle periodic points \( p_{11} \approx 0.408442 \) and \( p_{12} \approx 3.565982 \), so that \( \zeta_l(p_{11}) = 2.683285, \zeta_l(p_{12}) = -0.383357 \) and hence, \( \zeta_l(p_{11}) \zeta_l(p_{12}) = 1.00932 > 1 \) for \( \lambda = 18.5 \). It gives that the periodic 2-cycle of \( \zeta_{18.5}(x) \) is repelling. While, for \( \lambda = 18.44505 > \lambda^* \), \( \zeta_l(x) \) has 2-cycle periodic points \( s_{11} \approx 0.409865 \) and \( s_{12} \approx 3.55911 \), so that \( \zeta_l(s_{11}) = 2.60007 \), \( \zeta_l(s_{12}) = -0.384606 \) and then, \( \zeta_l(s_{11}) \zeta_l(s_{12}) = 1 \) for \( \lambda = 18.44505 \). Therefore, the periodic 2-cycle of \( \zeta_{18.44505}(x) \) is indifferent.

3 Bifurcation and chaos in the real dynamics of \( \zeta_l \) in \( \mathcal{M} \)

In this section, the bifurcation and chaos in the real dynamics of the functions \( \zeta_l(x), x \in \mathbb{R} \setminus \{ -1 \} \), are investigated.

Theorem 3.1. Let \( \zeta_l \in \mathcal{M} \).

(a) If \( 0 < \lambda < 1 \), \( \zeta_l^n(x) \to 0 \) as \( n \to \infty \) for \( x \in [(-\infty, -1) \cup (-1, r_2) \cup (r_2, \infty)] \setminus T_P \).

(b) If \( \lambda = 1 \), \( \zeta_l^n(x) \to 0 \) as \( n \to \infty \) for \( x \in [(-\infty, -1) \cup (1, \infty)] \setminus T_P \).

(c) If \( 1 < \lambda < \lambda^* \), \( \zeta_l^n(x) \to a_\lambda \) as \( n \to \infty \) for \( x \in [(-\infty, 1) \cup (1, 0) \cup (0, \infty)] \setminus T_P \).

(d) If \( \lambda = \lambda^* \), \( \zeta_l^n(x) \to \sqrt{2} \) as \( n \to \infty \) for \( x \in [(-\infty, -1) \cup (-1, 0) \cup (0, \infty)] \setminus T_P \).

(e) If \( \lambda > \lambda^* \), the orbits \( \{ \zeta_l^n(x) \} \) repel for all \( x \in \mathbb{R} \setminus T_P \).

Proof. Let \( t_\lambda(x) = \zeta_l(x) - x \) for \( x \in \mathbb{R} \setminus \{ -1 \} \). It is easily seen that \( t_\lambda(x) \) is continuously differentiable for \( x \in \mathbb{R} \setminus \{ -1 \} \). Observe that the fixed points of \( \zeta_l(x) \) are zeros of \( t_\lambda(x) \).

Using Equation (3.5), the dynamics of \( \zeta_l(x) \) is as follows: (i) For \( x \in (0, \infty) \), \( \zeta_l(x) < x \). Since \( \zeta_l(x) > 0 \) for \( x \in (0, \infty) \), the sequence \( \{ \zeta_l^n(x) \} \) is decreasing and bounded below by 0. Hence \( \zeta_l^n(x) \to 0 \) as \( n \to \infty \) for \( x \in (0, \infty) \). (ii) For \( x \in (r_2, 0) \), \( \zeta_l(x) > x \). Since \( \zeta_l(x) \) is negative and increasing for \( x \in (r_2, 0) \), the sequence \( \{ \zeta_l^n(x) \} \) is increasing and bounded above by 0. Hence \( \zeta_l^n(x) \to 0 \) as \( n \to \infty \) for \( x \in (r_2, 0) \). (iii) Since \( \zeta_l(x) \) maps \( (-\infty, -1) \) into \( (0, \infty) \), by (i), we get \( \zeta_l^n(x) \to 0 \) as \( n \to \infty \) for \( x \in (-\infty, -1) \). (iv) The forward orbit of \( \zeta_l(x) \) for each point in \( (-1, r_2) \setminus T_P \) is contained in \( (-\infty, -1) \). Thus, by (iii), \( \zeta_l^n(x) \to 0 \) as \( n \to \infty \) for \( x \in (-1, r_2) \setminus T_P \). Fig. 2(i) shows the web diagram of dynamics of \( \zeta_l(x) \) for \( 0 < \lambda < 1 \).
If $\lambda = 1$, by Theorem 2.1, $\zeta_2(x)$ has a rationally indifferent fixed point $0$. This gives $t'_2(0) = 0$ and $t''_2(0) < 0$, so that $t_2(x)$ has maxima at 0. But $t_2(0) = 0$ therefore $t_2(x) < 0$ in a neighbourhood of 0. By continuity of $t_2(x)$, for $\delta > 0$, $t_2(x) < 0$ in $(-\delta, 0) \cup (0, \delta)$. Since $t_2(x) \neq 0$ in $(-1, 0) \cup (0, \infty)$, it now follows that

$$t_2(x) = \zeta_2(x) - x < 0 \quad \text{for} \ x \in (-1, 0) \cup (0, \infty).$$

Now, by Equation (3.6), (i) for $x \in (0, \infty)$, $\zeta_2(x) < x$. Since $\zeta_2(x) > 0$ for $x \in (0, \infty)$, the sequence $\{\zeta^n_2(x)\}$ is decreasing and bounded below by 0. Hence $\zeta^n_2(x) \to 0$ as $n \to \infty$ for $x \in (0, \infty)$. (ii) By arguments similar to those used in the proof for (iii) of (a), it follows that $\zeta^n_2(x) \to 0$ as $n \to \infty$ for $x \in (-\infty, -1)$. (iii) The forward orbit of $\zeta_2(x)$ for each point in $(-1, 0) \setminus T_p$ is contained in $(-\infty, -1)$. Therefore, by (ii), $\zeta^n_2(x) \to 0$ as $n \to \infty$. The web diagram of dynamics of $\zeta_2(x)$, $\lambda = 1$, is given in Fig. 2(ii).

If $1 < \lambda < \lambda^*$, by Theorem 2.1, $\zeta_2(x)$ has an attracting fixed point $a_2$ and a repelling fixed point 0. The dynamics of $\zeta_2(x)$, for $1 < \lambda < \lambda^*$, is found to be as follows: (i) First let $x \in (-\frac{1+\sqrt{5}}{2}, a_2)$. Since $\zeta''_2(x) < 0$ for $x \in (-\infty, x_0)$, then $\zeta'_2(x)$ is decreasing for $x \in (-\infty, x_0)$, where $x_0 \approx 1.26953$ is a solution of the equation $x^3 + 2x^2 - x - 4 = 0$. Since $\zeta'_2(\frac{-1+\sqrt{5}}{2}) = 0$ and $-1 < \zeta'_2(a_2) < 0$, by Mean Value Theorem $|\zeta'_2(x) - a_2| < |x - a_2|$. Therefore, $\zeta''_2(x) \to a_2$ as $n \to \infty$ for $x \in (-\frac{1+\sqrt{5}}{2}, a_2)$. Next, for each $x \in (0, \frac{-1+\sqrt{5}}{2})$, there exits $n_0 \in \mathbb{N}$ such that $\zeta''_2(x) \in (-\frac{1+\sqrt{5}}{2}, a_2)$. It gives that $\zeta''_2(x) \to a_2$ as $n \to \infty$ for $x \in (0, \frac{-1+\sqrt{5}}{2})$. Further, for $x \in (\alpha_2, \infty)$, $\zeta_2(x)$ maps $\alpha_2$ into $0$, $\zeta''_2(x) \to a_2$ as $n \to \infty$ for $x \in (\alpha_2, \infty)$. Hence $\zeta''_2(x) \to a_2$ as $n \to \infty$ for $x \in (0, \infty)$. (ii) By arguments similar to those used for (iii) of (a), it follows that $\zeta''_2(x) \to a_2$ as $n \to \infty$. (iii) The forward orbit of $\zeta_2(x)$ for each point in $(-1, 0) \setminus T_p$ is contained in $(-\infty, -1)$. Therefore, by (ii), $\zeta''_2(x) \to a_2$ as $n \to \infty$. Fig. 3 gives the web diagram of dynamics of $\zeta_2(x)$ for $1 < \lambda < \lambda^*$.

If $\lambda = \lambda^*$, by Theorem 2.1, $\zeta_2(x)$ has a rationally indifferent fixed point $\sqrt{2}$ and a repelling fixed point 0. The dynamics of $\zeta_2(x)$, for $\lambda = \lambda^*$, is as follows: (i) First let $x \in (-\frac{1+\sqrt{5}}{2}, \sqrt{2})$, since $\zeta''_2(x) < 0$ for $x \in (-\infty, x_0)$,
\(\zeta_n'(x)\) is decreasing for \(x \in (-\infty, x_0)\), where \(x_0 (\approx 1.26953)\) is a solution of the equation \(x^3 + 2x^2 - x - 4 = 0\). The rest of proof of assertion is same as part (i) of (c). (ii) By arguments similar to those used in the proof for part (iii) of (a), it follows that \(\zeta_n'(x) \to \sqrt{2}\) as \(n \to \infty\). (iii) Since forward orbit of \(\zeta_n(x)\) for each point in \((-1, 0) \setminus T_p\) is contained in \((-\infty, -1)\), by (ii), \(\zeta_n'(x) \to \sqrt{2}\) as \(n \to \infty\). The web diagram of dynamics of \(\zeta_n(x)\), \(\lambda = \lambda^*\), is given by Fig. 4(i).

\[\begin{align*}
(\text{i}) \lambda &= 9.93026, n = 200, x_0 = 0.5 \\
(\text{ii}) \lambda &= 11, n = 50, x_0 = 0.2
\end{align*}\]

Figure 4: Web diagrams of \(\zeta_n(x)\)

(e) If \(\lambda > \lambda^*\), by Theorem 2.1, \(\zeta_n(x)\) has a repelling fixed point 0. Using arguments similar than above cases, the orbits \(\{\zeta_n'(x)\}\) repel for all \(x \in \mathbb{R} \setminus T_p\), Fig. 4(ii) shows the web diagram of dynamics of \(\zeta_n(x)\) for \(\lambda > \lambda^*\).

This completes the proof of theorem.

For \(\lambda > \lambda^*\), the dynamics of \(\zeta_n(x) \in \mathcal{M}\) is now described as follows: (i) Since, for \(n = 2\), \(\zeta_n(x)\) has an attracting, repelling or indifferent periodic cycle of period 2 and 0 is a repelling fixed point, the iterations either converge to attracting cycle or keep moving. Consequently, all orbits of points in \((0, \infty)\) will either attract, or repel, or keep moving indefinitely. Thus, in this case, the orbits \(\{\zeta_n'(x)\}\) either attract, or repel, or are chaotic for \(x \in (0, \infty)\). (ii) Since \(\zeta_n(x)\) maps each point in \((-\infty, -1)\) to a point of the interval \((0, \infty)\), by case (i), the orbits \(\{\zeta_n'(x)\}\) either attract, or repel, or are chaotic for \(x \in (-\infty, -1)\). (iii) The forward orbit of \(\zeta_n(x)\) for each point in \((-1, 0) \setminus T_p\) is contained in the interval \((-\infty, -1)\). Therefore, by case (ii), the orbits \(\{\zeta_n'(x)\}\) either attract, or repel, or chaotic for \(x \in (-1, 0) \setminus T_p\). Fig. 5 gives the web diagram of dynamics of \(\zeta_n(x)\) for \(\lambda > \lambda^*\).

\[\begin{align*}
\text{Figure 5: Web diagram of } \zeta_n(x) \text{ for } \lambda = 11, n = 50, x_0 = 1.6
\end{align*}\]

It observed that, by Theorem 3.1, bifurcations in the dynamics of the function \(\zeta_n(x), x \in \mathbb{R} \setminus \{-1\}\) occur at several parameter values like \(\lambda = 1\) and \(\lambda = (\sqrt{2} + 1)e^{\sqrt{2}}\). The bifurcation diagram of the function \(\zeta_n(x)\) for \(\lambda > 0\) is shown in Fig. 6. It is also seen that period-doubling occurs in the real dynamics of \(\zeta_n(x) \in \mathcal{M}\) which is a route to chaos in the real dynamics of \(\zeta_n(x) \in \mathcal{M}\).
In order to quantify the chaos in the real dynamics, the Lyapunov exponents of $\zeta_\lambda(x)$ for certain values of $\lambda$ are computed as

$$L(\zeta_\lambda) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \ln \left( \frac{\lambda \left| x_i^2 + x_i - \ln e^{x_i} \right|}{(x_i + 1)^2} \right)$$

Figure 7 shows the computed values of Lyapunov exponents of $\zeta_\lambda(x)$ for $10 \leq \lambda \leq 45$ with $x_0$ as a suitable point near a fixed point or periodic point and $k = 2000$. It is found that, for certain ranges of the values of the parameter $\lambda$, the Lyapunov exponents are positive. It gives that chaotic behaviour exists in the real dynamics of $\zeta_\lambda(x)$. While, for $\lambda = 2, 12, 18.5, 38$, these are negative. There is no chaotic behaviour.

**Remark 3.1.** Our results on complex dynamics in next sections are induced from the corresponding results found in this section on real dynamics of $\zeta_\lambda \in \mathcal{M}$. 
4 Complex dynamics of $\zeta_\lambda \in \mathcal{M}$

The chaotic behaviour of complex functions in our family $\mathcal{M}$ is shown by Julia sets. Using Theorem 2.1, the complex dynamics of $\zeta_\lambda \in \mathcal{M}$ for $0 < \lambda < 1$, $\lambda = 1$, $1 < \lambda < \lambda^*$, $\lambda = \lambda^*$ and $\lambda > \lambda^*$ are described in this section.

If $0 < \lambda < 1$, by Theorem 2.1, the function $\zeta_\lambda \in \mathcal{M}$ admits the basin of attraction $A(0) = \{z \in \mathbb{C} : \zeta_\lambda^n(z) \to 0 \text{ as } n \to \infty\}$ for attracting fixed point $0$. The following theorem gives the characterization of the Julia set $J(\zeta_\lambda)$, $0 < \lambda < 1$, as the complement of the basin of attraction $A(0)$:

**Theorem 4.1.** Let $\zeta_\lambda \in \mathcal{M}$, $0 < \lambda < 1$. Then, the Julia set $J(\zeta_\lambda) = \hat{\mathbb{C}} \setminus A(0)$.

**Proof.** Since $z = 0$ is an attracting fixed point of $\zeta_\lambda(z)$, its only asymptotic value at $z = 0$ lies in the basin of attraction $A(0)$. The critical values of $\zeta_\lambda(z)$ are only finitely many and all of these lie on the real line. By Theorem 3.1 (a), the forward orbit of every critical value tends to attracting fixed point 0 under iteration. Therefore, all singular values of $\zeta_\lambda(z)$ and their orbits lie in the same component of $A(0)$. Thus, it follows that $\zeta_\lambda(z)$ has no Siegel disk or Herman ring in $F(\zeta_\lambda)$. By Theorem 2.1, $\zeta_\lambda(z)$ has only one attracting fixed point and two repelling fixed points on the real axis so $U$ is not parabolic domain. Using Theorem 1.1 and Theorem 1.2, the Fatou set $F(\zeta_\lambda)$ has no wandering domains and no Baker domains since $\zeta_\lambda \in \mathcal{M}$ is critically finite meromorphic function. Consequently, the only possible stable component $U$ of $F(\zeta_\lambda)$ is the basin of attraction $A(0)$ for $0 < \lambda < 1$.

Since the Julia set is complement of the Fatou set $F(\zeta_\lambda) = A(0)$, the Julia set $J(\zeta_\lambda) = \hat{\mathbb{C}} \setminus A(0)$.

**Remark 4.1.** By Theorem 3.1(a), for $0 < \lambda < 1$, $\zeta_\lambda^n(x) \to 0$ as $n \to \infty$ for $[(-\infty, -1) \cup (-1, r_2) \cup (r_2, 0) \cup (0, \infty)] \setminus T_p$. Therefore, it follows that $\mathbb{R} \setminus \{T_p \cup \{r_2\}\}$ is contained in $A(0)$ for $0 < \lambda < 1$.

The following proposition shows that the Fatou set of $\zeta_\lambda \in \mathcal{M}$ contains certain intervals of real line for $\lambda = 1$:

**Proposition 4.1.** Let $\zeta_\lambda \in \mathcal{M}$, $\lambda = 1$. Then, $F(\zeta_\lambda)$ contains the intervals $(-\infty, -1), (-1, 0), (0, \infty)$.

**Proof.** By Theorem 3.1(b), $\zeta_\lambda^n(x) \to 0$ for $x \in [(-\infty, -1) \cup (-1, 0) \cup (0, \infty)] \setminus T_p$ if $\lambda = 1$. Therefore, the intervals $(-\infty, -1), (-1, 0) \setminus T_p$ and $(0, \infty)$ are contained in the Fatou set $F(\zeta_\lambda)$.

If $1 < \lambda < \lambda^*$, by Theorem 2.1, the function $\zeta_\lambda(z)$ admits the basin of attraction $A(a_\lambda) = \{z \in \mathbb{C} : \zeta_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}$ for its attracting fixed point $a_\lambda$. The following theorem shows the characterization of the Julia set $J(\zeta_\lambda)$ as the complement of the basin of attraction $A(a_\lambda)$:

**Theorem 4.2.** Let $\zeta_\lambda \in \mathcal{M}$, $1 < \lambda < \lambda^*$. Then, the Julia set $J(\zeta_\lambda) = \hat{\mathbb{C}} \setminus A(a_\lambda)$.

**Proof.** By taking the fixed point $a_\lambda$ instead of 0 and using Theorem 3.1(c) instead of Theorem 3.1 (a), the proof of theorem follows on the lines of proof similar to that of Theorem 4.1.

**Remark 4.2.** By Theorem 3.1(c), for $1 < \lambda < \lambda^*$, $\zeta_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $[(-\infty, -1) \cup (-1, 0) \cup (0, \infty)] \setminus T_p$. Therefore, it follows that $\mathbb{R} \setminus \{T_p \cup \{0\}\}$ is contained in $A(a_\lambda)$ for $1 < \lambda < \lambda^*$.

The following proposition shows that the Fatou set of $\zeta_\lambda \in \mathcal{M}$ for $\lambda = \lambda^*$ contains certain intervals of real line:

**Proposition 4.2.** Let $\zeta_\lambda \in \mathcal{M}$, $\lambda = \lambda^*$. Then, $F(\zeta_\lambda)$ contains the intervals $(-\infty, -1), (-1, 0) \setminus T_p, (0, \sqrt{2})$ and $(\sqrt{2}, \infty)$.

**Proof.** By taking the fixed point $a_\lambda$ instead of 0 and using Theorem 3.1(d) instead of Theorem 3.1(b), the proof of proposition follows on the lines of proof similar to that of Proposition 4.1.

Let $P_a$ be a set of attracting periodic orbits of $\zeta_\lambda(x)$. The following theorem describes the complex dynamics of $\zeta_\lambda(z)$ for $\lambda > \lambda^*$ showing that the Julia set $J(\zeta_\lambda)$ contains the subset of the real line consisting of the complement of attracting periodic orbits of $\zeta_\lambda(x)$:

**Theorem 4.3.** Let $\zeta_\lambda \in \mathcal{M}$, $\lambda > \lambda^*$. Then, $J(\zeta_\lambda)$ contains the set $\mathbb{R} \setminus P_a$.

**Proof.** For $\lambda > \lambda^*$, all points in the dynamics of $\zeta_\lambda(x)$ on $\mathbb{R} \setminus T_p$ either attract, or repel, or chaotic. Therefore, Julia set $J(\zeta_\lambda)$ contains the set $\mathbb{R} \setminus (T_p \cup P_a)$. Since pole and preimages of the pole are contained in $J(\zeta_\lambda)$ and attracting orbits are contained in the Fatou set, it follows that $J(\zeta_\lambda)$ contains the set $\mathbb{R} \setminus P_a$. 

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5 Simulation and comparisons

Using the characterizations of the Julia set $J(\zeta_\lambda)$ obtained in Theorems 4.1, 4.2 and 4.3, the images of $J(\zeta_\lambda)$ are generated. In the output images, the white points represent $J(\zeta_\lambda)$ and the red points represent the Fatou set of $\zeta_\lambda(z)$. The Julia sets of a function $\zeta_\lambda \in \mathcal{M}$ for $\lambda = 0.9$, $\lambda = 1.1$, $\lambda = 9.93$ and $\lambda = 9.94$ are generated in the rectangular domain $R = \{z \in \mathbb{C} : -1.0 \leq \Re(z) \leq 1.0, -1.0 \leq \Im(z) \leq 1.0\}$ upto iterations 250. The Julia set $J(\zeta_\lambda)$ for $\lambda = 0.9$ given by Fig. 8(a) has the same pattern as those of the Julia sets of $\zeta_\lambda(z)$ for any other $\lambda$ satisfying $0 < \lambda < 1$. This confirms to the result of Theorem 4.1. The nature of images of the Julia sets of $\zeta_\lambda(z)$ have the same pattern for all $\lambda$ satisfying $1 < \lambda < \lambda^*$. This is visualized in Fig. 8(b) for $\lambda = 1.1$ and is in conformity to the result of Theorem 4.2. The nature of image of $J(\zeta_\lambda)$ for $\lambda = 9.94$ is shown in Fig. 8(d). The Julia set $J(\zeta_\lambda)$ for any other $\lambda > \lambda^*$ remains the same as that of $J(\zeta_\lambda)$ for $\lambda = 9.94$. On comparison of Julia sets of $\zeta_\lambda(z)$ for $\lambda = 9.93(< \lambda^* = (\sqrt{2} + 1)e^{\sqrt{2}})$ and $\lambda = 9.94(> \lambda^* = (\sqrt{2} + 1)e^{\sqrt{2}})$ (Figs. 8(c) and (d)), it is observed that the phenomenon of chaotic burst is not visible here. The reason probably being just that the orbits are chaotic but these remain bounded after crossing the parameter value.

![Figure 8: Julia sets of $\zeta_\lambda(z)$ for (a) $\lambda = 0.9$, (b) $\lambda = 1.1$, (c) $\lambda = 9.93$, (d) $\lambda = 9.94$](image)

Finally, the comparison among dynamical properties of $\zeta_\lambda \in \mathcal{M}$ are found here with the dynamics of functions (i) $T_\lambda(z) = \lambda \tan z$, $\lambda \in \mathbb{C} \setminus \{0\}$ having polynomial Schwarzian derivative [5, 6, 24] (ii) $E_\lambda(z) = \frac{\lambda e^{z}}{1 - \lambda e^{z}}, \lambda > 0$ having transcendental meromorphic Schwarzian derivative [12] and (iii) $f_\lambda(z) = \lambda f(z)$, $\lambda > 0$, where $f(z)$ has only real critical values, and having rational Schwarzian derivative [21]. It is observed that although the nature of the functions in the families under comparison is widely different, the dynamical behaviour of the functions in all these families have an underlying differences in the nature of dynamics of functions in our family obtained in earlier sections, bifurcation occurs at different number of parameter values in our family of functions but the Julia sets for functions in the families $T_\lambda(z)$, $E_\lambda(z)$ and $f_\lambda = \lambda f(z)$, the Julia set does not contain whole real line in our family of functions but the Julia sets for functions in the families $T_\lambda(z)$, $E_\lambda(z)$ contain the whole real line and Julia set is not the whole complex plane for any value of parameter $\lambda$ for our family of functions but it is whole complex plane for the family of functions $T_\lambda(z)$ for $\lambda = i\pi$. Moreover, these families have an underlying similarity in dynamical behaviour with our family, all singular values of these families are bounded, the Fatou set equals the basin of attraction of the real attracting fixed point, the Julia set is the closure of escaping points and; Herman rings and wandering domains do not exist.

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