A Step-by-Step Algorithm for Plotting Local Bifurcation Diagram

Shahram Aghaei¹, Abolghasem Daeichian²

(1) Electrical and Computer Engineering Department, Yazd University, Yazd, Iran
(2) Electrical and Computer Engineering Department, Arak University, Arak, Iran

Abstract
This paper proposes a graphically step-by-step algorithm for plotting local bifurcation diagram as well as determining stability or instability of all branches. The presented method which consists of eight steps, is effortlessly Applicable to the first and second order complex nonlinear systems. It provides a visual sense for engineers to analyze and design systems. The method is validated by applying on several different examples.

Keywords: Bifurcation diagram, Dynamical systems, Equilibrium points.

1 Introduction

Bifurcation theory is a mathematical study of changes in the qualitative or topological structure of a given family such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system, causes a sudden qualitative or topological change in its behavior [1]. Bifurcations occur in both continuous systems (described by ODEs, DDEs or PDEs), and discrete systems (described by maps).

It is useful to divide bifurcations into two principal classes. Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds; and Global bifurcations, which often occur when larger invariant sets of the system collide with each other, or with equilibria of the system. They can not be detected purely by a stability analysis of the equilibria (fixed points) [2].

Local Bifurcation in continuous systems corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ODEs), this corresponds to a fixed point having a Floquet multiplier with modulus equal to one [3]. In both cases, the equilibrium is non-hyperbolic at the bifurcation point. The topological changes in the phase-portrait of the system can be confined to arbitrarily small neighborhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence local) [5]. Examples of local bifurcations include Saddle-node (fold) bifurcation, Transcritical bifurcation, Pitchfork bifurcation, Period-doubling (flip) bifurcation, Hopf bifurcation and Neimark (secondary Hopf) bifurcation.

A bifurcation diagram shows the possible long-term values (equilibria/fixed points or periodic orbits) of a system as a function of a bifurcation parameter in the system [4]. Bifurcation diagram is a powerful tool that visually gives
information about the system behavioral respect to the change of parameter. Bifurcation diagram is very helpful for engineers to control dynamical systems to prevent or control bifurcations; e.g. [7, 8]. Numerical and analytical methods are used for plotting this types of diagrams by machine and hand. For plotting by hand, the analytical method may increase the complexity of the context.

This paper presents a graphically step-by-step algorithm, based on the well-known root loci algorithm, originally introduced by Evans [5, 6], for plotting the local bifurcation diagram. The proposed method, which is simpler and more straightforward than the analytical methods, may be used as a manual step-by-step tool for plotting the bifurcation diagrams. It is Applicable to the first and second order complex nonlinear systems. Section 2 presents the problem statement. The algorithm is introduced in section 3. Section 4 is devoted to the validation check of the proposed method by some examples. The paper is concluded in section 5.

2 Problem statements

Consider the following dynamical system:

\[ \dot{x} = f(x) + \lambda g(x) \]  

(2.1)

where \( x \in \mathbb{R}^n \) is the state of system and \( \lambda \in \mathbb{R} \) is a variable parameter. The aim is to develop an algorithm to plot the bifurcation diagram respect to \( \lambda \). The equilibrium points of the system Eq.2.1 are roots of the following equation:

\[ f(x) + \lambda g(x) = 0 \]  

(2.2)

Obviously, the roots of the Eq.2.2 are depended on \( \lambda \) except those that are common roots of \( f(x) \) and \( g(x) \). Thus, Eq.2.2 can be rewritten as:

\[ h(x)(f_1(x) + \lambda g_1(x)) = 0 \]  

(2.3)

where \( f_1(x) \) and \( g_1(x) \) are coprime and \( h(x) \) is the common factor of \( f(x) \) and \( g(x) \) and its roots are independent of \( \lambda \).

**Remark 2.1.** The roots of \( h(x) \) are considered as constant equilibria of Eq.2.1. So, the following focuses on the roots of

\[ f_1(x) + \lambda g_1(x) = 0 \]  

(2.4)

as equilibria depended on \( \lambda \).

**Corollary 2.1.** If \( \lambda = 0 \) then the roots of \( f_1(x) \) are the equilibrium points of Eq.2.1.

**Proof.** considering Eq.2.4 with \( \lambda = 0 \), obviously, the proof is completed.

**Corollary 2.2.** If \( \lambda = \pm \infty \) then the roots of \( g_1(x) \) are the equilibrium points of Eq.2.1.

**Proof.** dividing Eq.2.4 by \( \lambda = 0 \) yields \( \frac{f_1(x)}{\lambda} + g_1(x) = 0 \). Then with \( \lambda = \pm \infty \), obviously, the proof is completed.

**Remark 2.2.** Based on root loci [5, 6], the Eq.2.4 can be rewritten as:

\[ 1 + \mu \frac{f_1(x)}{g_1(x)} = 0 \]  

(2.5)

in which the greatest degree of \( f_1(x) \) and \( g_1(x) \) have positive coefficients and \( \mu = \lambda \) or \( \mu = -\lambda \). The roots of \( f_1(x) \) and \( g_1(x) \) are called as poles and zeros, respectively.
3 Proposed Algorithm for Plotting the Bifurcation Diagram

In this section, an step by step algorithm is introduced for plotting the bifurcation diagram of dynamical nonlinear systems in the form of which has been considered in the previous section.

Algorithm:

Step 1: Following Eqs.2.1 to 2.4 then write Eq.2.5.

Step 2: Locate the poles and zeros on the vertical axis of bifurcation diagram. Sign the sections of vertical axis that above which there are odd number of poles and zeros as locus for $m > 0$ and the other sections as locus for $m < 0$.

Proof. Write the Eq.2.5 in the form of poles and zeros as follows:

$$\frac{\Pi(x-z_i)}{\Pi(x-p_i)} = -\frac{1}{\mu} \quad (3.6)$$

where $z_i$ and $p_i$ are zeros and poles, respectively. Thus, $\forall x \in \mathbb{R}$ if $\Sigma \angle(x-z_i) - \Sigma \angle(x-p_i) = 0 \pm 2k\pi$, then there exist a $\mu > 0$ that $x$ lies on roots of Eq.3.6.

$$\Sigma \angle(x-z_i) - \Sigma \angle(x-p_i) = \Sigma_{\phi_i > x} \angle(x-z_i) - \Sigma_{\phi_i < x} \angle(x-p_i)$$

Providing $m + n = 2k$ in which $m$ and $n$ are the numbers of $z_i > x$ and $p_i > x$, respectively. Repeating the above procedure lead to $m + n = 2k + 1$ for $\mu < 0$. The step 2 is proved. 

Step 3: Consider a vertical linear asymptote at finite value $\mu = \lim_{x \to \infty} \frac{f^1(x)}{g^1(x)}$ if there is any.

Proof. It’s clear from Eq.2.5.

Step 4: Consider horizontal linear asymptotes at finite zeros.

Proof. It’s result of corollary 2.1.

Step 5: Sketch the branches of bifurcation diagram, starting from poles, and proceed to asymptotes in each section, considering the sign of $\mu$.

Proof. See corollaries 2.1 and 2.2.

Step 6: Sketch horizontal lines corresponding to the roots of $h(x)$ in Eq.2.3 as the branches of bifurcation diagram independent of $\lambda$ (or equivalently $\mu$).

Proof. See remark 2.1.

Step 7: If $\mu = -\lambda$, then flip the bifurcation diagram horizontally.

Definition 3.1. A branch is a segment of bifurcation diagram along which there is no change in the sign of slope.

Step 8: If $f(x) + \lambda g(x) > 0$ for $x \gg \max\{z_i, p_i\}$ then the upper branch is unstable, next one is stable and so on; otherwise the upper branch is stable, next one is unstable and so on.

Proof. The proof is trivial considering the diagram of $f(x) + \lambda g(x)$ versus $x$ for any $\lambda$.

Remark 3.1. The coordinates of branch junction points can be easily calculating by $\frac{d[(f^1(x))/(g^1(x))]}{dx} = 0$. Those roots of this equation are acceptable that lead to admissible $\lambda = -\frac{f_1(x)}{g_1(x)}$. 

International Scientific Publications and Consulting Services
Remark 3.2. To apply the proposed algorithm for the non-polynomial nonlinear dynamical systems, Taylor expansion can be used.

Remark 3.3. for plotting the bifurcation diagram of 2-dimensional dynamical systems, one can first rewrite it in the polar coordination, if possible, and then applies the proposed algorithm considering the r-equation.

In the sequel, some basic examples shows the performance of the proposed step by step handy algorithm.

4 Case Studies

Here, some basic dynamical systems taught in dynamical systems courses [6] are considered for validation of the proposed algorithm. In any case, the goal is to plot the bifurcation diagram of the given dynamical system. Please note that some steps of the algorithm corresponding to each example will be shown by diagram.

Example 4.1. Consider the one-dimensional system $\dot{x} = \lambda x - x^3$.

Step 1: $x(1 - \lambda \frac{1}{x}) = x(1 + \mu \frac{1}{x})$, where $\mu = -\lambda$. There are two poles at $x = 0$, no zeros, and a constant root at $x = 0$.

Step 2: See Fig. 4.a.

Step 3: There is no finite value vertical asymptote.

Step 4: There is no zeros.

Step 5: See Fig. 4.b.

Step 6: See Fig. 4.c.

Step 7: See Fig. 4.d.

Step 8: See Fig. 4.e.

Example 4.2. Consider dynamical system $\dot{x} = c + dx - x^3$.

a) Plotting the bifurcation diagram for $d = 1 + 2c$.

Step 1: $c + (1 + 2c)x - x^3 \Rightarrow 1 - c \frac{2x + 1}{x^2 - x} = 1 + \mu \frac{2x + 1}{x(x - 1)}$, where $\mu = -c$. There are three poles at $x = 0, 1, -1$, and
one zero at $x = -\frac{1}{2}$.
Step 2: See Fig.4.a.
Step 3: There is no finite value vertical asymptote.
Step 4: Consider a horizontal asymptote at $x = -0.5$.
Step 5: See Fig.4.b.
Step 7: See Fig.4.c.
Step 8: See Fig.4.d.

Figure 2: Example 2-a

b) Plotting the bifurcation diagram for $d = 1 + 0.5c$. See Fig.4.

Figure 3: Example 2-b

Example 4.3. Consider dynamical system
\[
\begin{align*}
\dot{x} &= y + x[\lambda - \eta(x^2 + y^2) + (x^2 + y^2)^2] \\
\dot{y} &= -x + y[\lambda - \eta(x^2 + y^2) + (x^2 + y^2)^2]
\end{align*}
\]
Plotting the bifurcation diagram for $\lambda = \eta$.

Rewrite the system in polar coordination with $\lambda = \eta$:

$$
\dot{r} = r(\lambda - \lambda r^2 + r^4) \\
\dot{\theta} = -1
$$

Step 1: $\dot{r} = r[1 - \frac{\lambda}{r^2} - 1] = r[1 + \mu \frac{\lambda}{r^2}]$, where $\mu = -c$.

Step 2: See Fig.4.a.

Step 3: There is no finite value vertical asymptote.

Step 4: Consider two horizontal asymptotes at $x = -1, 1$.

Step 5: See Fig.4.b.

Step 6: See Fig.4.c.

Step 7: See Fig.4.d.

Figure 4: Example 3

5 Conclusion

The proposed algorithm prepares a simple and fast method to plot the local bifurcation diagram. This algorithm can be applied to dynamical systems introduced by Eq.1. It consists of some handy steps. Some examples show its performance and properties in the sense of simplification and quickness.

References


https://doi.org/10.1007/978-1-4612-4426-4


https://doi.org/10.1109/T-AIEE.1948.5059708

https://doi.org/10.1109/T-AIEE.1950.5060121

https://doi.org/10.1016/j.amc.2015.12.015

[8] Najmeh Zamani, Mohammad Ataei, Mehdi Niroomand, Analysis and control of chaotic behavior in boost con-
verter by ramp compensation based on Lyapunov exponents assignment: theoretical and experimental investiga-
https://doi.org/10.1016/j.chaos.2015.08.010