Journal Nonlinear Analysis and Application 2017 No.2 (2017) 132-143

Available online at www.ispacs.com/jnaa

Volume 2017, Issue 2, Year 2017 Article ID jnaa-00314, 12 Pages

doi:10.5899/2017/jnaa-00314

Research Article

Multidimensional coincidence point results for generalized $(\psi,\,\theta,\,\varphi)$ -contraction on ordered metric spaces

Bhavana Deshpande[1]*, Amrish Handa[2]

(1) *Department of Mathematics, Govt. B.S.P.G. College Jaora, Dist. Ratlam (M. P.) India*

(2) *Department of Mathematics, Govt. P.G. Arts & Science College, Ratlam (M. P.) India*

Abstract

The main objective of this research article is to establish some coincidence point theorem for g-non-decreasing mappings under generalized $(\psi,\,\theta,\,\varphi)$-contraction on a partially ordered metric space. Furthermore, we show how multidimensional results can be seen as a simple consequences of our unidimensional coincidence point theorem. Our results modify, improve, sharpen, enrich and generalize various known results.

Keywords: Generalized $(\psi,\,\theta,\,\varphi)$ -contraction, coincidence point, partially ordered metric space, O -compatible.

Mathematics Subject Classification: 47H10, 54H25.

1 Introduction

In [12], Guo and Lakshmikantham introduced the notion of coupled fixed point and initiated the investigation of multidimensional fixed point theory. Following this initial paper Gnana-Bhaskar and Lakshmikantham [6] obtained some coupled fixed point theorems for mapping $F : X \times X \to X$ (where X is a partially ordered metric space) by defining the notion of mixed monotone mapping. After that, Lakshmikantham and Ciric [22] proved coupled fixed/coincidence point theorems for mappings $F : X \times X \to X$ and $g : X \to X$ by introducing the concept of the mixed g-monotone property. They also illustrated these results by proving the existence and uniqueness of the solution for periodic boundary value problems. A large number of authors established coupled fixed/coincidence point theorems by using this notion in different context, (see [1], [2], [3], [8], [9], [10], [15], [16], [23], [24], [28], [33], [35], [36]).
Inspired by these papers, Berinde and Borcut [4] defined tripled fixed points and established some tripled fixed point theorems. As a natural extension, Karapinar [17] studied the quadruple case. Very recently, the concept of multidimensional fixed/coincidence point introduced by Roldan et al. in [29], which is an extension of Berzig and Samet's notion given in [5], which extended and generalized the mentioned fixed point results to higher dimensions. However, they used permutations of variables and distinguished between the first and the last variables. For more details one can refer ([11], [18], [19], [20], [21], [22], [26], [30], [31], [32], [34], [38]).
In this research paper, we obtain some coincidence point theorem for g-non-decreasing mappings under generalized $(\psi,\,\theta,\,\varphi)$-contraction on a partially ordered metric space. Furthermore, we show how multidimensional results can

*Corresponding author. Email address: bhavnadeshpande@yahoo.com

be seen as simple consequences of our unidimensional coincidence point theorem. We modify, improve, sharpen, enrich and generalize the results of Alotaibi and Alsulami [1], Alsulami [2], Gnana-Bhaskar and Lakshmikantham [6], Harjani and Sadarangani [13], Harjani et al. [14], Lakshmikantham and Ciric [22], Luong and Thuan [23], Nieto and Rodriguez-Lopez [26], Ran and Reurings [27], Razani and Parvaneh [28] and many other famous results in the literature.

2 Preliminaries

In order to fix the framework needed to state our main results, we recall the following notions. For simplicity, we denote from now on $X\times X\times ...\times X$ (n times) by X^n, where $n\in\mathbb{N}$ with $n\geq 2$ and X is a non-empty set. Let $\{A, B\}$ be a partition of the set $\Lambda_n=\{1, 2, ..., n\}$, that is, A and B are nonempty subsets of Λ_n such that $A\cup B=\Lambda_n$ and $A\cap B=\emptyset$. We will denote $\Omega_{A,B}=\{\sigma:\Lambda_n\to\Lambda_n:\sigma(A)\subseteq A,\ \sigma(B)\subseteq B\}$ and $\Omega'_{A,B}=\{\sigma:\Lambda_n\to\Lambda_n:\sigma(A)\subseteq B,\ \sigma(B)\subseteq A\}$. Henceforth, let $\sigma_1, \sigma_2, ..., \sigma_n$ be n mappings from Λ_n into itself and let Υ be the $n-$tuple $(\sigma_1, \sigma_2, ..., \sigma_n)$. Let $F:X^n\to X$ and $g:X\to X$ be two mappings. For brevity, we denote $g(x)$ by gx where $x\in X$.
A partial order $\preceq$ on X can be extended to a partial order $\sqsubseteq$ on X^n in the following way. If $(X, \preceq)$ be a partially ordered space, $x, y\in X$ and $i\in\Lambda_n$, we will use the following notations:

$$x\preceq_i y\Rightarrow\begin{cases} x\preceq y, \text{ if } i\in A,\\ x\succeq y, \text{ if } i\in B.\end{cases}\tag{2.1}$$

Consider on the product space X^n the following partial order: for $Y=(y_1, y_2, ..., y_i, ..., y_n)$, $V=(v_1, v_2, ..., v_i, ..., v_n)\in X^n$,

$$Y\sqsubseteq V\Leftrightarrow y_i\preceq_i v_i.\tag{2.2}$$

Notice that $\sqsubseteq$ depends on A and B. We say that two points Y and V are comparable, if $Y\sqsubseteq V$ or $V\sqsubseteq Y$. Obviously, $(X^n, \sqsubseteq)$ is a partially ordered set.

Definition 2.1. *([21], [29], [32]). A point $(x_1, x_2, ..., x_n)\in X^n$ is called a $\Upsilon-$coincidence point of the mappings $F:X^n\to X$ and $g:X\to X$ if*

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ..., x_{\sigma_i(n)})=gx_i, \textit{ for all } i\in\Lambda_n.\tag{2.3}$$

If g is the identity mapping on X, then $(x_1, x_2, ..., x_n)\in X^n$ is called a $\Upsilon-$fixed point of the mapping F.
It is clear that the previous definition extends the notions of coupled, tripled, and quadruple fixed points. In fact, if we represent a mapping $\sigma:\Lambda_n\to\Lambda_n$ throughout its ordered image, that is, $\sigma=(\sigma(1), \sigma\ (2), ..., \sigma\ (n))$, then

(i) *Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n=2$, $\sigma_1=(1, 2)$ and $\sigma_2=(2, 1)$,*

(ii) *Berinde and Borcut's tripled fixed points are associated with $n=3$, $\sigma_1=(1, 2, 3)$, $\sigma_2=(2, 1, 2)$ and $\sigma_3=(3, 2, 1)$,*

(iii) *Karapinar's quadruple fixed points are considered when $n=4$, $\sigma_1=(1, 2, 3, 4)$, $\sigma_2=(2, 3, 4, 1)$, $\sigma_3=(3, 4, 1, 2)$ and $\sigma_4=(4, 1, 2, 3)$.*

These cases consider A as the odd numbers in $\{1, 2, ..., n\}$ and B as its even numbers. However, Berzig and Samet [5] use $A=\{1, 2, ..., m\}$, $B=\{m+1, ..., n\}$ and arbitrary mappings.

Definition 2.2. *[29]. Let $(X, \preceq)$ be a partially ordered space. We say that F has the mixed $(g, \preceq)$-monotone property if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, that is, for all $x_1, x_2, ..., x_n, y, z\in X$ and all i,*

$$gy\preceq gz\Rightarrow F(x_1,...,x_{i-1},y,x_{i+1},...,x_n)\preceq_i F(x_1,...,x_{i-1},z,x_{i+1},...,x_n).\tag{2.4}$$

Remark 2.1. *[20]. In order to ensure the existence of Υ-coincidence/fixed points, it is very important to assume that the mixed g-monotone property is compatible with the permutation of the variables, that is, the mappings of $\Upsilon=(\sigma_1, \sigma_2, ..., \sigma_n)$ should verify $\sigma_i\in\Omega_{A,B}$ if $i\in A$ and $\sigma_i\in\Omega'_{A,B}$ if $i\in B$.*

Definition 2.3. *([32] ,[37]). Let (X, d) be a metric space and define $\Delta_n, \rho_n : X^n \times X^n \to [0, +\infty)$, for $Y = (y_1, y_2, ..., y_n)$, $V = (v_1, v_2, ..., v_n) \in X^n$, by*

$$\Delta_n(Y, V) = \frac{1}{n}\sum_{i=1}^{n} d(y_i, v_i) \text{ and } \rho_n(Y, V) = \max_{1\le i\le n} d(y_i, v_i). \tag{2.5}$$

Then Δ_n and ρ_n are metric on X^n and (X, d) is complete if and only if (X^n, Δ_n) and (X^n, ρ_n) are complete. It is easy to see that

$$\begin{aligned} \Delta_n(Y^k, Y) &\to 0 \text{ (as } k \to \infty) \Leftrightarrow d(y_i^k, y_i) \to 0 \text{ (as } k \to \infty), \\ \text{and } \rho_n(Y^k, Y) &\to 0 \text{ (as } k \to \infty) \Leftrightarrow d(y_i^k, y_i) \to 0 \text{ (as } k \to \infty),\ i \in \Lambda_n, \end{aligned} \tag{2.6}$$

where $Y^k = (y_1^k, y_2^k, ..., y_n^k)$ and $Y = (y_1, y_2, ..., y_n) \in X^n$.

Lemma 2.1. *([32] ,[37], [38]). Let $(X, d, \preceq)$ be an ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Define $F_\Upsilon, G : X^n \to X^n$, for all $y_1, y_2, ..., y_n \in X$, by*

$$\begin{aligned} F_\Upsilon(y_1, y_2, ..., y_n) &= \begin{pmatrix} F(y_{\sigma_1(1)}, y_{\sigma_1(2)}, \dots, y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)}, y_{\sigma_2(2)}, \dots, y_{\sigma_2(n)}), \\ \dots, F(y_{\sigma_n(1)}, y_{\sigma_n(2)}, \dots, y_{\sigma_n(n)}) \end{pmatrix}, \\ \text{and } G(y_1, y_2, ..., y_n) &= (gy_1, gy_2, ..., gy_n). \end{aligned} \tag{2.7}$$

(1) *If F has the mixed $(g, \preceq)-$monotone property, then F_Υ is monotone $(G, \sqsubseteq)-$non-decreasing.*

(2) *If F is $d-$continuous, then F_Υ is Δ_n-continuous and ρ_n-continuous.*

(3) *If g is $d-$continuous, then G is Δ_n-continuous and ρ_n-continuous.*

(4) *A point $(y_1, y_2, ..., y_n) \in X^n$ is a $\Upsilon-$fixed point of F if and only if $(y_1, y_2, ..., y_n)$ is a fixed point of F_Υ.*

(5) *A point $(y_1, y_2, ..., y_n) \in X^n$ is a $\Upsilon-$coincidence point of F and g if and only if $(y_1, y_2, ..., y_n)$ is a coincidence point of F_Υ and G.*

(6) *If $(X, d, \preceq)$ is regular, then $(X^n, \Delta_n, \sqsubseteq)$ and $(X^n, \rho_n, \sqsubseteq)$ are also regular.*

The commutativity and compatibility of the mappings are defined as follows.

Definition 2.4. *[30]. We will say that two mappings $T, g : X \to X$ are commuting if $gTx = Tgx$ for all $x \in X$. We will say that $F : X^n \to X$ and $g : X \to X$ are commuting if $gF(x_1, x_2, ..., x_n) = F(gx_1, gx_2, ..., gx_n)$ for all $x_1, x_2, ..., x_n \in X$.*

The following notion was introduced in order to avoid the necessity of commutativity.

Definition 2.5. *([7], [16], [24], [25]). Let $(X, d, \preceq)$ be an ordered metric space. Two mappings $T, g : X \to X$ are said to be O-compatible if*

$$\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0, \tag{2.8}$$

provided that $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is $\preceq$-monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X. \tag{2.9}$$

The natural extension to an arbitrary number of variables is the following one.

Definition 2.6. *[25]. Let $(X, d, \preceq)$ be an ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. We will say that (F, g) is a $(O, \Upsilon)-$compatible pair if, for all $i \in \Lambda_n$,*

$$\lim_{m\to\infty} d(gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, ..., x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, ..., gx_m^{\sigma_i(n)})) = 0, \tag{2.10}$$

whenever $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$ are sequences in X such that $\{gx_m^1\}, \{gx_m^2\}, ..., \{gx_m^n\}$ are $\preceq -$monotone and

$$\lim_{m\to\infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, ..., x_m^{\sigma_i(n)}) = \lim_{n\to\infty} gx_m^i \in X \text{ for all } i \in \Lambda_n. \tag{2.11}$$

Lemma 2.2. *[25]. If F and g are (O, Υ)-compatible, then F_Υ and G are O-compatible.*

The following definitions are usual in the field of fixed point theory.

Definition 2.7. *[6]. An ordered metric space $(X, d, \preceq)$ is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all n, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all n. $(X, d, \preceq)$ is said to be regular if it is both non-decreasing-regular and non-increasing-regular.*

Definition 2.8. *[11]. Let$(X, \preceq)$ be a partially ordered set and let $T, g : X \to X$ be two mappings. We say that T is $(g, \preceq)$-non-decreasing if $Tx \preceq Ty$ for all $x, y \in X$ such that $gx \preceq gy$. If g is the identity mapping on X, we say that T is $\preceq$-non-decreasing.*

Lemma 2.3. *[11]. If T is $(g, \preceq)$-non-decreasing and $gx = gy$, then $Tx = Ty$. It follows that*

$$\begin{aligned} gx &= gy \Rightarrow \left\{ \begin{array}{c} gx \preceq gy, \\ gy \preceq gx \end{array} \right\} \\ &\Rightarrow \left\{ \begin{array}{c} Tx \preceq Ty, \\ Ty \preceq Tx \end{array} \right\} \Rightarrow Tx = Ty. \end{aligned} \tag{2.12}$$

3 Main results

Definition 3.1. *An altering distance function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:*

(i_ψ) *ψ is continuous and non-decreasing,*

(ii_ψ) *$\psi(t) = 0$ if and only if $t = 0$.*

Theorem 3.1. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $T, g : X \to X$ be two mappings such that the following properties are fulfilled:*

(i) *$T(X) \subseteq g(X)$,*

(ii) *T is $(g, \preceq)$-non-decreasing,*

(iii) *there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$,*

(iv) *there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

$$\psi(d(Tx, Ty)) \leq \theta(d(gx, gy)) - \varphi(d(gx, gy)),$$

for all $x, y \in X$ such that $gx \preceq gy$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Also assume that, at least, one of the following conditions holds.

(a) (X, d) *is complete,* T *and* g *are continuous and the pair* (T, g) *is O-compatible,*

(b) (X, d) *is complete,* T *and* g *are continuous and commuting,*

(c) $(g(X), d)$ *is complete and* $(X, d, \preceq)$ *is non-decreasing-regular,*

(d) (X, d) *is complete,* $g(X)$ *is closed and* $(X, d, \preceq)$ *is non-decreasing-regular,*

(e) (X, d) *is complete,* g *is continuous, the pair* (T, g) *is O-compatible and* $(X, d, \preceq)$ *is non-decreasing-regular.*

Then T *and* g *have, at least, a coincidence point.*

Proof. We divide the proof into four steps.

Step 1.
We claim that there exists a sequence $\{x_n\} \subseteq X$ such that $\{gx_n\}$ is $\preceq$-non-decreasing and $gx_{n+1} = Tx_n$, for all $n \geq 0$. Starting from $x_0 \in X$ given in (iii) and taking into account that $Tx_0 \in T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Then $gx_0 \preceq Tx_0 = gx_1$. Since T is $(g, \preceq)$-non-decreasing, $Tx_0 \preceq Tx_1$. Now $Tx_1 \in T(X) \subseteq g(X)$, so there exists $x_2 \in X$ such that $Tx_1 = gx_2$. Then $gx_1 = Tx_0 \preceq Tx_1 = gx_2$. Since T is $(g, \preceq)$-non-decreasing, $Tx_1 \preceq Tx_2$. Repeating this argument, there exists a sequence $\{x_n\}_{n\geq 0}$ such that $\{gx_n\}$ is $\preceq$-non-decreasing, $gx_{n+1} = Tx_n \preceq Tx_{n+1} = gx_{n+2}$ and

$$gx_{n+1} = Tx_n \text{ for all } n \geq 0. \tag{3.13}$$

Step 2.
We claim that $\{d(gx_n, gx_{n+1})\} \to 0$. Now, by contractive condition (iv) and (i_ψ), we have

$$\begin{aligned} & \psi(d(gx_{n+1}, gx_{n+2})) \\ = \ & \psi(d(Tx_n, Tx_{n+1})) \\ \leq \ & \theta(d(gx_n, gx_{n+1})) - \varphi(d(gx_n, gx_{n+1})), \end{aligned} \tag{3.14}$$

but we have $\psi(d(gx_n, gx_{n+1})) - \theta(d(gx_n, gx_{n+1})) + \varphi(d(gx_n, gx_{n+1})) > 0$. Then

$$\frac{\psi(d(gx_{n+1}, gx_{n+2}))}{\psi(d(gx_n, gx_{n+1}))} \leq \frac{\theta(d(gx_n, gx_{n+1})) - \varphi(d(gx_n, gx_{n+1}))}{\psi(d(gx_n, gx_{n+1}))} < 1.$$

Thus

$$\psi(d(gx_{n+1}, gx_{n+2})) < \psi(d(gx_n, gx_{n+1})). \tag{3.15}$$

Since ψ is non-decreasing, therefore

$$d(gx_{n+1}, gx_{n+2}) < d(gx_n, gx_{n+1}). \tag{3.16}$$

This shows that the sequence $\{\delta_n\}_{n=0}^{\infty}$ defined by $\delta_n = d(gx_n, gx_{n+1})$, is a decreasing sequence of positive numbers. Then there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(gx_n, gx_{n+1}) = \delta. \tag{3.17}$$

We shall prove that $\delta = 0$. Suppose to the contrary that $\delta > 0$. Taking $n \to \infty$ in (3.14), by using the property of ψ, θ, φ and (3.17), we obtain

$$\psi(\delta) \leq \theta(\delta) - \varphi(\delta),$$

so

$$\psi(\delta) - \theta(\delta) + \varphi(\delta) \leq 0,$$

which is a contradiction. Thus, by (3.17), we get

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{3.18}$$

Step 3.
We claim that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, and

$$d(gx_{n(k)},\ gx_{m(k)}) \geq \varepsilon \text{ for } n(k) > m(k) > k.$$

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d(gx_{n(k)-1},\ gx_{m(k)}) < \varepsilon.$$

Now, by triangle inequality, we have

$$\begin{aligned}
\varepsilon &\leq d(gx_{n(k)},\ gx_{m(k)}) \\
&\leq d(gx_{n(k)},\ gx_{n(k)-1}) + d(gx_{n(k)-1},\ gx_{m(k)}) \\
&< d(gx_{n(k)},\ gx_{n(k)-1}) + \varepsilon,
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using (3.18), we have

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)}) = \varepsilon. \tag{3.19}$$

By the triangle inequality, we have

$$\begin{aligned}
d(gx_{n(k)+1},\ gx_{m(k)+1}) &\leq d(gx_{n(k)+1},\ gx_{n(k)}) + d(gx_{n(k)},\ gx_{m(k)}) \\
&\quad + d(gx_{m(k)},\ gx_{m(k)+1}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (3.18) and (3.19), we have

$$\lim_{k\to\infty} d(gx_{n(k)+1},\ gx_{m(k)+1}) = \varepsilon. \tag{3.20}$$

As $n(k) > m(k)$, $gx_{n(k)} \succeq gx_{m(k)}$, by using contractive condition (iv), we have

$$\begin{aligned}
& \psi\left(d(gx_{n(k)+1},\ gx_{m(k)+1})\right) \\
= \ & \psi\left(d(Tx_{n(k)},\ Tx_{m(k)})\right) \\
\leq \ & \theta\left(d(gx_{n(k)},\ gx_{m(k)})\right) - \varphi\left(d(gx_{n(k)},\ gx_{m(k)})\right).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (3.19), (3.20), we have

$$\psi(\varepsilon) \leq \theta(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction due to $\varepsilon > 0$. This shows that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X.

Step 4.
We claim that T and g have a coincidence point distinguishing between cases $(a)-(e)$.

Suppose now that (a) holds, that is, $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is O-compatible. Since $(X,\ d)$ is complete, therefore there exists $z \in X$ such that $\{gx_n\} \to z$. Now $Tx_n = gx_{n+1}$ for all n, we also have that $\{Tx_n\} \to z$. As T and g are continuous, then $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Taking into account that the pair $(T,\ g)$ is O-compatible, we deduce that $\lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0$. In such a case, we conclude that $d(gz,\ Tz) = \lim_{n\to\infty} d(ggx_{n+1},\ Tgx_n) = \lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0$, that is, z is a coincidence point of T and g.
Suppose now that (b) holds, that is, $(X,\ d)$ is complete, T and g are continuous and commuting. It is obvious because (b) implies (a).
Suppose now that (c) holds, that is, $(g(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular. As $\{gx_n\}$ is a Cauchy sequence in the complete space $(g(X),\ d)$, so there exist $y \in g(X)$ such that $\{gx_n\} \to y$. Let $z \in X$ be any

point such that $y = gz$. In this case $\{gx_n\} \to gz$. Indeed, as $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to gz, we deduce that $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (iv),

$$\psi(d(gx_{n+1}, Tz)) = \psi(d(Tx_n, Tz)) \leq \theta(d(gx_n, gz)) - \varphi(d(gx_n, gz)).$$

On taking $n \to \infty$ in the above inequality, we get $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g.
Suppose now that (d) holds, that is, (X, d) is complete, $g(X)$ is closed and $(X, d, \preceq)$ is non-decreasing-regular. It follows from the fact that a closed subset of a complete metric space is also complete. Then, $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular. Thus (c) is applicable.
Suppose now that (e) holds, that is, (X, d) is complete, g is continuous, the pair (T, g) is O-compatible and $(X, d, \preceq)$ is non-decreasing-regular. As (X, d) is complete, so there exists $z \in X$ such that $\{gx_n\} \to z$. Since $Tx_n = gx_{n+1}$ for all n, we also have that $\{Tx_n\} \to z$. As g is continuous, then $\{ggx_n\} \to gz$. Furthermore, since the pair (T, g) is O-compatible, we have $\lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$. As $\{ggx_n\} \to gz$, the previous property means that $\{Tgx_n\} \to gz$.
Indeed, as $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to z, we deduce that $gx_n \preceq z$ for all $n \geq 0$. Applying the contractive condition (iv), we get

$$\psi(d(Tgx_n, Tz)) \leq \theta(d(ggx_n, gz)) - \varphi(d(ggx_n, gz)).$$

Taking $n \to \infty$ in the above inequality, we get $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g. □

If we take $\psi(t) = \theta(t)$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $T, g : X \to X$ be two mappings satisfying $(i) - (iii)$ of Theorem 3.1 and there exist an altering distance function ψ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

$$\psi(d(Tx, Ty)) \leq \psi(d(gx, gy)) - \varphi(d(gx, gy)),$$

for all $x, y \in X$ such that $gx \preceq gy$, where $\varphi(0) = 0$. Also assume that, at least, one of the conditions $(a) - (e)$ of Theorem 3.1 holds. Then T and g have, at least, a coincidence point.

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Theorem 3.1, we get the following corollary.

Corollary 3.2. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $T, g : X \to X$ be two mappings satisfying $(i) - (iii)$ of Theorem 3.1 and*

$$d(Tx, Ty) \leq kd(gx, gy),$$

for all $x, y \in X$ such that $gx \preceq gy$, where $k < 1$. Also assume that, at least, one of the conditions $(a) - (e)$ of Theorem 3.1 holds. Then T and g have, at least, a coincidence point.

Example 3.1. *Let $X = \mathbb{R}$ be a metric space with the metric $d : X \times X \to [0, +\infty)$ defined by $d(x, y) = |x - y|$, for all $x, y \in X$, with the natural ordering of real numbers $\leq$. Let $T, g : X \times X \to X$ be defined as*

$$Tx = \frac{x^2}{3} \text{ and } gx = x^2 \text{ for all } x \in X.$$

Clearly, T and g satisfied the contractive condition of Theorem 3.1 with $\psi(t) = \theta(t) = t$ and $\varphi(t) = \frac{2t}{3}$ for $t \geq 0$. In addition, all the other conditions of Theorem 3.1 are satisfied and $z = 0$ is a coincidence point of T and g.

Next we give an $n-$dimensional fixed point theorem for mixed monotone mappings. For brevity, $(y_1, y_2, ..., y_n)$, $(v_1, v_2, ..., v_n)$ and $(y_0^1, y_0^2, ..., y_0^n)$ will be denoted by Y, V and Y_0 respectively. Consider the mappings $F_\Upsilon, G : X^n \to X^n$ defined by

$$\begin{aligned} F_\Upsilon(Y) &= \begin{pmatrix} F(y_{\sigma_1(1)}, y_{\sigma_1(2)}, ..., y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)}, y_{\sigma_2(2)}, ..., y_{\sigma_2(n)}), \\ ..., F(y_{\sigma_n(1)}, y_{\sigma_n(2)}, ..., y_{\sigma_n(n)}) \end{pmatrix}, \\ \text{and } G(Y) &= (gy_1, gy_2, ..., gy_n), \text{ for } Y \in X^n. \end{aligned} \tag{3.21}$$

Under these conditions, the following properties hold:

Lemma 3.1. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Then*

(1) *If there exist $y_0^1, y_0^2, ..., y_0^n \in X$ verifying $gy_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, ..., y_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$, then there exists $Y_0 \in X^n$ such that $G(Y_0) \sqsubseteq F_\Upsilon(Y_0)$.*

(2) *If there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

$$\begin{aligned} & \psi(d(F(y_1, y_2, ..., y_n), F(v_1, v_2, ..., v_n))) \\ \leq \quad & \theta(\max_{1\leq i\leq n} d(gy_i, gv_i)) - \varphi(\max_{1\leq i\leq n} d(gy_i, gv_i)), \end{aligned} \tag{3.22}$$

for which $y_i, v_i \in X$ such that $gy_i \preceq_i gv_i$ for all $i \in \Lambda_n$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$, then

$$\begin{aligned} & \psi(\rho_n(F_\Upsilon(Y), F_\Upsilon(V))) \\ \leq \quad & \theta(\rho_n(G(Y), G(V))) - \varphi(\rho_n(G(Y), G(V))), \end{aligned} \tag{3.23}$$

for all $Y, V \in X^n$ with $G(Y) \sqsubseteq G(V)$.

Proof.

(1) is obvious.

(2) Suppose that $G(Y) \sqsubseteq G(V)$ for $Y, V \in X^n$. For fixed $i \in A$, we have $gy_{\sigma_i(t)} \preceq_t gv_{\sigma_i(t)}$ for $t \in \Lambda_n$. From (3.22), we have

$$\begin{aligned} & \psi(d(F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, ..., y_{\sigma_i(n)}), F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, ..., v_{\sigma_i(n)}))) \\ \leq \quad & \theta(\max_{1\leq i\leq n} d(gy_i, gv_i)) - \varphi(\max_{1\leq i\leq n} d(gy_i, gv_i)), \end{aligned} \tag{3.24}$$

for all $i \in A$. Similarly, for fixed $i \in B$, we have $gy_{\sigma_i(t)} \succeq_t gv_{\sigma_i(t)}$ for $t \in \Lambda_n$. It follows from (3.22) that

$$\begin{aligned} & \psi(d(F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, ..., y_{\sigma_i(n)}), F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, ..., v_{\sigma_i(n)}))) \\ = \quad & \psi(d(F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, ..., v_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, ..., y_{\sigma_i(n)}))) \\ \leq \quad & \theta\left(\max_{1\leq i\leq n} d(gy_i, gv_i)\right) - \varphi\left(\max_{1\leq i\leq n} d(gy_i, gv_i)\right), \end{aligned} \tag{3.25}$$

for all $i \in B$. By (2.2), (3.21), (3.24), (3.25) and ψ is non-decreasing, we have

$$\psi(\rho_n(F_\Upsilon(Y), F_\Upsilon(V))) \leq \theta(\rho_n(G(Y), G(V))) - \varphi(\rho_n(G(Y), G(V))),$$

for all $Y, V \in X^n$ with $G(Y) \sqsubseteq G(V)$. □

Theorem 3.2. *Let$(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings and $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Suppose that the following properties are fulfilled:*

(i) *$F(X^n) \subseteq g(X)$,*

(ii) *F has the mixed g-monotone property,*

(iii) *there exist $y_0^1, y_0^2, ..., y_0^n \in X$ verifying $gy_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, ..., y_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$,* *(iv) there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying* (3.22). *Also assume that at least one of the following conditions holds,*

(a) (X, d) is complete, F and g are continuous and the pair (F, g) is (O, Υ)-compatible,

(b) (X, d) is complete, F and g are continuous and commuting,

(c) $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular,

(d) (X, d) is complete, $g(X)$ is closed and $(X, d, \preceq)$ is regular,

(e) (X, d) is complete, g is continuous, the pair (F, g) is (O, Υ)-compatible and $(X, d, \preceq)$ is non-decreasing-regular.

Then F and g have, atleast, a $\Upsilon-$coincidence point.

Proof. It is only necessary to apply Theorem 3.1 to the mappings $T = F_\Upsilon$ and $g = G$ in the ordered metric space $(X^n, \rho_n, \sqsubseteq)$ taking into account all items of Lemma 2.1 and Lemma 3.1. □

In a similar way, we may state the results analog of Corollary 3.1 and Corollary 3.2.

References

[1] A. Alotaibi, S. M. Alsulami, Coupled coincidence points for monotone operators in partially ordered metric spaces, Fixed Point Theory Appl, 44 (2011).
https://doi.org/10.1186/1687-1812-2011-44

[2] S. M. Alsulami, Some coupled coincidence point theorems for a mixed monotone operator in a complete metric space endowed with a partial order by using altering distance functions, Fixed Point Theory Appl, 194 (2013).
https://doi.org/10.1186/1687-1812-2013-194

[3] H. Aydi, E. Karapinar, W. Shatanawi, Coupled fixed point results for $(\psi, \varphi)-$weakly contractive condition in ordered partial metric spaces, Comput. Math. Appl, 62 (12) (2011) 4449-4460.
https://doi.org/10.1016/j.camwa.2011.10.021

[4] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal, 74 (2011) 4889-4897.
https://doi.org/10.1016/j.na.2011.03.032

[5] M. Berzig, B. Samet, An extension of coupled fixed point's concept in higher dimension and applications, Comput. Math. Appl, 63 (8) (2012) 1319-1334.
https://doi.org/10.1016/j.camwa.2012.01.018

[6] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal, 65 (7) (2006) 1379-1393.
https://doi.org/10.1016/j.na.2005.10.017

[7] B. S. Choudhury, A. Kundu, A coupled coincidence point results in partially ordered metric spaces for compatible mappings, Nonlinear Anal, 73 (2010) 2524-2531.
https://doi.org/10.1016/j.na.2010.06.025

[8] B. Deshpande, A. Handa, Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations, Afr. Mat, 26 (3-4) (2015) 317-343.
https://doi.org/10.1007/s13370-013-0204-0

[9] B. Deshpande, A. Handa, Application of coupled fixed point technique in solving integral equations on modified intuitionistic fuzzy metric spaces, Adv. Fuzzy Syst, Volume 2014 (2014), Article ID 348069, 11 pages.
http://dx.doi.org/10.1155/2014/348069

[10] H. S. Ding, L. Li, S. Radenovic, Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces, Fixed Point Theory Appl, 96 (2012).
https://doi.org/10.1186/1687-1812-2012-96

[11] I. M. Erhan, E. Karapinar, A. Roldan, N. Shahzad, Remarks on coupled coincidence point results for a generalized compatible pair with applications, Fixed Point Theory Appl, 207 (2014).
https://doi.org/10.1186/1687-1812-2014-207

[12] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal, 11 (5) (1987) 623-632.
https://doi.org/10.1016/0362-546X(87)90077-0

[13] J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal, 72 (3-4) (2010) 1188-1197.
https://doi.org/10.1016/j.na.2009.08.003

[14] J. Harjani, B. Lopez, K. Sadarangani, Fixed point theorems for mixed monotone operators and applications to integral equations, Nonlinear Anal, 74 (2011) 1749-1760.
https://doi.org/10.1016/j.na.2010.10.047

[15] X. Q. Hu, Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces, Fixed Point Theory Appl, (2011), Article ID 363716.
https://doi.org/10.1155/2011/363716

[16] N. M. Hung, E. Karapinar, N. V. Luong, Coupled Coincidence Point Theorem in Partially Ordered Metric Spaces via Implicit Relation, Abstr. Appl. Anal, Volume 2012 (2012), Article ID 796964, 14 pages.
http://dx.doi.org/10.1155/2012/796964

[17] E. Karapinar, N. V. Luong, Quadruple fixed point theorems for nonlinear contractions, Comput. Math. Appl, 64 (6) (2012) 1839-1848.
https://doi.org/10.1016/j.camwa.2012.02.061

[18] E. Karapinar, A. Roldán, A note on n-Tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces, J. Inequal. Appl, (2013), Article ID 567.
https://doi.org/10.1186/1029-242X-2013-567

[19] E. Karapinar, A. Roldán, C. Roldán, J. Martínez-Moreno, A note on N-Fixed point theorems for nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl, (2013).
https://doi.org/10.1186/1687-1812-2013-310

[20] E. Karapinar, A. Roldán, J. Martínez-Moreno, C. Roldán, Meir-Keeler type multidimensional fixed point theorems in partially ordered metric spaces, Abstr. Appl. Anal, Volume 2013 (2013), Article ID 406026, 9 pages.
http://dx.doi.org/10.1155/2013/406026

[21] E. Karapinar, A. Roldán, N. Shahzad, W. Sintunavarat, Discussion on coupled and tripled coincidence point theorems for $\phi-$contractive mappings without the mixed g-monotone property, Fixed Point Theory Appl, (2014).
https://doi.org/10.1186/1687-1812-2014-92

[22] V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal, 70 (12) (2009) 4341-4349.
https://doi.org/10.1016/j.na.2008.09.020

[23] N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal, 74 (2011) 983-992.
https://doi.org/10.1016/j.na.2010.09.055

[24] N. V. Luong, N. X. Thuan, Coupled points in ordered generalized metric spaces and application to integro-differential equations, Comput. Math. Appl, 62 (11) (2011) 4238-4248.
https://doi.org/10.1016/j.camwa.2011.10.011

[25] S. A. Al-Mezel, H. Alsulami, E. Karapinar, A. Roldán, Discussion on multidimensional coincidence points via recent publications, Abstr. Appl. Anal. Volume 2014 (2014), Article ID 287492, 13 pages.
http://dx.doi.org/10.1155/2014/287492

[26] J. J. Nieto, R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 22 (2005) 223-239.
https://doi.org/10.1007/s11083-005-9018-5

[27] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc, 132 (2004) 1435-1443.
https://doi.org/10.1090/S0002-9939-03-07220-4

[28] A. Razani, V. Parvaneh, Coupled coincidence point results for (ψ, α, β)-weak contractions in partially ordered metric spaces, J. Appl. Math, (2012), Article ID 496103.
https://doi.org/10.1155/2012/496103

[29] A. Roldán, J. Martínez-Moreno, C. Roldán, Multidimensional fixed point theorems in partially ordered metric spaces, J. Math. Anal. Appl, 396 (2012) 536-545.
https://doi.org/10.1016/j.jmaa.2012.06.049

[30] A. Roldán, E. Karapinar, Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under $(\psi, \varphi)-$contractivity conditions, Fixed Point Theory Appl, 2013, Article ID 158.
https://doi.org/10.1186/1687-1812-2013-158

[31] A. Roldán, J. Martínez-Moreno, C. Roldán, E. Karapinar, Multidimensional fixed-point theorems in partially ordered complete partial metric spaces under $(\psi, \varphi)-$contractivity conditions, Abstr. Appl. Anal, 2013, Article ID 634371.
https://doi.org/10.1155/2013/634371

[32] A. Roldán, J. Martínez-Moreno, C. Roldán, E. Karapinar, Some remarks on multidimensional fixed point theorems, Fixed Point Theory, 15 (2) (2014) 545-558.

[33] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal, 72 (12) (2010) 4508-4517.
https://doi.org/10.1016/j.na.2010.02.026

[34] B. Samet, E. Karapinar, H. Aydi, V. C. Rajic, Discussion on some coupled fixed point theorems, Fixed Point Theory Appl, (2013).
https://doi.org/10.1186/1687-1812-2013-50

[35] F. Shaddad, M. S. M. Noorani, S. M Alsulami, H. Akhadkulov, Coupled point results in partially ordered metric spaces without compatibility, Fixed Point Theory Appl, (2014).
https://doi.org/10.1186/1687-1812-2014-204

[36] W. Shatanawi, B. Samet, M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, Math. Comput. Modelling, 55 (3-4) (2012) 680-687.
https://doi.org/10.1016/j.mcm.2011.08.042

[37] S. Wang, Coincidence point theorems for G-isotone mappings in partially ordered metric spaces, Fixed Point Theory Appl, (2013).
https://doi.org/10.1186/1687-1812-2013-96

[38] S. Wang, Multidimensional fixed point theorems for isotone mappings in partially ordered metric spaces, Fixed Point Theory Appl, (2014).
https://doi.org/10.1186/1687-1812-2014-137