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Tripled Fixed Point in Ordered Multiplicative Metric Spaces

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Abstract
In this paper, we present some triple fixed point theorems in partially ordered multiplicative metric spaces depended on another function. Our results generalise the results of [6] and [5].

Keywords: Multiplicative metric space, Tripled fixed point, Partially ordered set.

1 Introduction and Preliminaries

Definition 1.1. *Let X be a non empty set. A mapping $d : X \times X \to \mathbb{R}$ is said to be a metric if it satisfies the following conditions:*

(i) $d(x,y) \geq 0$ *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$.

(ii) $d(x,y) = d(y,x)$ *for all* $x,y \in X$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x,y,z \in X$ *(additive triangle inequality)*

Also, (X,d) *is called a metric space.*

F. Frechet first introduced metric space in the year 1906. Till now, we have so many spaces which are generalized from metric space. Among the generalized metric space, multiplicative metric space is one. The definition of multiplicative metric space was given by Michael Grossman and Robert Katz in 1967-1970.
In the paper of Ozavsar and Cevikel, they proved all the necessary definitions and theorems which are true in metric space to multiplicative metric space and also introduced multiplicative contractions mappings. With this multiplicative contraction mapping, the proved the famous Banach fixed point theorems on multiplicative metric space.
The definition of coupled fixed point was first given by Guo and Lakshmikantham in the year 1987. Bhaskar and Lakshmikantham proved a coupled fixed point theorem for a partial ordered having mixed monotone mapping in a metric space by using a weak contractivity type assumption with applications.
In 2011, V. Berinde and M. Borcut introduced the concept of triple fixed point for non linear mappings in partially ordered complete metric spaces and obtain existence and uniqueness theorems for contractive type mappings.
The aim of this paper is to introduced the concept of tripled fixed point in the context of partially ordered multiplicative metric space.

Definition 1.2. *[1] Let X be a non empty set. A mapping $d : X \times X \to \mathbb{R}$ is said to be a multiplicative metric if it satisfies the following conditions:*

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(i) $d(x,y) \geq 1$ *for all* $x,y \in X$ *and* $d(x,y) = 1$ *if and only if* $x = y$.

(ii) $d(x,y) = d(y,x)$ *for all* $x,y \in X$

(iii) $d(x,z) \leq d(x,y).d(y,z)$ *for all* $x,y,z \in X$ *(multiplicative triangle inequality)*

Also, (X,d) *is called a multiplicative metric space.*

Definition 1.3. *[2] Let* (X,d) *be a multiplicative metric space,* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *If for every multiplicative open ball*

$$\mathrm{B}_\varepsilon(x) = \{y \in X | d(x,y) < \varepsilon\},\ \varepsilon > 1$$

there exists a natural number N *such that* $n \geq \mathrm{N}$, *then* $x_n \in \mathrm{B}_\varepsilon(x)$. *The sequence* $\{x_n\}$ *is said to be multiplicative convergent to x, denoted by* $x_n \to x \ (n \to \infty)$

Definition 1.4. *[2] Let* (X,d) *be a multiplicative metric space and* $\{x_n\}$ *be a sequence in* X. *The sequence is called a multiplicative Cauchy sequence if it holds that for all* $\varepsilon > 1$, *there exists* $\mathrm{N} \in \mathbb{N}$ *such that* $d(x_n,x_m) < \varepsilon$ *for all* $m,n > \mathrm{N}$.

Definition 1.5. *[2] Let* (X,d) *be a multiplicative metric space. A mapping* $f : X \to X$ *is called a multiplicative contraction if there exists a real constant* $\lambda \in [0,1)$ *such that* $d(f(x), f(y)) \leq d(x,y)^\lambda$ *for all* $x,y \in X$.

Definition 1.6. *[2] We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergent to* $x \in X$.

Definition 1.7. *[4] Let* $(X,\preceq)$ *be a partially ordered set and* $S : X \times X \to X$. *The mapping S is said to has the mixed monotone property if S is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any* $x,y \in X$,

$$x_1,x_2 \in X, x_1 \preceq x_2 \Rightarrow S(x_1,y) \preceq S(x_2,y),$$

$$y_1,y_2 \in X, y_1 \preceq y_2 \Rightarrow S(x,y_1) \succeq S(x,y_2)$$

Definition 1.8. *[4] An element* $(x,y) \in X \times X$ *is called a coupled fixed point of the mapping* $S : X \times X \to X$ *if*

$$S(x,y) = x,\ S(y,x) = y.$$

Definition 1.9. *[5] Let* $(X,\preceq)$ *be a partially ordered set and* $F : X \times X \times X \to X$. *The mapping* F *is said to has the mixed monotone property if for any* $x,y,z \in X$

$$x_1,x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1,y,z) \preceq F(x_2,y,z)$$

$$y_1,y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x,y_1,z) \succeq F(x,y_2,z)$$

$$z_1,z_2 \in X, z_1 \preceq z_2 \Rightarrow F(x,y,z_1) \preceq F(x,y,z_2)$$

Definition 1.10. *[5] Let* $F : X \times X \times X \to X$. *An element* (x,y,z) *is called a triple fixed point of* F *if*

$$F(x,y,z) = x,\ F(y,x,y) = y \text{ and } F(z,y,x) = z$$

Theorem 1.1. *[5] Let* $(X,\preceq,d)$ *be a partially ordered set and suppose there is a multiplicative metric d on X such that* (X,d) *is a complete multiplicative metric space. Suppose* $F : X \times X \times X \to X$ *such that* F *has the mixed monotone property and there exists* $j,r,l \geq 0$ *with* $j+r+l < 1$ *such that*

$$d(F(x,y,z),F(u,v,w)) \leq jd(x,u) + rd(y,v) + ld(z,w) \tag{1.1}$$

for any $x,y,z \in X$ *for which* $x \preceq u,\ v \preceq y$ *and* $z \preceq w$. *Suppose either* F *is continuous, or* X *has the following property:*

1. *if non-decreasing sequence* $x_n \to x$, *then* $x_n \preceq x$ *for all n,*

2. *if non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$, i.e. F has a tripled fixed point.

Definition 1.11. *[6] Let (X, d) be a multiplicative metric space. A mapping $T : X \to X$ is said to be ICS if T is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.*

Let Φ be the set of all functions $\phi : (0, \infty) \to (0, \infty)$ such that

1. *ϕ is non-decreasing,*

2. *$\phi(t) < t$ for all $t > 0$,*

3. *$\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$*

2 Main section

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a multiplicative metric d on X such that (X, d) is a complete multiplicative metric space. Suppose $T : X \to X$ is an ICS mapping and $F : X \times X \times X \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$d(TF(x, y, z), TF(u, v, w)) \leq \phi(\max\{d(Tx, Tu), d(Ty, Tv), d(Tz, Tw)\}) \tag{2.2}$$

for any $x, y, z \in X$ for which $x \preceq u$, $v \preceq y$ and $z \preceq w$. Suppose either
(a) F is continuous, or
(b) X has the following property:

1. *if non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \preceq x$ (respectively, $z_n \preceq z$) for all n,*

2. *if non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \preceq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$, that is, F has a tripled fixed point.
Suppose that for all $(x, y, z), (u, v, r) \in X \times X \times X$, there exists $(a, b, c) \in X \times X \times X$ such that $(F(a, b, c), F(b, a, b), F(c, b, a))$ is comparable to $(F(x, y, z), F(y, x, y), F(z, y, x))$ and $(F(u, v, r), F(v, u, v), F(r, v, u))$. Then, F has a unique tripled fixed point (x, y, z).

Proof. Let $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$. Set

$$x_1 = F(x_0, y_0, z_0), y_1 = F(y_0, x_0, y_0) \text{ and } z_1 = F(z_0, y_0, x_0) \tag{2.3}$$

Continuing this process, we can construct sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, y_n) \text{ and } z_{n+1} = F(z_n, y_n, x_n) \tag{2.4}$$

Since F has the mixed monotone property, then using mathematical induction, we have

$$x_n \preceq x_{n+1}, y_{n+1} \preceq y_n, z_n \preceq z_n \text{ for } n = 0, 1, 2, ... \tag{2.5}$$

Assume for some $n \in \mathbb{N}$,

$$x_n = x_{n+1}, y_n = y_{n+1} \text{ and } z_n = z_{n+1}$$

then, by (2.4), (x_n, y_n, z_n) is a tripled fixed point of F. From now on, assume for any $n \in \mathbb{N}$, that at least

$$x_n \neq x_{n+1} \text{ or } y_n \neq y_{n+1} \text{ or } z_n \neq z_{n+1} \tag{2.6}$$

Since T is injective, then by (2.6), for any $n \in \mathbb{N}$

$$\max\{d(Tx_n,Tx_{n+1}),d(Ty_n,Ty_{n+1}),d(Tz_n,Tz_{n+1})\} > 1$$

Due to (2.2) and (2.4), we have

$$\begin{aligned} d(Tx_n,Tx_{n+1}) &= d(T(F(x_{n-1},y_{n-1},z_{n-1})),T(F(x_n,y_n,z_n))) \\ &\leq \phi(\max\{d(Tx_{n-1},Tx_n),d(Ty_{n-1},Ty_n),d(Tz_{n-1},Tz_n)\}) \end{aligned} \tag{2.7}$$

$$\begin{aligned} d(Ty_{n+1},Ty_n) &= d(T(F(y_n,x_n,y_n)),T(F(y_{n-1},x_{n-1},y_{n-1}))) \\ &\leq \phi(\max\{d(Ty_{n-1},Ty_n),d(Tx_{n-1},Tx_n),d(Ty_{n-1},Ty_n)\}) \\ &= \phi(\max\{d(Ty_{n-1},Ty_n),d(Tx_{n-1},Tx_n)\}) \\ &\leq \phi(\max\{d(Tz_{n-1},Tz_n),d(Ty_{n-1},Ty_n),d(Tx_{n-1},Tx_n)\}) \end{aligned} \tag{2.8}$$

$$\begin{aligned} d(Tz_n,Tz_{n+1}) &= d(T(F(z_{n-1},y_{n-1},x_{n-1})),T(F(z_n,y_n,x_n))) \\ &\leq \phi(\max\{d(Tz_{n-1},Tz_n),d(Ty_{n-1},Ty_n),d(Tx_{n-1},Tx_n)\}) \end{aligned} \tag{2.9}$$

Having in mind that $\phi(t) < t$ for all $t > 0$, so from (2.7)-(2.9), we obtain that

$$\begin{aligned} 1 &\leq \max\{d(Tx_n,Tx_{n+1}),d(Ty_n,Ty_{n+1}),d(Tz_n,Tz_{n+1})\} \\ &\leq \phi(\max\{d(Tz_{n-1},Tz_n),d(Ty_{n-1},Ty_n),d(Tx_{n-1},Tx_n)\}) \\ &< \max\{d(Tz_{n-1},Tz_n),d(Ty_{n-1},Ty_n),d(Tx_{n-1},Tx_n)\} \end{aligned} \tag{2.10}$$

It follows that

$$\begin{aligned} \max\{d(Tx_n,Tx_{n+1}),d(Ty_n,Ty_{n+1}),d(Tz_n,Tz_{n+1})\} < \max\{&d(Tz_{n-1},Tz_n), \\ &d(Ty_{n-1},Ty_n),d(Tx_{n-1},Tx_n)\} \end{aligned}$$

Thus, $\{\max\{d(Tx_n,Tx_{n+1}),d(Ty_n,Ty_{n+1}),d(Tz_n,Tz_{n+1})\}\}$ is a positive decreasing sequence. Hence, there exists $r \geq 1$ such that

$$\lim_{n\to+\infty}\max\{d(Tx_n,Tx_{n+1}),d(Ty_n,Ty_{n+1}),d(Tz_n,Tz_{n+1})\} = r$$

Suppose that $r > 1$. Letting $n \to +\infty$ in (2.10), we obtain that

$$\begin{aligned} 1 < r &\leq \lim_{n\to\infty}\max\{d(Tx_n,Tx_{n+1}),d(Ty_n,Ty_{n+1}),d(Tz_n,Tz_{n+1})\} \\ &= \lim_{t\to r^+}\phi(t) < r \end{aligned} \tag{2.11}$$

which is a contradiction. We deduce that

$$\lim_{n\to+\infty}\max\{d(Tx_n,Tx_{n+1}),d(Ty_n,Ty_{n+1}),d(Tz_n,Tz_{n+1})\} = 1 \tag{2.12}$$

We shall show that $\{Tx_n\}$,$\{Ty_n\}$ and $\{Tz_n\}$ are Cauchy sequences. Assume the contrary, i.e. $\{Tx_n\}$,$\{Ty_n\}$ or $\{Tz_n\}$ is not a Cauchy sequences that is

$$\lim_{n,m\to+\infty} d(Tx_m,Tx_n) \neq 1, \text{ or } \lim_{n,m\to+\infty} d(Ty_m,Ty_n) \neq 1, \text{ or } \lim_{n,m\to+\infty} d(Tz_m,Tz_n) \neq 1.$$

This means that there exists $\varepsilon > 1$ for which we can find subsequences of integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$\max\{d(Tx_{m_k},Tx_{n_k}),d(Ty_{m_k},Ty_{n_k}),d(Tz_{m_k},Tz_{n_k})\} \geq \varepsilon \tag{2.13}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.13). Then

$$\max\{d(Tx_{m_k},Tx_{n_k-1}),d(Ty_{m_k},Ty_{n_k-1}),d(Tz_{m_k},Tz_{n_k-1})\} < \varepsilon \tag{2.14}$$

By triangular inequality and (2.14), we have

$$\begin{aligned} d(Tx_{m_k},Tx_{n_k}) &\leq d(Tx_{m_k},Tx_{n_k-1}).d(Tx_{n_k-1},Tx_{n_k}) \\ &< \varepsilon.d(Tx_{n_k-1},Tx_{n_k}) \end{aligned}$$

Thus, by (2.12), we obtain

$$\lim_{k\to\infty} d(Tx_{m_k},Tx_{n_k}) \leq \lim_{k\to\infty} d(Tx_{m_k},Tx_{n_k-1}) \leq \varepsilon \tag{2.15}$$

Similarly, we have

$$\lim_{k\to\infty} d(Ty_{m_k},Ty_{n_k}) \leq \lim_{k\to\infty} d(Ty_{m_k},Ty_{n_k-1}) \leq \varepsilon \tag{2.16}$$

$$\lim_{k\to\infty} d(Tz_{m_k},Tz_{n_k}) \leq \lim_{k\to\infty} d(Tz_{m_k},Tz_{n_k-1}) \leq \varepsilon \tag{2.17}$$

Again by (2.14), we have

$$\begin{aligned} d(Tx_{m_k},Tx_{n_k}) &\leq d(Tx_{m_k},Tx_{m_k-1}).d(Tx_{m_k-1},Tx_{n_k-1}).d(Tx_{n_k-1},Tx_{n_k}) \\ &\leq d(Tx_{m_k},Tx_{m_k-1}).d(Tx_{m_k-1},Tx_{m_k}). \\ &\quad d(Tx_{m_k},Tx_{n_k-1}).d(Tx_{n_k-1},Tx_{n_k}) \\ &< d(Tx_{m_k},Tx_{m_k-1}).d(Tx_{m_k-1},Tx_{m_k}).\varepsilon.d(Tx_{n_k-1},Tx_{n_k}) \end{aligned}$$

Letting $k \to +\infty$ and using (2.12), we get

$$\lim_{k\to+\infty} d(Tx_{m_k},Tx_{n_k}) \leq \lim_{k\to+\infty} d(Tx_{m_k-1},Tx_{n_k-1}) \leq \varepsilon \tag{2.18}$$

$$\lim_{k\to+\infty} d(Ty_{m_k},Ty_{n_k}) \leq \lim_{k\to+\infty} d(Ty_{m_k-1},Ty_{n_k-1}) \leq \varepsilon \tag{2.19}$$

$$\lim_{k\to+\infty} d(Tz_{m_k},Tz_{n_k}) \leq \lim_{k\to+\infty} d(Tz_{m_k-1},Tz_{n_k-1}) \leq \varepsilon \tag{2.20}$$

Using (2.13) and (2.18)-(2.20), we have

$$\begin{aligned} \lim_{k\to+\infty}\{d(Tx_{m_k},Tx_{n_k}),d(Ty_{m_k},Ty_{n_k}),d(Tz_{m_k},Tz_{n_k})\} = \lim_{k\to+\infty}\{d(Tx_{m_k},Tx_{n_k}), \\ d(Ty_{m_k},Ty_{n_k}),d(Tz_{m_k},Tz_{n_k})\} = \varepsilon \end{aligned} \tag{2.21}$$

Now, using inequality (2.2), we obtain

$$\begin{aligned} d(Tx_{m_k},Tx_{n_k}) &= d(T(F(x_{m_k-1},y_{m_k-1},z_{m_k-1})),T(F(x_{n_k-1},y_{n_k-1},z_{n_k-1}))) \\ &\leq \phi(\max\{d(Tx_{m_k-1},Tx_{n_k-1}),d(Ty_{m_k-1},Ty_{n_k-1}), \\ &\quad d(Tz_{m_k-1},Tz_{n_k-1})\}) \end{aligned} \tag{2.22}$$

$$\begin{aligned} d(Ty_{m_k},Ty_{n_k}) &= d(T(F(y_{m_k-1},x_{m_k-1},y_{m_k-1})),T(F(y_{n_k-1},x_{n_k-1},y_{n_k-1}))) \\ &\leq \phi(\max\{d(Ty_{m_k-1},Ty_{n_k-1}),d(Tx_{m_k-1},Tx_{n_k-1}) \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} d(Tz_{m_k},Tz_{n_k}) &= d(T(F(z_{m_k-1},y_{m_k-1},x_{m_k-1})),T(F(z_{n_k-1},y_{n_k-1},x_{n_k-1}))) \\ &\leq \phi(\max\{d(Tx_{m_k-1},Tx_{n_k-1}),d(Ty_{m_k-1},Ty_{n_k-1}), \\ &\quad d(Tz_{m_k-1},Tz_{n_k-1})\}) \end{aligned} \tag{2.24}$$

We deduce from (2.22)-(2.24) that

$$\begin{aligned}\max\{d(Tx_{m_k},Tx_{n_k}),d(Ty_{m_k},Ty_{n_k}),d(Tz_{m_k},Tz_{n_k})\} \le \phi(\max\{d(Tx_{m_k},Tx_{n_k}),\\ d(Ty_{m_k},Ty_{n_k}),d(Tz_{m_k},Tz_{n_k})\}) = \varepsilon\end{aligned} \tag{2.25}$$

Letting $k \to +\infty$ in (2.25) and having in mind (2.21), we get that

$$1 < \varepsilon \le \lim_{t\to\varepsilon^+} \phi(t) < \varepsilon,$$

which is a contradiction. Thus $\{Tx_n\}$,$\{Ty_n\}$ and $\{Tz_n\}$ are Cauchy sequences in (X,d). Since X is a complete multiplicative metric space, $\{Tx_n\}$,$\{Ty_n\}$ and $\{Tz_n\}$ are convergent sequences.
Since T is an ICS mapping, there exists $x,y,z \in X$ such that

$$\lim_{n\to+\infty} x_n = x, \lim_{n\to+\infty} y_n = y, \text{ and } \lim_{n\to+\infty} z_n = z. \tag{2.26}$$

Since T is continuous, we have

$$\lim_{n\to+\infty} Tx_n = Tx, \lim_{n\to+\infty} Ty_n = Ty, \text{ and } \lim_{n\to+\infty} Tz_n = Tz. \tag{2.27}$$

Suppose now the assumption (a) holds, that is F is continuous. By (2.4), (2.26) and (2.27), we obtain

$$\begin{aligned}
x = \lim_{n\to+\infty} x_{n+1} &= \lim_{n\to+\infty} F(x_n,y_n,z_n)\\
&= F(\lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n, \lim_{n\to+\infty} z_n) = F(x,y,z)\\
y = \lim_{n\to+\infty} y_{n+1} &= \lim_{n\to+\infty} F(y_n,x_n,y_n)\\
&= F(\lim_{n\to+\infty} y_n, \lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n) = F(y,x,y)\\
z = \lim_{n\to+\infty} z_{n+1} &= \lim_{n\to+\infty} F(z_n,y_n,x_n)\\
&= F(\lim_{n\to+\infty} z_n, \lim_{n\to+\infty} y_n, \lim_{n\to+\infty} x_n) = F(z,y,x)
\end{aligned}$$

We have proved that F has a triple fixed point.
Suppose now the assumption (b) holds. Since $\{x_n\}$,$\{z_n\}$ are non-decreasing with $x_n \to x$, $z_n \to z$ and also $\{y_n\}$ is non-increasing with $y_n \to y$, then by assumption (b), we have

$$x_n \preceq x, y_n \succeq y \text{ and } z_n \preceq z,$$

for all n. Consider now

$$\begin{aligned}
d(Tx,TF(x,y,z)) &\le d(Tx,Tx_{n+1}).d(Tx_{n+1},TF(x,y,z))\\
&= d(Tx,Tx_{n+1}).d(TF(x_n,y_n,z_n),TF(x,y,z))\\
&\le d(Tx,Tx_{n+1}).\phi(\max\{d(Tx_n,Tx),d(Ty_n,Ty),d(Tz_n,Tz)\})
\end{aligned} \tag{2.28}$$

Taking $n \to \infty$ and using (2.27), the right-hand side of (2.28) tends to 1, so we get that $d(Tx,TF(x,y,z)) = 1$. Thus, $Tx = TF(x,y,z))$ and since T is injective, we get that $x = F(x,y,z))$. Analogously, we can find that $y = F(y,x,y))$ and $z = F(z,y,x))$.
Thus we proved that F has a tripled fixed point.

Uniqueness of tripled fixed point:
Let (x,y,z) and (u,v,r) are two triple fixed point of F, that is,

$$\begin{aligned}
F(x,y,z) &= x, F(y,x,y) = y, F(z,x,y) = z\\
F(u,v,r) &= u, F(v,u,v) = v, F(r,v,u) = r
\end{aligned}$$

We have to show that (x,y,z) and (u,v,r) are equal. By assumption, there exists $(a,b,c) \in X \times X \times X$ such that $F(a,b,c),F(b,a,b),F(c,b,a)$ is comparable to $F(x,y,z),F(y,x,y),F(z,y,x)$ and $F(u,v,r),F(v,u,v),F(r,v,u)$.
Define sequences $\{a_n\}$,$\{b_n\}$ and $\{c_n\}$ such that $a_0 = a, b_0 = b, c_0 = c$, and for any $n \geq 1$

$$\begin{aligned} a_n &= F(a_{n-1},b_{n-1},c_{n-1}) \\ b_n &= F(b_{n-1},a_{n-1},b_{n-1}) \\ c_n &= F(c_{n-1},b_{n-1},a_{n-1}) \end{aligned} \tag{2.29}$$

for all n. Further, set $x_0 = x, y_0 = y, z_0 = z$ and $u_0 = u, v_0 = v, r_0 = r$, and on the same way define the sequences $\{x_n\}$,$\{y_n\}$, $\{z_n\}$ and $\{u_n\}$,$\{v_n\}$, $\{r_n\}$. Then, it is easy that

$$\begin{aligned} F(x,y,z) &= x_n, F(y,x,y) = y_n, F(z,x,y) = z_n \\ F(u,v,r) &= u_n, F(v,u,v) = v_n, F(r,v,u) = r_n \end{aligned} \tag{2.30}$$

for all $n \geq 1$.
Since $(F(x,y,z),F(y,x,y),F(z,y,x)) = (x_1,y_1,z_1) = (x,y,z)$ is comparable to $(F(a,b,c),F(b,a,b),F(c,b,a)) = (a_1,b_1,c_1)$, then it is easy to show $(x,y,z) \succeq (a_1,b_1,c_1)$.
Recursively, we get that

$$(x,y,z) \succeq (a_n,b_n,c_n) \text{ for all } n \tag{2.31}$$

By (2.31) and (2.2), we have

$$\begin{aligned} d(Tx,Ta_{n+1}) &= d(T(F(x,y,z)),T(F(a_n,b_n,c_n))) \\ &\leq \phi(\max\{d(Tx,Ta_n),d(Ty,Tb_n),d(Tz,Tc_n)\}) \end{aligned} \tag{2.32}$$

$$\begin{aligned} d(Tb_{n+1},Ty) &= d(T(F(b_n,a_n,b_n)),T(F(y,x,y))) \\ &\leq \phi(\max\{d(Tb_n,Ty),d(Ta_n,Tx),d(Tc_n,Tz)\}) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} d(Tz,Tc_{n+1}) &= d(T(F(z,y,x)),T(F(c_n,b_n,a_n))) \\ &\leq \phi(\max\{d(Tz,Tc_n),d(Ty,Tb_n),d(Tx,Ta_n)\}) \end{aligned} \tag{2.34}$$

It follows from (2.32)-(2.34) that

$$\max\{d(Tz,Tc_{n+1}),d(Ty,Tb_{n+1}),d(Tx,Ta_{n+1})\} \leq \phi(\max\{d(Tz,Tc_n), d(Ty,Tb_n),d(Tx,Ta_n)\})$$

Therefore, for each $n \geq 1$,

$$\max\{d(Tz,Tc_n),d(Ty,Tb_n),d(Tx,Ta_n)\} \leq \phi^n(\max\{d(Tz,Tc_0),d(Ty,Tb_0),d(Tx,Ta_0)\}) \tag{2.35}$$

It is known that $\phi(t) < t$ and

$$\lim_{r\to t^+} \phi(r) < t \Rightarrow \lim_{n\to\infty} \phi^n(t) = 1$$

for each $t > 1$. Thus, from (2.35),

$$\lim_{n\to\infty} \max\{d(Tz,Tc_n),d(Ty,Tb_n),d(Tx,Ta_n)\} = 1$$

This yields that

$$\begin{aligned} \lim_{n\to\infty} d(Tx,Ta_n) &= 1 \\ \lim_{n\to\infty} d(Ty,Tb_n) &= 1 \\ \lim_{n\to\infty} d(Tz,Tc_n) &= 1 \end{aligned} \tag{2.36}$$

Analogously, we show that

$$\begin{array}{rcl}\lim_{n\to\infty} d(Tu,Ta_n) &=& 1\\ \lim_{n\to\infty} d(Tv,Tb_n) &=& 1\\ \lim_{n\to\infty} d(Tr,Tc_n) &=& 1\end{array} \tag{2.37}$$

Combining (2.36) and (2.37) yields that (Tx,Ty,Tz) and (Tu,Tv,Tw) are equal. Since T is injective gives us $x = u, y = v$ and $z = w$ □

Corollary 2.1. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a multiplicative metric d on X such that (X,d) is a complete multiplicative metric space. Suppose $T : X\times X$ is an ICS mapping and $F : X\times X\times X \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi\in\Phi$ such that*

$$d(TF(x,y,z),TF(u,v,w)) \le \phi(\{d(Tx,Tu),d(Ty,Tv),d(Tz,Tw)\}^{\frac{1}{3}})$$

for any $x,y,z\in X$ for which $x\preceq u$, $v\preceq y$ and $z\preceq w$. Suppose either
(a) F is continuous, or
(b) X has the following property:

1. *if non-decreasing sequence $x_n\to x$ (respectively, $z_n\to z$), then $x_n\preceq x$ (respectively, $z_n\preceq z$) for all n,*
2. *if non-increasing sequence $y_n\to y$, then $y_n\succeq y$ for all n*

If there exists $x_0,y_0,z_0\in X$ such that $x_0\preceq F(x_0,y_0,z_0)$, $y_0\succeq F(y_0,x_0,y_0)$ and $z_0\preceq F(z_0,y_0,x_0)$, then there exist $x,y,z\in X$ such that $F(x,y,z)=x$, $F(y,x,y)=y$ and $F(z,y,x)=z$, that is, F has a triple fixed point.

Proof. It suffices to remark that

$$\frac{d(Tx,Tu)+d(Ty,Tv)+d(Tz,Tw)}{3} \le \max\{d(Tx,Tu),d(Ty,Tv),d(Tz,Tw)\}$$

Then we apply Theorem 2.1 □

Corollary 2.2. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a multiplicative metric d on X such that (X,d) is a complete multiplicative metric space. Suppose $T : X\times X\times X$ is an ICS mapping and $F : X\times X\times X\to X$ is such that F has the mixed monotone property. Assume that there exists $k\in[0,1)$ such that*

$$d(TF(x,y,z),TF(u,v,w)) \le \{d(Tx,Tu),d(Ty,Tv),d(Tz,Tw)\}^{k}$$

for any $x,y,z\in X$ for which $x\preceq u$, $v\preceq y$ and $z\preceq w$. Suppose either
(a) F is continuous, or
(b) X has the following property:

1. *if non-decreasing sequence $x_n\to x$ (respectively, $z_n\to z$), then $x_n\preceq x$ (respectively, $z_n\preceq z$) for all n,*
2. *if non-increasing sequence $y_n\to y$, then $y_n\succeq y$ for all n*

If there exist $x_0,y_0,z_0\in X$ such that $x_0\preceq F(x_0,y_0,z_0)$, $y_0\succeq F(y_0,x_0,y_0)$ and $z_0\preceq F(z_0,y_0,x_0)$, then there exist $x,y,z\in X$ such that $F(x,y,z)=x$, $F(y,x,y)=y$ and $F(z,y,x)=z$, that is, F has a tripled fixed point.

Proof. Take $\phi(t)=t^k$ and apply Theorem 2.1. □

Corollary 2.3. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a multiplicative metric d on X such that (X,d) is a complete multiplicative metric space. Suppose $T : X\times X\times X$ is an ICS mapping and $F : X\times X\times X\to X$ is such that F has the mixed monotone property. Assume that there exists $k\in[0,1)$ such that*

$$d(TF(x,y,z),TF(u,v,w)) \le \{d(Tx,Tu),d(Ty,Tv),d(Tz,Tw)\}^{\frac{k}{3}} \tag{2.38}$$

for any $x,y,z\in X$ for which $x\preceq u$, $v\preceq y$ and $z\preceq w$. Suppose either
(a) F is continuous, or
(b) X has the following property:

1. *if non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \preceq x$ (respectively, $z_n \preceq z$) for all n,*

2. *if non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$, that is, F has a tripled fixed point.

Proof. Take $\phi(t) = t^k$ and apply Corollary 2.1. □

Example 2.1. *Let $X = [\frac{1}{2}, 64]$ with the multiplicative metric $d(x,y) = a^{|x-y|}, a > 1$ for all $x, y \in X$ and the usual ordering $\preceq$. Clearly, (X, d) is a complete multiplicative metric space.*
Let $T : X \to X$ and $F : X \times X \times X \to X$ be defined by

$$Tx = ln(x) + 1 \text{ and } F(x,y,z) = 8(\frac{\sqrt{xz}}{y})^{\frac{1}{6}}, \forall x, y, z \in X$$

Here, condition for T should be ICS mapping, F has the mixed monotone property and continuous holds true.
Set $k = \frac{1}{2}$. Taking $x, y, z, u, v, w \in X$ for which $x \preceq u, y \succeq v$ and $z \preceq w$, we have

$$\begin{aligned} d(TF(x,y,z), TF(u,v,w)) &= a^{|\{ln8 + \frac{1}{12}(lnx+lnz) - \frac{1}{6}lny + 1\} - \{ln8 + \frac{1}{12}(lnu+lnw) - \frac{1}{6}lnv + 1\}|} \\ &= a^{|\frac{1}{12}(lnx+lnz-2lny) - \frac{1}{12}(lnu+lnw-2lnv)|} \\ &\leq a^{\frac{1}{12}|lnx-lnu| + \frac{1}{6}|lny-lnv| + \frac{1}{12}|lnz-lnw|} \\ &\leq a^{\frac{1}{6}\{|lnx-lnu| + |lny-lnv| + |lnz-lnw|\}} \\ &= a^{\frac{1}{6}|(lnx-lnu)|}.a^{\frac{1}{6}|(lny-lnv)|}.a^{\frac{1}{6}|(lnz-lnw)|} \\ &= a^{\frac{k}{3}|(lnx-lnu)|}.a^{\frac{k}{3}|(lny-lnv)|}.a^{\frac{k}{3}|(lnz-lnw)|} \\ &= \{d(Tx,Tu).d(Ty,Tv).d(Tz,Tw)\}^{\frac{k}{3}} \end{aligned}$$

which is a contractive condition of (2.38). Moreover, taking $x_0 = 1 = z_0$ and $y_0 = 64$, we have

$$x_0 \preceq F(x_0, y_0, z_0), y_0 \succeq F(y_0, x_0, y_0) \text{ and } z_0 \preceq F(z_0, y_0, x_0)$$

Therefore, all conditions of Corollary 2.3. hold and (8,8,8) is the unique triple fixed point of F, since also the hypothesis of theorem 2.1 hold.

3 Conclusion

In this work we present some triple fixed point theorems results in partially ordered multiplicative metric spaces depended on another function which generalise results of Hassen Aydi et. al. and Berinde et. al.

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