Two general fixed point principles and applications

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Abstract
We present a couple of general fixed point principles using the constructive approach and derive some interesting well-known fixed point theorems in a metric space and a partially ordered metric space as corollaries. Our general fixed point principles include more than 100 fixed point theorems in different metric spaces as special cases.

Keywords: Metric space, Contraction mapping, Monotone mapping, Hybrid fixed point theorem.

1 Introduction
Nonlinear analysis is a multi-stored building and Schauder, Banach and Tarski have provided the foundation stones called the three basic pillars for this monument. The very purpose of this subject is to solve or analyze the nonlinear equations that we encounter in several processes of the real world problems. If we need some extra information about the behavior, then the above three basic principles are not enough to describe the nature of underlined phenomena. Therefore, an attempt is made to develop a fourth pillar for nonlinear analysis by mixing the arguments from above three principles and the present author gathered such results under the title hybrid fixed point theory and is the forth pillar of nonlinear analysis. There are mainly two basic approaches to the problems of nonlinear analysis, namely, (1) theoretical and (2) constructive. The theoretical approach is followed by Schauder [10] and his successors on the lines of measure of noncompactness as given in Appell [1] whereas the constructive approach is followed by Banach [2] and his successors on the lines of Kannan [8]. We prefer the constructive approach, because it has some advantages over the theoretical one in that the constructive approach provides a way to find the approximate or numerical solutions of the nonlinear problems. In this note we state a couple of general fixed point principles of constructive nonlinear analysis which include more than 100 of fixed point theorems as special cases.

Given a non-empty set \( X \), let \( \mathcal{T} : X \to X \) be a mapping and consider the mapping equation \( \mathcal{T}x = x \). It is well-known that a solution to this mapping equation is called a fixed point and any mathematical statement that guarantees the existence of such a fixed point is called a fixed point principle or theorem. By a sequence of successive iterations of \( \mathcal{T} \) at a point \( x \in X \) we mean a sequence \( \{x_n\} \) of points in \( X \) defined by \( x_0 = x, x_{n+1} = \mathcal{T}x_n, n = 0, 1, ... \). We often write this sequence in the form \( \{\mathcal{T}^n x\} \) and call the sequence of Picard iterations when \( X \) is a metric space. We may consider other types of iterations, like Krasnoselski or Mann or Ishikawa iterations in a linear or more specifically in a Banach space \( X \). We restrict our attention to Picard’s iterations only, because they are better to handle and more useful from the point of view of applications than any other iterations known to us. However, the subsequent discussion is also true for other types of iteration in an appropriate abstract space. Another type of iterations that we consider here are the Dhage iterations \( \{\mathcal{T}^n x_0\} \) in a partially ordered sets \( (X, \leq) \) which begins with a lower or an

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upper solution $x_0$ of the mapping equation $\mathcal{T}x = x$. An element $x_0 \in X$ is called a lower or an upper solution of the equation $\mathcal{T}x = x$ if the inequality $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$ holds. In the following section we state the first general fixed point principle in a metric space $X$ and derive some interesting consequences of it.

2 First general fixed point principle

We need the following definitions in what follows. Given a metric space $(X, d)$, by an orbit of the mapping $\mathcal{T}$ at a point $x \in X$ we mean a set $\mathcal{O}(x, \mathcal{T})$ of iterations of $\mathcal{T}$ at $x$ defined as

$$\mathcal{O}(x, \mathcal{T}) = \{x, \mathcal{T}x, \mathcal{T}^2 x, \cdots \} = \{\mathcal{T}^n x\}_{n=0}^{\infty}.$$ (2.1)

The mapping $\mathcal{T}$ is called $\mathcal{T}$-orbitally continuous at $x \in X$ if $\mathcal{T}^n x \to x^*$ implies that $\mathcal{T}(\mathcal{T}^n x) \to \mathcal{T} x^*$. $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous on $X$ if $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous at every point $x \in X$. Similarly $X$ is called $\mathcal{T}$-orbitally complete at $x \in X$ if the Cauchy sequence of the form $\mathcal{T}^n x$ converges to a point in $X$. Again, $X$ is called $\mathcal{T}$-orbitally complete if every Cauchy sequence $(\mathcal{T}^n x)$, $x \in X$ converges to a point in $X$.

**Theorem 2.1.** Let $(X, d)$ be a metric space and let $\mathcal{T} : X \to X$ be a mapping satisfying the following conditions:

(a) There exists an element $x \in X$ such that Picard iterations $\{\mathcal{T}^n x\}$ converges to $x^*$, and

(b) $\mathcal{T}$ is continuous at $x^*$.

Then $x^*$ a fixed point of $\mathcal{T}$.

**Proof.** The proof is obvious. \qed

Now we find the equivalent or sufficient conditions that guarantee the hypotheses (a) and (b) of Theorem 2.1. Below we list all such results according to their degree of generality.

**Corollary 2.1.** Let $(X, d)$ be a metric space and let $\mathcal{T} : X \to X$ be a mapping satisfying the following conditions:

(a) There exists an element $x \in X$ such that Picard iterations $\{\mathcal{T}^n x\}$ converges to $x^*$, and

(b) $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous at $x$.

Then $x^*$ a fixed point of $\mathcal{T}$.

**Corollary 2.2.** Let $(X, d)$ be a metric space and let $\mathcal{T} : X \to X$ be a mapping satisfying the following conditions:

(a) There exists an element $x \in X$ such that $\{\mathcal{T}^n x\}$ is Cauchy sequence and $X$ is $\mathcal{T}$-orbitally complete, and

(b) $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous at $x$.

Then $\mathcal{T}$ has a fixed point $x^*$ and the sequence of Picard iterations $\{\mathcal{T}^n x\}$ converges to $x^*$.

**Corollary 2.3.** Let $(X, d)$ be a metric space and let $\mathcal{T} : X \to X$ be a mapping satisfying the following conditions:

(a) $\{\mathcal{T}^n x\}$ is Cauchy sequence for each $x \in X$ and $X$ is complete, and

(b) $\mathcal{T}$ is continuous on $\{\mathcal{T}^n x\}'$,

where $\{\mathcal{T}^n x\}'$ is the set of all limit points of the set $\{\mathcal{T}^n x\}$ in $X$. Then $\mathcal{T}$ has a fixed point $x^*$ and the sequence of Picard iterations $\{\mathcal{T}^n x\}$ converges to $x^*$ for each $x \in X$.

The following interesting geometric sufficient condition called contraction guaranteeing the validity of hypotheses (a) and (b) of Corollary 2.3 is given in Banach [2] which has numerous applications to nonlinear equations for proving the existence and uniqueness of the solutions.
Corollary 2.4. Let \((X, d)\) be a complete metric space and let \(T : X \rightarrow X\) be a mapping satisfying
\[
d(Tx, Ty) \leq \lambda d(x, y)
\]
for all \(x, y \in X\), where \(0 < \lambda < 1\). Then \(T\) has a unique fixed point \(x^*\) and the sequence of Picard iterations \(\{T^n x\}\)
converges to \(x^*\) for each \(x \in X\).

There are more than 50 generalizations and extensions of Corollary 2.3 on the lines of Kannan [8] and so Theorem 2.1 includes more than 50 fixed point theorems in a complete metric space as special cases. Furthermore, all the statements made above are also true if we replace a metric space with a \(T\)-metric or a 2-metric or a bi-metric or a convex-metric or any other metric space with appropriate modifications. See Dhage [3, 5] and the references therein.

3 Second general fixed point principle

In the following section we state the second general fixed point principle in a partially ordered metric space and derive some interesting consequences of it. Unless otherwise mentioned, throughout this paper that follows, let \((X, \leq, d)\) denote a partially ordered metric space with an order relation \(\leq\) and the metric \(d\). Two elements \(x\) and \(y\) in \(E\) are said to be comparable if either the relation \(x \leq y\) or \(y \leq x\) holds. A non-empty subset \(C\) of \(E\) is called a chain or totally ordered if all the elements of \(C\) are comparable. It is known that \(X\) is regular if \(\{x_n\}_{n \in \mathbb{N}}\) is a nondecreasing (resp. nonincreasing) sequence in \(X\) such that \(x_n \rightarrow x^*\) as \(n \rightarrow \infty\), then \(x_n \leq x^*\) (resp. \(x_n \geq x^*\)) for all \(n \in \mathbb{N}\). Clearly, the partially ordered Banach space \(C(I, \mathbb{R})\) is regular and the conditions guaranteeing the regularity of any partially ordered metric space \(X\) may be found in Heikkilä and Lakshmikantham [7] and the references therein.

We need the following preliminary definitions given in Dhage [3, 4] in what follows.

A mapping \(T : X \rightarrow X\) is called isotone or nondecreasing if it preserves the order relation \(\leq\), that is, if \(x \leq y\) implies \(Tx \leq Ty\) for all \(x, y \in X\). A mapping \(T : X \rightarrow X\) is called partially continuous at a point \(a \in X\) if for \(\varepsilon > 0\) there exists a \(\delta > 0\) such that \(d(Tx, Ta) < \varepsilon\) whenever \(x\) is comparable to \(a\) and \(d(x, a) < \delta\). \(T\) is called partially continuous on \(X\) if it is partially continuous at every point of it. It is clear that if \(T\) is partially continuous on \(X\), then it is continuous on every chain \(C\) contained in \(X\). A non-empty subset \(S\) of the partially ordered Banach space \(X\) is called partially bounded if every chain \(C\) in \(S\) is bounded. An operator \(T : X \rightarrow X\) is called partially bounded if every chain \(C\) in \(T(X)\) is bounded by a unique constant. A non-empty subset \(S\) of the partially ordered metric space \(X\) is called partially compact if every chain \(C\) in \(S\) is a relatively compact subset of \(X\). An operator \(T : X \rightarrow X\) is called partially compact if every chain or totally ordered set \(C\) in \(T(X)\) is a relatively compact subset of \(X\). \(T\) is called uniformly partially compact if \(T(X)\) is a uniformly partially bounded and partially compact on \(X\). \(T\) is called partially totally bounded if for any bounded subset \(S\) of \(X\), \(T(S)\) is a partially relatively compact subset of \(X\). If \(T\) is partially continuous and partially totally bounded, then it is called partially completely continuous on \(X\). Note that if \(T\) is nondecreasing, then it is partially bounded or partially compact provided \(T(C)\) is bounded or relatively compact for each chain \(C\) in \(X\).

Definition 3.1 (Dhage [4]). The order relation \(\leq\) and the metric \(d\) on a non-empty set \(X\) are said to be compatible if \(\{x_n\}\) is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in \(X\) and if a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) converges to \(x^*\) implies that the original sequence \(\{x_n\}\) converges to \(x^*\). A subset \(S\) of \(X\) is called Janhavi if the order relation \(\leq\) and the metric \(d\) are compatible in it. In particular, if \(S = X\), then \(X\) is called a Janhavi metric space.

Definition 3.2 (Dhage and Dhage [6]). A partially ordered metric space \((X, \leq, d)\) is said to satisfy Condition D if every sequence \(\{x_n\}\) in \(X\) whose consecutive terms are comparable has a monotone, i.e., nondecreasing or nonincreasing subsequence.

There do exist sequences in \(X\) with Condition D. For example, if we consider \(X = \mathbb{R}\), then the sequence \(\{x_n\}\) in \(\mathbb{R}\) defined by \(x_n = \frac{(-1)^{n+1}}{n}\) has two subsequences, one is nondecreasing another is nonincreasing. Again, the sequence \(\{1, \frac{1}{2}, 3, \frac{1}{4}, \ldots\}\) in \(\mathbb{R}\) satisfies the property that mentioned in Condition D. Our main general hybrid fixed point theorem is as follows.
Theorem 3.1. Let \((X, \preceq, d)\) be a regular partially ordered metric space satisfying Condition D and let \(T : X \to X\) be a mapping which maps comparable elements into comparable elements. Suppose that there exists an element \(x \in X\) comparable to \(T x\) such that the sequence \(\{T^n x\}\) of iterates converges to \(x'\). If \(T\) is partially continuous at \(x'\), then \(x'\) is a fixed point of \(T\) and there is a subsequence \(\{T^k x\}\) of \(\{T^n x\}\) which converges monotonically to \(x'\).

Proof. Let \(x \in X\) be an element which is comparable to \(T x\), that is, \(x \preceq T x\) or \(x \succeq T x\). Let \(x_0 = x\) and define a sequence \(\{x_n\}\) by
\[
x_{n+1} = T x_n, \quad n = 1, 2, \ldots.
\]
(3.3)

By hypothesis, \(\{x_n\}\) is a sequence in \(X\) whose every consecutive terms are comparable. Since \(X\) satisfies Condition D, the sequence \(\{x_n\}\) has a monotone subsequence, say \(\{x_{n_k}\}\). As \(x_n \to x'\), every subsequence and consequently \(\{x_{n_k}\}\) converges to \(x'\). By regularity of \(X\), one has \(x_{n_k} \preceq x'\) for all \(k \in \mathbb{N}\). Finally, by partial continuity of \(T\), we obtain
\[
T x' = T \left( \lim_{k \to \infty} x_{n_k} \right) = \lim_{k \to \infty} x_{n_k+1} = x'.
\]

Hence \(\{T^n x\}\) converges monotonically to the fixed point of \(T\). This completes the proof.

Remark 3.1. The regularity of the partially ordered metric space \(X\) in above Theorem 3.1 may be replaced with a stronger continuity condition of the mapping \(T\) on \(X\).

In view of above remark we obtain the following version of Theorem 3.1.

Theorem 3.2. Let \((X, \preceq, d)\) be a partially ordered metric space satisfying Condition D and let \(T : X \to X\) be a continuous mapping which maps comparable elements into comparable elements. Suppose that there exists an element \(x \in X\) comparable to \(T x\) such that the sequence \(\{T^n x\}\) of iterates converges to \(x'\). Then \(x'\) is a fixed point of \(T\) and there is a subsequence \(\{T^k x\}\) of \(\{T^n x\}\) which converges monotonically to \(x'\).

The above two hybrid fixed point principles include more than a 50 fixed point theorems in the literature. The following are the root fixed point theorems from two streams of analysis and topology which are very important from the point of view of applications to nonlinear equations for proving the different aspects of the solutions.

Corollary 3.1 (Ran and Reurings [9]). Let \((X, \preceq)\) be a partially ordered set and suppose that there is a metric \(d\) in \(X\) such that \((X, d)\) is a complete metric space. Let \(T : X \to X\) be a monotone nondecreasing mapping such that there exists a constant \(k \in (0, 1)\) such that
\[
d(T x, T y) \leq kd(x, y)
\]
(3.4)
for all elements \(x, y \in X\), \(x \geq y\). Assume that either \(T\) is continuous or \(X\) is such that if \(\{x_n\}\) is a nondecreasing sequence with \(x_n \to x\) in \(X\), then \(x_n \preceq x\) for all \(n \in \mathbb{N}\). Further if there is an element \(x_0 \in X\) satisfying \(x_0 \preceq T x_0\) or \(x_0 \succeq T x_0\), then \(T\) has a fixed point \(x^*\) and the Dhage iterations \(\{T^n x_0\}\) converges to \(x^*\). Moreover, \(x^*\) is unique provided “every pair of elements in \(X\) has a lower and an upper bound.”.

Corollary 3.2 (Dhage [3, 4]). Let \((X, \preceq, d)\) be a regular partially ordered complete metric space such that the every compact chain \(C\) in \(X\) is Janhavi. Suppose that \(T : X \to X\) is a partially continuous, nondecreasing and partially compact mapping. If there exists an element \(x_0 \in X\) such that \(x_0 \preceq T x_0\) or \(x_0 \succeq T x_0\), then \(T\) has a fixed point \(x^*\) and the sequence \(\{T^n x_0\}\) of successive iterations converges monotonically to \(x^*\).

Remark 3.2. Finally we remark that if \(X\) is a partially ordered linear or more specifically a Banach space and \(T : X \to X\), then all the results of this section are also true for Dhage iterations of Krasnoselskii type or Dhage iterations of Mann type etc., that is, Krasnoselskii or Mann iterations which again begin with a lower or an upper solutions \(x_0\) of the operator equation \(T x = x\).
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