Some Results of Fixed Points in Generalized Metric Space by Methods of Suzuki and Samet

Hojjat Afshari¹, Hajar Sharghi²

(1) Faculty of Basic Science, University of Bonab, Bonab, Iran
(2) Department of Mathematics, Payame Noor University, Tehran, Iran

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Abstract
In 1992 Dhage introduced the notion of generalized metric or D-metric spaces and claimed that D-metric convergence define a Hausdorff topology and that D-metric is sequentially continuous in all the three variables. Many authors have taken these claims for granted and used them in proving fixed point theorems in D-metric spaces. In 1996 Rhoades generalized Dhages contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self map in D-metric space. Recently motivated by the concept of compatibility for metric space. In 2002 Sing and Sharma introduced the concept of D-compatibility of maps in D-metric space and proved some fixed point theorems using a contractive condition. In this paper, we prove some fixed point theorems and common fixed point theorems in D-complete metric spaces under particular conditions among weak compatibility. Also by Using method of Suzuki and Samet we prove some theorems in generalised metric spaces.

Keywords: Generalized D-metric space, fixed point, weak compatible.
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1 Introduction
Let X be a nonempty set. A generalized D'-metric on X is a function, D’ : X³ → ℜ⁺ that satisfies the following conditions for all x, y, z, a ∈ X,
(D1) D’(x, y, z) ≥ 0,
(D2) D’(x, y, z) = 0 if and only if x = y = z,
(D3) D’(x, y, z) = D’(p{x, y, z}), (symmetry) where p is a permutation function,
(D4) D’(x, y, z) ≤ D’(x, a, a) + D’(a, y, z),
the function D’ is called a generalized D'-metric and the pair (X, D') is called a generalized D'-metric space. Note that every D'-metric on X induces a metric dD on X defined by

\[ d_{D'}(x, y) = D'(x, y, y) + D'(y, x, x), \quad \forall x, y \in X. \]  (1.1)

Remark 1.1. In a D'-metric space, we prove that D'(x, x, y) = D'(x, y, y)
(i) D’(x, x, y) ≤ D’(x, x, x) + D’(x, y, y) = D’(x, y, y),

*Corresponding author. Email address: hojat.afshari@yahoo.com; hojat.afshari@bonabu.ac.ir
\[(ii)\ D^*(y,y,x) \leq D^*(y,y,y) + D^*(y,x,x) = D^*(y,x,x),\]

Hence by (i), (ii) we get \(D^*(x,x,y) = D^*(x,y,x)\).

**Definition 1.1.** [10] Let \((X, D^*)\) be a \(D^*\)-metric space, and let \(\{x_n\}\) be a sequence of points of \(X\). We say that \(\{x_n\}\) is \(D^*\)-convergent to \(x \in X\) if

\[
\lim_{n,m \to \infty} D^*(x_n, x_m) = 0.
\]

That is, for any \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(D^*(x_n, x_m) < \epsilon\) for all \(n, m \geq N\).

**Proposition 1.1.** [10] Let \((X, D^*)\) be a \(D^*\)-metric space. The following are equivalent

(i) \(\{x_n\}\) is \(D^*\)-convergent to \(x\),

(ii) \(D^*(x_n, x_n, x) \to 0\) as \(n \to \infty\),

(iii) \(D^*(x_n, x, x) \to 0\) as \(n \to \infty\),

(iv) \(D^*(x_n, x_n, x) \to 0\) as \(n, m \to \infty\).

**Definition 1.2.** [10] Let \((X, D^*)\) be a \(D^*\)-metric space. A sequence \(\{x_n\}\) is called a \(D^*\)-Cauchy sequence if for any \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(D^*(x_n, x_m, x_l) < \epsilon\) for all \(m, n, l \geq N\), that is, \(D^*(x_n, x_m, x_l) \to 0\) as \(n, m, l \to \infty\).

**Proposition 1.2.** [10] Let \((X, D^*)\) be a \(D^*\)-metric space. Then the following are equivalent

(1) the sequence \(\{x_n\}\) is \(D^*\)-Cauchy,

(2) for any \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(D^*(x_n, x_m, x_l) < \epsilon\), for all \(m, n \geq N\).

**Definition 1.3.** [10] A \(D^*\)-metric space \((X, D^*)\) is called \(D^*\)-complete if every \(D^*\)-Cauchy sequence is \(D^*\)-convergent in \((X, D^*)\).

Note that in \(D^*\)-metric space a nonempty set \(A \subset X\) is \(D^*\)-closed in the \(D^*\)-metric space \((X, D^*)\) if \(A = \overline{A}\).

**Proposition 1.3.** Let \((X, D^*)\) be a \(D^*\)-metric space and \(A\) be a nonempty subset of \(X\). \(A\) is \(D^*\)-closed if for any \(D^*\)-convergent sequence \(\{x_n\}\) in \(A\) with limit \(x\), one has \(x \in A\).

**Definition 1.4.** [1] A \(D^*\)-metric space \(X\) is said to be compact if every \(\tau\)-open cover of \(X\) has a finite subcover.

**Theorem 1.1.** [1] In a \(D^*\)-metric space \(X\), the following statement are equivalent.

(a) \(X\) is compact,

(b) \(X\) is countably compact,

(c) \(X\) has Bolzano-Weierstrass property,

(d) \(X\) is sequentially compact.

**Theorem 1.2.** [1] In a \(D^*\)-metric space \(X\),

(a) a compact subset of a \(D^*\)-metric space is closed and bounded,

(b) a \(D^*\)-metric space \(X\) is a compact if and only if it is complete and totally bounded,

(c) a subset \(S\) of a complete \(D^*\)-metric space is compact if and only if it is closed and totally bounded.

**Theorem 1.3.** [1] Every real-valued continuous function on a compact \(D^*\)-metric space \(X\) is bounded and attains its supremum and infimum on \(X\).
2 Main results

Definition 2.1. Let $T : X \to X$ and $\alpha : X \times X \times X \to [0, +\infty)$. We say that $T$ is $\alpha$-admissible mapping if

$$x, y \in X, \quad \alpha(x, y, z) \geq 1 \implies \alpha(Tx, Ty, Tz) \geq 1.$$ 

Denote with $\Psi$ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) < t$.

Lemma 2.1. For every function $\psi : [0, +\infty) \to [0, +\infty)$ the following holds:

If $\psi$ is nondecreasing then for each $t > 0$, $\lim_{n \to +\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.

Theorem 2.1. Let $(X, D^*)$ be a complete $D^*$-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty $D^*$-closed subsets of $X$. Let $Y = \bigcup_{j=1}^m A_j$ and $T : Y \to Y$ be a $\alpha$–admissible mapping satisfying

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \ldots, m, \quad \alpha_{m+1} = A_1.$$ 

If there exist two functions $\alpha : Y \times Y \times Y \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y, Tz)D^*(Tx, Ty, Tz) \leq \psi(D^*(x, y, z))$$

holds for all $x \in A_j$ and $y, z \in A_{j+1}$, $j = 1, \ldots, m$, and there exist $x_0 \in Y$ such that $\alpha(x_0, T_{x_0}, T^2x_0) \geq 1$, then $T$ has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Proof. Let $x_0 \in Y$ such that $\alpha(x_0, T_{x_0}, T^2x_0) \geq 1$ and without loss of generality assume that $x_0 \in A_1$. Define the sequence $\{x_n\}$ in $Y$ as follows

$$x_n = Tx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$ 

Since $T$ is cyclic, $x_0 \in A_1$, $x_1 = T(x_0) \in A_2$, and so on. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then obviously, the fixed point of $T$ is $x_{n_0}$. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $T$ is $\alpha$–admissible, we have

$$\alpha(x_0, x_1, x_2) = \alpha(x_0, Tx_0, T^2x_0) \geq 1 \implies \alpha(Tx_0, Tx_1, Tx_2) = \alpha(x_1, x_2, x_3) \geq 1.$$ 

By induction, we get

$$\alpha(x_{n-1}, x_n, x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}. \quad (2.3)$$ 

Applying the inequality (2.2) with $x = x_{n-1}$ and $y = z = x_n$, and using (2.3), we obtain

$$0 \leq D^*(x_n, x_{n+1}, x_{n+1}) = D^*(Tx_{n-1}, Tx_n, Tx_n) \leq \alpha(x_{n-1}, x_n, Tx_n)D^*(Tx_{n-1}, Tx_n, Tx_n) \leq \psi(D^*(x_{n-1}, x_n, x_n)).$$

Therefore, by repetition of the above inequality, we have that

$$D^*(x_n, x_{n+1}, x_{n+1}) \leq \psi^n(D^*(x_0, x_1, x_1)), \quad \text{for all } n \in \mathbb{N}. \quad (2.4)$$ 

Fix $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \geq n(\varepsilon)} \psi^n(D^*(x_0, x_1, x_1)) < \varepsilon$. On the other hand, by symmetry (D3) and the rectangle inequality (D4), we have

$$D^*(x, y, y) = D^*(y, y, x) \leq D^*(y, x, x) + D^*(x, y, y) = 2D^*(y, x, x)$$

(2.5)

The inequality (2.5) with $x = x_n$ and $y = x_{n-1}$ becomes

$$D^*(x_n, x_{n-1}, x_{n-1}) \leq 2D^*(x_{n-1}, x_n, x_n) \quad \text{for all } n \in \mathbb{N}. \quad (2.6)$$

Letting $n \to \infty$ in (2.6), we get

$$\lim_{n \to \infty} D^*(x_n, x_{n-1}, x_{n-1}) = 0.$$
We show next that the sequence \( \{x_n\} \) is a Cauchy sequence in the metric space \((X, d_{DP'})\), where \(d_{DP'}\) is given in (1.1). Let \(n, l \in \mathbb{N} \) with \(n > l > n(\varepsilon) \) we obtain
\[
d_{DP'}(x_n, x_l) \leq d_{DP'}(x_n, x_{n-1}) + d_{DP'}(x_{n-1}, x_{n-2}) + \cdots + d_{DP'}(x_{l+1}, x_l)
\]
\[
= D^*(x_n, x_{n-1}, x_{n-1}) + D^*(x_{n-1}, x_{n-2}) + \cdots + D^*(x_{l+1}, x_{l+1})
\]
\[
= \sum_{i=l+1}^{n} [D^*(x_i, x_{i-1}, x_{i-1}) + D^*(x_{i-1}, x_{i})],
\]
(2.7)

By using (2.4) and (2.6) we obtain
\[
0 \leq d_{DP'}(x_n, x_l) \leq \sum_{i=l+1}^{n} [2D^*(x_{i-1}, x_i, x_i) + D^*(x_{i-1}, x_i, x_i)]
\]
\[
 \leq \sum_{i=l+1}^{n} 3\psi^{-1}(D^*(x_0, x_1, x_1))
\]
\[
 \leq \sum_{l>n(\varepsilon)} 3\psi(D^*(x_0, x_1, x_1)) < \varepsilon.
\]

Thus we proved that \( \{x_n\} \) is a Cauchy sequence in the \((X, d_{DP'})\). Since the space \((X, D^*)\) is \(D^*\)-complete then \((X, d_{DP'})\) is complete (see Proposition 10 in [10]) and hence, \( \{x_n\} \) converges to a number say, \(u \in X\). Moreover, \( \{x_n\} \) is \(D^*\)-Cauchy in \((X, D^*)\) (see Proposition 9 in [10]). Now we show that \(u \in \bigcap_{j=1}^{m} A_j\). If \(x_0 \in A_1\), then the subsequence \(\{x_{m(n-1)+1}\}_{n=1}^{\infty} \in A_1\), the subsequence \(\{x_{m(n-1)+1}\}_{n=1}^{\infty} \in A_2\), and continuing in this way, the subsequence \(\{x_{m(n-1)+1}\}_{n=1}^{\infty} \in A_m\). All the \(m\) subsequences are \(D^*\)-convergent and hence, they all converge to the same limit \(u\). In addition, the sets \(A_j\) are \(D^*\)-closed, thus the limit \(u \in \bigcap_{j=1}^{m} A_j\). We show that \(u \in X\) is a fixed point of \(T\). Consider now (1.1 ) and (2.2 ) with \(x = x_n, y = z = Tu\) and suppose that \(u \neq Tu\) or \(d_{DP'}(u, Tu) > 0\), then we have,
\[
0 \leq d_{DP'}(x_n, Tu) = D^*(x_n, Tu, Tu) + D^*(Tu, x_n, x_n)
\]
\[
= D^*(Tx_{n-1}, Tu, Tu) + D^*(Tu, Tx_{n-1}, Tx_{n-1})
\]
\[
\leq D^*(Tx_{n-1}, Tu, Tu) + 2D^*(Tx_{n-1}, Tu, Tu)
\]
\[
\leq \alpha(x_{n-1}, u, Tu)3D^*(Tx_{n-1}, Tu, Tu)
\]
\[
\leq 3\psi(D^*(x_{n-1}, u, u)).
\]
(2.8)

Passing to Limit as \(n \to \infty\), we end up with \(0 \leq d_{DP'}(u, Tu) \leq 0\) which contradicts the assumption \(d_{DP'}(u, Tu) > 0\). Hence \(u = Tu\). therefore \(u \in X\) is a fixed point of \(T\). To prove the uniqueness, We assume that \(v \in X\) is another fixed point of \(T\) such that \(v \neq u\). Both \(u\) and \(v\) lie in \(\bigcap_{j=1}^{m} A_j\), thus we can substitute \(x = u\) and \(y = v\) in (2.2). This yields
\[
D^*(Tu, Tv, Tv) \leq \alpha(u, v, v)D^*(Tu, Tv, Tv) \leq \psi(D^*(u, v, v)).
\]

From lemma (2.1) and \(v = Tv\) we have \(D^*(u, v, v) \leq \alpha(u, v, v)D^*(Tu, Tv, Tv) < D^*(u, v, v)\) which is contradiction, Thus \(u = v\), and the fixed point of \(T\) is unique.

\(\square\)

**Example 2.1.** Let \(X = [-1, 1]\) and let \(T : X \to X\) be given as \(Tx = -\frac{x}{2}\). Let \(A = [-1, 0]\) and \(B = [0, 1]\). Define the function \(G : X \times X \times X \to [0, \infty)\) as
\[
G(x, y, z) = |x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|
\]
Clearly the function \(G\) is a \(D^*\)-metric on \(X\). Define also \(\psi : [0, \infty) \to [0, \infty)\) and \(\alpha : Y \times Y \to [0, \infty)\) as
\[
\psi(t) = \frac{t}{2}
\]
\[
\alpha(x, y, z) = \begin{cases} 
1 & x, y, z \in [-1, 1] \\
0 & \text{otherwise}
\end{cases}
\]

\(T\) satisfies conditions of the previous theorem. Obviously, The map \(T\) has a unique fixed point \(x = 0 \in A \cap B\).
Theorem 2.2. Let \( (X, D^*) \) be a generalized compact \( D^* \)-metric space and \( T : X \to X \) be a given mapping. Assume that

\[
\frac{1}{2} D^*(x, x, Tx) \leq D^*(x, y, z) \quad \text{implies} \quad D^*(Tx, Ty, Tz) < D^*(x, y, z)
\]  

(2.9)

for all \( x, y, z \in X \), Then \( T \) has a unique fixed point.

Proof. We put

\[
\beta = \inf \{ D^*(x, x, Tx) : x \in X \}
\]

and choose a sequence \( \{x_n\} \) in \( X \) satisfying

\[
\lim_{n \to \infty} D^*(x_n, x_n, T(x_n)) = \beta.
\]

Since \( X \) is compact, without loss of generality, we may assume that \( \{x_n\} \) and \( \{T(x_n)\} \) converge to some elements \( v, w \in X \), respectively. We shall show \( \beta = 0 \). Arguing by contradiction, we assume \( \beta > 0 \). We have

\[
\lim_{n \to \infty} D^*(x_n, x_n, w) = D^*(v, v, w) = \lim_{n \to \infty} D^*(x_n, x_n, T(x_n)) = \beta.
\]

We can choose \( n_0 \in \mathbb{N} \) such that

\[
\frac{2}{3} \beta < D^*(x_n, x_n, w) \quad \text{and} \quad D^*(x_n, x_n, T(x_n)) < \frac{4}{3} \beta
\]

for \( n \in \mathbb{N} \) with \( n \geq n_0 \). Thus, \( \frac{1}{4} D^*(x_n, x_n, T(x_n)) < D^*(x_n, x_n, w) \) for \( n \geq n_0 \). By Assumption (2.9), \( D^*(T(x_n, T(x_n, T(w))) < D^*(x_n, x_n, w) \) holds for \( n \geq n_0 \). This implies

\[
D^*(w, w, Tw) = \lim_{n \to \infty} D^*(T(x_n, T(x_n, T(w)) \leq \lim_{n \to \infty} D^*(x_n, x_n, w) = \beta.
\]

From the definition of \( \beta \), we obtain \( D^*(w, w, Tw) = \beta \). Since \( \frac{1}{4} D^*(w, w, Tw) < D^*(w, w, Tw) \), we have

\[
D^*(Tw, Tw, T^2 w) < D^*(w, w, Tw) = \beta,
\]

which contradicts the definition of \( \beta \). Therefore we obtain \( \beta = 0 \). We next show that \( T \) has a fixed point. Arguing by contradiction, we assume that \( T \) does not have a fixed point. Then we note that \( D^*(T(x_n, T(x_n, T^2 x_n)) < D^*(x_n, x_n, T(x_n)) \) holds for every \( n \in \mathbb{N} \) because \( 0 < \frac{1}{4} D^*(x_n, x_n, T(x_n)) < D^*(x_n, x_n, T(x_n)) \). We have

\[
\lim_{n \to \infty} D^*(v, v, T(x_n)) = D^*(v, v, T(x_n)) = \lim_{n \to \infty} D^*(x_n, x_n, T(x_n)) = \beta = 0,
\]

Which implies that \( \{T(x_n)\} \) also converges to \( v \). We also have

\[
\lim_{n \to \infty} D^*(v, v, T^2 x_n) \leq \lim_{n \to \infty} (D^*(v, v, T(x_n)) + D^*(T(x_n, T^2 x_n, T^2 x_n)))
\]

\[
\leq \lim_{n \to \infty} (D^*(v, v, T(x_n)) + D^*(x_n, x_n, T(x_n))) = 0.
\]

Thus, \( \{T^2 x_n\} \) converges to \( v \). If

\[
D^*(x_n, x_n, v) \leq \frac{1}{2} D^*(x_n, x_n, T(x_n)) \quad \text{and} \quad D^*(T(x_n, T(x_n, T^2 x_n)) \leq \frac{1}{2} D^*(T(x_n, T(x_n, T^2 x_n)),
\]

Then we have

\[
D^*(T(x_n, T(x_n, T^2 x_n)) + D^*(x_n, x_n, T(x_n)) < 2D^*(x_n, x_n, T(x_n))
\]

\[
\leq 2D^*(x_n, x_n, v) + 2D^*(v, T(x_n, T(x_n)) \leq D^*(x_n, x_n, T(x_n)) + D^*(T(x_n, T(x_n, T^2 x_n)),
\]

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which is a contradiction. Hence for every $n \in \mathbb{N}$, either

\[
\frac{1}{2}D^r(x_n, x_{n+1}, T x_n) < D^r(x_n, x_{n+1}) \quad \text{or} \quad \frac{1}{2}D^r(T x_n, T x_{n+1}, T^2 x_n) < D^r(T x_n, T x_{n+1}, v)
\]

holds. By assumption (2.9), either

\[
D^r(T^2 x_n, T^2 x_{n+1}, T v) < D^r(T x_n, T x_{n+1}, v) \quad \text{or} \quad D^r(T x_n, T x_{n+1}, T v) < D^r(x_n, x_{n+1}, v)
\]

holds. Hence one of the following holds:

(I) There exists an infinite subset $I$ of $\mathbb{N}$ such that $D^r(T^2 x_n, T^2 x_{n+1}, T v) < D^r(T x_n, T x_{n+1}, v)$ for all $n \in I$.

(II) There exists an infinite subset $J$ of $\mathbb{N}$ such that $D^r(T x_n, T x_{n+1}, T v) < D^r(x_n, x_{n+1}, v)$ for all $n \in J$.

In the first case, we obtain

\[
D^r(v, v, T v) = \lim_{n \in J, n \to \infty} D^r(T^2 x_n, T^2 x_{n+1}, T v) \leq \lim_{n \in J, n \to \infty} D^r(T x_n, T x_{n+1}, v) = 0,
\]

which implies $T v = v$. Also, in the second case, we obtain

\[
D^r(v, v, T v) = \lim_{n \in J, n \to \infty} D^r(T x_n, T x_{n+1}, T v) \leq \lim_{n \in J, n \to \infty} D^r(x_n, x_{n+1}, v) = 0.
\]

Hence, we have shown that $v$ is a fixed point of $T$ in both cases. This is a contradiction. Therefore there exists $z \in X$ such that $T z = z$. Fix $y \in X$ with $y \neq z$. Then since $\frac{1}{2}D^r(\cdot, z, T z) = 0 < D^r(\cdot, z, y)$, we have

\[
D^r(z, z, T y) = D^r(z, T z, T y) < D^r(z, z, y),
\]

and hence $y$ is not a fixed point of $T$. Therefore the fixed point $z$ of $T$ is unique. \hfill \blacksquare

**Theorem 2.3.** Let $(X, D^r)$ be a $D^r$-complete metric space and $T : X \to X$ be a map such that

\[
D^r(T x, T^2 y, T^2 z) \leq aD^r(x, y, T z) + bD^r(x, T^2 y, T^2 z)
\]

for all $x, y, z \in X$, where $3a + 2b < 1$, and $a, b \geq 0$. Then $T$ has a unique fixed point.

**Proof.** Let $x_0 \in X$ be a fixed arbitrary element. Define the sequence $\{x_n\}$ in $X$ as $x_{n+1} = T x_n$, for $n = 0, 1, 2, \ldots$. Then we have

\[
D^r(x_n, x_{n+1}, x_{n+2}) = D^r(T x_n, x_{n+1}, T x_{n+1}) = D^r(T x_n, x_{n+1}, T^2 x_{n+1})
\]

\[
\leq aD^r(x_n, x_{n+1}, T x_{n+1}) + bD^r(T x_n, T^2 x_{n+1}, T^2 x_{n+1})
\]

\[
\leq aD^r(x_n, x_{n+1}, T^2 x_{n+1}) + bD^r(T x_n, T^2 x_{n+1}, T^2 x_{n+1})
\]

\[
\leq aD^r(x_n, x_{n+1}, x_{n+2}) + aD^r(T x_n, T^2 x_{n+1}, T^2 x_{n+1}) + aD^r(x_{n+1}, x_{n+2}, x_{n+1})
\]

\[
+ bD^r(x_n, x_{n+1}, x_{n+2}) + bD^r(T x_n, T^2 x_{n+1}, T^2 x_{n+1})
\]

\[
= aD^r(x_n, x_{n+1}, x_{n+2}) + aD^r(x_n, x_{n+1}, x_{n+2}) + aD^r(x_{n+1}, x_{n+2}, x_{n+1})
\]

\[
+ bD^r(x_n, x_{n+1}, x_{n+2}) + bD^r(T x_n, T^2 x_{n+1}, T^2 x_{n+1})
\]

Hence

\[
D^r(x_n, x_{n+1}, x_{n+2}) \leq \frac{a + b}{1 - 2a - b} D^r(x_n, x_{n+1}, x_{n+2}).
\]

If we let $\frac{a + b}{1 - 2a - b} = q$, then by hypothesis we have $0 \leq q < 1$

\[
D^r(x_n, x_{n+1}, x_{n+2}) \leq q D^r(x_n, x_{n+1}, x_{n+2}) \leq \cdots \leq q^n D^r(x_0, x_1) \to 0 \text{ as } n \to \infty.
\]
For $m \geq n$ we have

\[
D^s(x_n, x_n, x_m) \leq D^s(x_n, x_n, x_{n+1}) + D^s(x_{n+1}, x_{n+1}, x_{n+1}) + \ldots + D^s(x_{m-1}, x_{m-1}, x_m)
\leq (q^n + q^{n+1} + \ldots + q^{m-1}) D^s(x_0, x_1, x_1)
\leq \frac{q^n}{1-q} D^s(x_0, x_1, x_1).
\]

which implies that

\[
\lim_{n,m \to \infty} D^s(x_n, x_n, x_m) = 0.
\]

Hence $\{x_n\}$ is a Cauchy sequence in $D^s$-complete metric space, Thus $x_n \to x$ in $X$.

Now we prove $Tx = x$. Suppose $Tx \neq x$, then

\[
D^s(x, x, Tx) = \lim_{n \to \infty} D^s(x_{n+2}, x_{n+2}, Tx) = \lim_{n \to \infty} D^s(Tx, T^2x_n, T^2x_n)
\leq \lim_{n \to \infty} (aD^s(x, Tx_n, Tx_n) + bD^s(Tx, T^2x_n, T^2x_n))
= \lim_{n \to \infty} (aD^s(x, x_{n+1}, x_{n+1}) + bD^s(x, x_{n+2}, x_{n+2})) = 0
\]

Thus, $x = Tx$. Finally to prove uniqueness, let $x \neq y$ be another fixed point of $T$. Then

\[
D^s(x, y, y) = D^s(Tx, T^2y, T^2y) \leq aD^s(x, Ty, Ty) + bD^s(Ty, T^2y, T^2y)
= aD^s(x, y, y) + bD^s(x, y, y) = (a + b)D^s(x, y, y) < D^s(x, y, y),
\]

Which is contradiction. Thus, $T$ has a unique fixed point.

By using similar proof in previous theorem we obtain corollaries.

**Corollary 2.1.** Let $(X, D^s)$ be a $D^s$-complete metric space and $T : X \to X$ be a map such that

\[
D^s(Tx, T^2y, T^2z) \leq aD^s(x, Ty, Tz) + bD^s(Tx, y, Tz) + cD^s(Tx, Ty, z)
\]

for all $x, y, z \in X$, where $a + b + c \leq 1$, and $a, b, c \geq 0$. Then $T$ has a unique fixed point.

**Corollary 2.2.** Let $(X, D^s)$ be a $D^s$-complete metric space and $T : X \to X$ be a map such that

\[
D^s(Tx, T^2y, T^2z) \leq aD^s(x, y, Tz) + bD^s(Tx, y, z) + cD^s(x, Ty, z)
\]

for all $x, y, z \in X$, where $a + b + c \leq 1$, and $a, b, c \geq 0$. Then $T$ has a unique fixed point.

**Theorem 2.4.** Let $(X, D^s)$ be a generalized $D^s$-metric space. Assume that the pair of maps $S, T : X \to X$ satisfies the following condition

\[
D^s(Tx, Ty, Tz) \leq \alpha_1 D^s(Sx, Sy, Sz) + \frac{\alpha_2}{2} \{D^s(Sy, Tx, Ty) + D^s(Sx, Tz, Ty)\}
+ \frac{\alpha_3}{2} \{D^s(Tx, Sz, Sx) + D^s(Ty, Sx, Sz)\}
\]

(2.10)

for all $x, y, z \in X$ where $\alpha_1, \alpha_2, \alpha_3 \geq 0$, and $0 < \alpha_1 + \frac{\alpha_2}{2} + \frac{\alpha_3}{3} < 1$ such that $T(X) \subset S(X)$, and $T(X) or S(X)$ is $D^s$-complete subspace of $X$. Then $S$ and $T$ have a unique coincidence point in $X$. Moreover if $S, T$ are weakly compatible then $S, T$ have a unique common fixed point.
Proof. Let \( x_0 \in X \) be arbitrary, there exists \( x_1 \in X \) such that \( T x_0 = S x_1 \), in this way we have sequence \( \{ S x_n \} \) with \( T x_{n-1} = S x_n \) for all \( n > 0 \). Then from the inequality (2.10), we have

\[
D^* (S x_n, S x_{n+1}, S x_{n+1}) = D^* (T x_{n-1}, T x_n, T x_n)
\]

\[
\leq a_1 D^* (S x_{n-1}, S x_n, S x_n) + \frac{a_2}{2} \{ D^* (S x_n, T x_{n-1}, T x_n) + D^* (S x_{n-1}, T x_n, T x_n) \}
+ \frac{a_3}{2} \{ D^* (T x_{n-1}, S x_{n-1}, S x_n) + D^* (T x_n, S x_{n-1}, S x_n) \}
\leq a_1 D^* (S x_{n-1}, S x_n, S x_n) + \frac{a_2}{2} \{ D^* (S x_n, S x_n, S x_{n+1}) + D^* (S x_{n-1}, S x_n, S x_{n+1}) \}
+ \frac{a_3}{2} \{ D^* (S x_n, S x_{n-1}, S x_n) + D^* (S x_{n+1}, S x_{n-1}, S x_n) \}
\leq a_1 D^* (S x_{n-1}, S x_n, S x_n) + \frac{a_2}{2} \{ D^* (S x_n, S x_n, S x_{n+1}) + D^* (S x_{n-1}, S x_n, S x_{n+1}) \}
+ \frac{a_3}{2} \{ D^* (S x_n, S x_{n-1}, S x_n) + D^* (S x_{n+1}, S x_{n-1}, S x_n) \}
+ D^* (S x_n, S x_{n+1}, S x_{n+1})
\]

\[
= (a_1 + \frac{a_2}{2} + a_3) D^* (S x_{n-1}, S x_n, S x_n) + (a_2 + \frac{a_3}{2}) D^* (S x_n, S x_n, S x_{n+1})
\]

Hence

\[
D^* (S x_n, S x_{n+1}, S x_{n+1}) \leq q D^* (S x_{n-1}, S x_n, S x_n),
\]

(2.11)

Where \( q = \frac{a_1 + \frac{a_2}{2} + a_3}{1 - a_2 - \frac{a_3}{2}} < 1 \). Obviously \( 0 \leq q < 1 \).

Repeating the above inequality (2.11) \( n \) times, we get

\[
D^* (S x_n, S x_{n+1}, S x_{n+1}) \leq q D^* (S x_{n-1}, S x_n, S x_n) \leq \ldots \leq q^n D^* (S x_0, S x_1, S x_1).
\]

(2.12)

Next, we shall prove that \( \{ x_n \} \) is \( D^* \)-cauchy sequence in \( X \). In fact for each \( n, m \in \mathbb{N}, n < m \), from (D4) and (2.12), we have

\[
D^* (S x_n, S x_m, S x_m) \leq D^* (S x_n, S x_n, S x_{n+1}) + D^* (S x_{n+1}, S x_{n+1}, S x_{n+2})
+ D^* (S x_{n+2}, S x_{n+2}, S x_{n+3}) + \ldots + D^* (S x_{m-1}, S x_{m-1}, S x_m)
\leq D^* (S x_n, S x_n, S x_{n+1}) + D^* (S x_{n+1}, S x_{n+1}, S x_{n+2})
+ D^* (S x_{n+2}, S x_{n+2}, S x_{n+3}) + \ldots + D^* (S x_{m-1}, S x_{m-1}, S x_m)
\leq (q^n + q^{n+1} + \ldots + q^{m-1}) D^* (S x_0, S x_1, S x_1)
\leq \frac{q^m}{1-q} D^* (S x_0, S x_1, S x_1).
\]

which implies that

\[
\lim_{n,m \to \infty} D^* (S x_n, S x_m, S x_m) = 0.
\]

Hence \( \{ x_n \} \) is \( D^* \)-cauchy sequence in \( X \). Noting that \( S(X) \) is \( D^* \)-complete, there exist \( u \in S(X) \) such that \( \{ S x_n \} \to u \) as \( n \to \infty \), and there exist \( p \in X \) such that \( S p = u \). According assumption if \( T(X) \) is \( D^* \)-complete, then there exist \( u \in T(X) \) such that \( S x_n \to u \), as \( T(X) \subset S(X) \) we have \( u \in S(X) \), then there exist \( p \in X \) such that \( S p = u \). We claim
that \( Tp = u \)

\[
D^*(Tp, Tp, u) = D^*(Tp, Tp, u) \leq D^*(Tp, Tx_n, u, u) + D^*(Tx_n, u, u)
\]

\[
\leq a_1 D^*(Sp, Sp, Sx_n) + \frac{a_2}{2} \{ D^*(Sp, Tp, Tp) + D^*(Sp, Tp, Tp) \}
\]

\[
+ \frac{a_3}{2} \{ D^*(Tp, Sp, Sx_n) + D^*(Tp, Sp, Sx_n) \} + D^*(Tx_n, u, u)
\]

\[
= a_1 D^*(u, u, Sx_n) + \frac{a_2}{2} \{ D^*(u, Tp, Tp) + D^*(u, Tp, Tp) \}
\]

\[
+ \frac{a_3}{2} \{ D^*(Tp, u, Sx_n) + D^*(Tp, u, Sx_n) \} + D^*(Sx_{n+1}, u, u).
\]

Taking limit as \( n \to \infty \) we get \( Sx_n \to u \), hence we have

\[
(1 - a_2 - a_3)D^*(u, u, Tp) \leq 0
\]

Thus \( Tp = u \), therefore \( Tp = Sp \) and \( p \) is a coincidence point of \( S \) and \( T \). Now we show that \( S \) and \( T \) have a unique coincidence point. For this, assume that there exists a point \( q \) in \( X \) such that \( Sq = Tq \). Now

\[
D^*(Tp, Tp, Tq) \leq a_1 D^*(Sp, Sp, Sq) + \frac{a_2}{2} \{ D^*(Sp, Tp, Tp) + D^*(Sp, Tp, Tp) \}
\]

\[
+ \frac{a_3}{2} \{ D^*(Tp, Sp, Sq) + D^*(Tp, Sp, Sq) \}
\]

\[
= a_1 D^*(Tp, Tp, Tq) + \frac{a_2}{2} \{ D^*(Tp, Tp, Tp) + D^*(Tp, Tp, Tp) \}
\]

\[
+ \frac{a_3}{2} \{ D^*(Tp, Tp, Tq) + D^*(Tp, Tp, Tq) \}.
\]

Hence

\[
D^*(Tp, Tp, Tq) < \frac{a_1}{1 - a_3} D^*(Tp, Tp, Tq).
\]

Since \( \frac{a_1}{1 - a_3} < 1 \) therefore we have \( Tp = Tq \). By proposition (1.4) of [7], \( S \) and \( T \) have a unique common fixed point. \( \square \)

**Example 2.2.** Let \((X, D^*)\) be a \( D^*\)-complete metric space, where \( X = [0, 2] \) and

\[
D^*(x, y, z) = |x - y| + |y - z| + |z - x|
\]

Define self-maps \( S \) and \( T \) on \( X \) as follows: \( Sx = \frac{x + 1}{2} \) and \( Tx = \frac{x + 5}{2} \) for all \( x \in X \). If we put \( 0 < a_1 < 1 \) and \( a_2 = a_3 = 0 \), then conditions of theorem (2.4) are satisfied, and \( 1 \) is the unique common fixed point of \( S \) and \( T \).

**References**


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