On Best Proximity Points In b-Metric Space

Reny George¹, Cihangir Alaca²*, K. P. Reshma³,⁴

(¹) Department of Mathematics and Computer Science St. Thomas College, Raubandha, Bhilai, Durg, Chhattisgarh State, 490006, India.
(²) Department of Mathematics, Faculty of Science and Arts, Celal Bayar University, 45140 Manisa, Turkey.
(³) Department of Mathematics, Rungta College of Engineering and Technology, Bhilai Chhattisgarh State, 490006, India.
(⁴) Research Center: Department of Mathematics, Govt. V.Y.T.P.G. Autonomous College Durg, Chhattisgarh State, India.

Abstract
Eldred and Veeramani proved a theorem which ensures the existence of a best proximity point of cyclic contraction in metric space. In this paper we prove the existence of best proximity points of cyclic contractions and generalize proximal contractions of first and second kinds in the setting of complete b-metric space, thereby ascertaining an optimal approximate solution to the equation \( Tx = x \) in a b-metric space.

Keywords: b-metric space, proximal contractions, cyclic contraction.

1 Introduction
Cyclic contraction and best proximity points are at present popular topic in fixed point theory. The first result in this area was reported in 2003, by Kirk et al. [13] as a generalization of the usual contraction and fixed point. Since then many researchers continued investigation in this direction and obtained many result. Later in 2006, Eldered and Veeramani [1] proved some results about best proximity points of cyclic contraction maps.

Let \( (X, d) \) be a metric space and let \( A \) and \( B \) be non-empty subsets of \( X \). A mapping \( T \) on \( A \cup B \) is called a cyclic mapping if \( T(A) \subset B \) and \( T(B) \subset A \). \( x \in A \cap B \) is called a best proximity point if \( d(Ty, Tx) = d(A, B) \) is satisfied, where

\[
d(A, B) = \inf \{ d(x, y) : x \in A, y \in B \}.
\]

Eldred and Veeramani [1] proved the following existence theorem:

**Theorem 1.1.** Let \( A \) and \( B \) be non-empty closed convex subsets of a uniformly convex Banach space. Suppose \( T \) is a cyclic contraction on \( A \cup B \), that is \( T(A) \subset B, T(B) \subset A \) and there exists \( r \in (0, 1) \), such that

\[
d(Ty, Tx) \leq rd(x, y) + (1 - r)d(A, B)
\]

for all \( x \in A \) and \( y \in B \). Then \( T \) has a unique best proximity point \( z \) in \( A \) and \( \{ T^{2n}x \} \) converges to \( z \) for all \( x \in A \).

*Corresponding author. Email address: cihangiralaca@yahoo.com.tr, Tel:+902362013209
The existence of a best proximity point for proximal pointwise contractions has been proved by Anuradha and Veeramani [3]. Furthermore, many best proximity point theorems have been analyzed in [2] [5] [6] [10] [12]. Sadiq Basha and Shahzad in [9] proved best proximity point theorems for generalized proximal contractions of the first and the second kinds in the setting of complete metric space, thereby ascertaining an optimal approximate solution to the equation $Tx = x$, where $T : A \to B$ is a generalized proximal contraction of the first kind or a generalized proximal contraction of the second kind. Recently many generalised metric spaces with non Hausdorff topology has come into existence. One such generalised metric space with non Hausdorff topology called b-metric space was introduced by Bakhtin [4]. Note that spaces with non Hausdorff topology plays an important role in Tarskian approach to programming language semantics used in computer science (For some details see [7]). It is interesting to note that unlike fixed points, best proximity points are not universal. More precisely if $x$ is a best proximity point of mapping $T$ defined in a b-metric space $(X, b)$, $x$ may not be a best proximity point of the same mapping $T$ when defined in a metric space $(X, d)$ and vice versa. (See Example 3.17 below). In view of the above it becomes important to investigate the existence of best proximity points of mappings defined in various generalised spaces. In this paper we prove the existence of best proximity points of cyclic contractions and generalize proximal contractions of first and second kinds in the setting of complete b-metric space.

2 Preliminaries

Definition 2.1. ([4]) Let $X$ be a non-empty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:

(bM1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(bM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(bM3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then $d$ is called a b-metric on $X$ and $(X, d)$ is called a b-metric space (in short bMS) with coefficient $s$. Note that every metric space is a b-metric space. However the converse is not necessarily true. For any $x \in X$, the open ball with centre $x$ and radius $r > 0$ is given by

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$  

The open balls in bMS are not necessarily open. Let $\mathcal{W}$ be the collection of all subsets $\mathcal{W}$ of $X$ satisfying the condition that for all $x \in \mathcal{W}$ there exists $r > 0$ such that $B_r(x) \subseteq \mathcal{W}$. Then $\mathcal{W}$ defines a topology for the bMS $(X, d)$, which is not necessarily Hausdorff.

Definition 2.2. ([4]) Let $(X, d)$ be a b-metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent in $(X, d)$ and converges to $x$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n \to \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in $(X, d)$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$ or equivalently, if $\lim_{n \to \infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(c) $(X, d)$ is said to be a complete b-metric space if every Cauchy sequence in $X$ converges to some $x \in X$.

Now we give some basic definitions and concepts which are useful and related to the context of our results.

$$\text{dist}(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$$

$A_0 = \{x \in A : d(x, y) = \text{dist}(A, B) \text{ for some } y' \in B\}$

$B_0 = \{y \in B : d(x', y) = \text{dist}(A, B) \text{ for some } x' \in A\}$.

Definition 2.3. ([11]) A subset $K$ of a metric space $X$ is boundedly compact if each bounded sequence in $K$ has a subsequence converging to a point in $K$.  

International Scientific Publications and Consulting Services
Definition 2.4. ([9]) Let \( A \) and \( B \) be non-empty subsets of a metric space \( (X, d) \). A mapping \( T : A \to B \) is said to be a generalized proximal contraction of the first kind if there exist non-negative numbers \( \alpha, \beta, \gamma, \delta \) with \( \alpha + \beta + \gamma + 2\delta < 1 \) such that the conditions
\[
d(u_1, T x_1) = d(A, B) \text{ and } d(u_2, T x_2) = d(A, B)
\]
implies the inequality that
\[
d(u_1, u_2) \leq \alpha d(x_1, x_2) + \beta d(x_1, u_1) + \gamma d(x_2, u_2) + \delta [d(x_1, u_2) + d(x_2, u_1)]
\]
for all \( u_1, u_2, x_1, x_2 \in A \). If \( T \) is a self-mapping on \( A \), then the requirement in the preceding definition reduces to the condition
\[
d(T x_1, T x_2) \leq \alpha d(x_1, x_2) + \beta d(x_1, x_2) + \gamma d(x_2, x_2) + \delta [d(x_1, x_2) + d(x_2, x_1)]
\]
Definition 2.5. ([9]) Let \( A \) and \( B \) be non-empty subsets of a metric space \( (X, d) \). A mapping \( T : A \to B \) is said to be a generalized proximal contraction of the second kind if there exist non-negative numbers \( \alpha, \beta, \gamma, \delta \) with \( \alpha + \beta + \gamma + 2\delta < 1 \) such that the conditions
\[
d(u_1, T x_1) = d(A, B) \text{ and } d(u_2, T x_2) = d(A, B)
\]
implies the inequality that
\[
d(T u_1, T u_2) \leq \alpha d(T x_1, T x_1) + \beta d(T x_1, T u_1) + \gamma d(T x_2, T u_2) + \delta [d(T x_1, T u_2) + d(T x_2, T u_1)]
\]
for all \( u_1, u_2, x_1, x_2 \in A \).

Definition 2.6. ([9]) The set \( B \) is said to be approximately compact with respect to \( A \) if every sequence \( \{y_n\} \) of \( B \) satisfying the condition that \( d(x, y_n) \to d(x, B) \) for some \( x \) in \( A \) has a convergent subsequence.

It is obvious that any compact set is approximatively compact, and that any set is approximatively compact with respect to itself.

3 Main Results

3.1 Existence of best proximity points in b-metric space

Definition 3.1. Let \( (X, d) \) be a b-metric space with \( s \geq 1 \) and \( A \) and \( B \) be non-empty closed subsets of \( X \). \( T \) is a cyclic contraction on \( A \cup B \), if \( T(A) \subset B \) and \( T(B) \subset A \) and there exists \( r \in (0, \frac{1}{\sqrt{s}}) \), such that
\[
d(T y, T x) \leq rd(x, y) + (1 - r) \text{dist}(A, B)
\]
for all \( x \in A \) and \( y \in B \).

Note that (3.8) implies that \( T \) satisfies
\[
d(T y, T x) \leq d(x, y)
\]
for all \( x \in A \) and \( y \in B \), also (3.8) can be rewritten as
\[
d(T y, T x) - \text{dist}(A, B) \leq r[d(x, y) - \text{dist}(A, B)]
\]
for all \( x \in A \) and \( y \in B \).

Proposition 3.1. Let \( A \) and \( B \) be non-empty subsets of a b-metric space \( (X, d) \) with \( s \geq 1 \). Suppose \( T : A \cup B \to A \cup B \) is a cyclic contraction map. Then starting with any \( x_0 \) in \( A \cup B \) we have \( d(x_n, T x_n) \to \text{dist}(A, B) \) where \( x_{n+1} = T x_n, n = 0, 1, 2, \ldots \).
Proof.

\[ d(x_n, x_{n+1}) \leq rd(x_{n-1}, x_n) + (1-r)dist(A,B) \]
\[ \leq r[rd(x_{n-1}, x_{n-2}) + (1-r)dist(A,B)] \]
\[ + (1-r)dist(A,B) \]
\[ = r^2d(x_{n-1}, x_{n-2}) + (1-r^2)dist(A,B) \]

Inductively we have \( d(x_n, x_{n+1}) \leq r^n d(x_1, x_0) + (1-r^n)dist(A,B) \). Therefore \( d(x_n, x_{n+1}) \to dist(A,B) \).

\[ \square \]

Proposition 3.2. Let \( A \) and \( B \) be non-empty closed subsets of a complete \( b \)-metric space \((X,d)\) with \( s \geq 1 \). Let \( T : A \cup B \to A \cup B \) be a cyclic contraction map, that is \( T(A) \subseteq B \), \( T(B) \subseteq A \) and there exists \( r \in (0, \frac{1}{\sqrt{s}}) \), such that

\[ d(Ty,Tx) \leq rd(x,y) + (1-r)d(A,B). \quad (3.11) \]

Let \( x_0 \in A \) and define \( x_{n+1} = Tx_n \). Suppose \( \{x_{2n}\} \) has a convergent subsequence in \( A \). Then there exists \( x \) in \( A \) such that \( d(x,Tx) = dist(A,B) \).

\[ \square \]

Proof. Let \( \{x_{2n}\} \) be a subsequence of \( \{x_{2n}\} \) converging to some \( x \in A \). By Proposition 3.1

\[ d(x,x_{2n-1}) = d(x_{2n}, x_{2n-1}) \to dist(A,B) \]

Hence

\[ d(x,x_{2n}) \to dist(A,B) \]
\[ d(x_{2n}, Tx) \to dist(A,B) \]

Proposition 3.3. Let \( A \) and \( B \) be non-empty subsets of a \( b \)-metric space \((X,d)\) with \( s \geq 1 \). Let \( T : A \cup B \to A \cup B \) be a cyclic contraction map. Then for \( x_0 \in A \cup B \) and \( x_{n+1} = Tx_n, n = 0, 1, 2, 3, \ldots \), the sequences \( \{x_{2n}\} \) and \( \{x_{2n+1}\} \) are bounded.

\[ \square \]

Proof. Suppose \( x_0 \in A \). By Proposition 3.1, \( d(x_{2n}, x_{2n+1}) \to dist(A,B) \). It is enough to prove that \( \{x_{2n+1}\} \) is bounded. Suppose \( \{x_{2n+1}\} \) is not bounded, and let

\[ M > \max\{\frac{2s^2d(x_0,Tx_0)}{1-r} + \frac{1-r^2}{1-r^2s}dist(A,B), d(T^2x_0,Tx_0)\} \]

Since the sequence is not bounded there exists \( N_0 \) such that

\[ d(T^{2N_0+1}x_0,T^{2N_0+1}x_0) > M \text{ and } d(T^2x_0,T^{2N_0+1}x_0) \leq M. \]
By the cyclic contraction property of $T$

$$d(T^{2N_0}x_0, T^{2N_0+1}x_0) - dist(A,B) \leq r[d(Tx_0, T^{2N_0}x_0) - dist(A,B)]$$

$$\leq r[kd(x_0, T^{2N_0-1}x_0) + (1-r)dist(A,B)] - r^2 dist(A,B)$$

$$= r^2[d(x_0, T^{2N_0-1}x_0) - dist(A,B)]$$

$$d(T^{2N_0}x_0, T^{2N_0+1}x_0) - dist(A,B) \leq d(x_0, T^{2N_0-1}x_0)$$

$$\leq s[d(x_0, T^2x_0) + d(T^2x_0, T^{2N_0-1}x_0)]$$

$$\leq s[s[d(x_0, Tx_0) + d(Tx_0, T^2x_0)] + sM]$$

$$\leq s[s[d(x_0, Tx_0) + d(x_0, Tx_0)] + sM]$$

$$M - dist(A,B) \leq 2s^2 d(x_0, Tx_0) + sM$$

$$M < 2s^2 d(x_0, Tx_0) + \frac{1}{s} - s^2 d(A,B)$$

a contradiction. Hence $\{x_{2n+1}\}$ is bounded.

**Theorem 3.1.** Let $A$ and $B$ be non-empty closed subsets of a $b$-metric space $(X,d)$ with $s \geq 1$. Let $T : A \cup B \rightarrow A \cup B$ be a cyclic contraction map. If either $A$ or $B$ is boundedly compact then there exists $x$ in $A \cup B$ with $d(x, Tx) = dist(A,B)$.

Proof follows from Proposition 3.2 and Proposition 3.3.

**Remark 3.1.** Taking $s = 1$ in Theorem 3.1 we get a result of Eldred and Veeramani [1] in metric space.

Now we define the generalized proximal contraction of the first kind and second kind in $b$-metric space.

**Definition 3.2.** Let $A$ and $B$ be non-empty subsets of a $b$-metric space $(X,d)$ with $s \geq 1$. A mapping $T : A \rightarrow B$ is said to be a generalized proximal contraction of the first kind if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $s(\alpha + \beta) + s(s+1)\delta + \gamma < 1$ such that the conditions

$$d(u_1, Tx_1) = d(A,B) \text{ and } d(u_2, Tx_2) = d(A,B) \quad (3.12)$$

imply the inequality that

$$d(u_1, u_2) \leq \alpha d(x_1, x_2) + \beta d(x_1, u_1) + \gamma d(x_2, u_2) + \delta [d(x_1, u_2) + d(x_2, u_1)] \quad (3.13)$$

for all $u_1, u_2, x_1, x_2$ in $A$. If $T$ is a self-mapping on $A$, then the requirement in the preceding definition reduces to the condition that

$$d(Tx_1, Tx_2) \leq \alpha d(x_1, x_2) + \beta d(x_1, Tx_1) + \gamma d(x_2, Tx_2) + \delta [d(x_1, Tx_2) + d(x_2, Tx_1)] \quad (3.14)$$

**Definition 3.3.** Let $A$ and $B$ be non-empty subsets of a $b$-metric space $(X,d)$ with $s \geq 1$. A mapping $T : A \rightarrow B$ is said to be a generalized proximal contraction of the second kind if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $s(\alpha + \beta) + s(s+1)\delta + \gamma < 1$ such that the conditions

$$d(u_1, Tx_1) = d(A,B) \text{ and } d(u_2, Tx_2) = d(A,B) \quad (3.15)$$
implies the inequality that

\[ d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2) + \beta d(Tx_1, Tu_1) + \gamma d(Tx_2, Tu_2) + \delta [d(Tx_1, Tu_2) + d(Tx_2, Tu_1)] \quad (3.16) \]

for all \( u_1, u_2, x_1, x_2 \in A \).

**Remark 3.2.** For \( s = 1 \) the above definitions reduces to the definition of generalized proximal contraction of the first kind and second kind in metric space.

**Theorem 3.2.** Let \( A \) and \( B \) be non-empty, closed subsets of a complete b-metric space \((X, d)\) with \( s \geq 1 \) such that \( B \) is approximately compact with respect to \( A \). Also suppose that \( A_0 \) and \( B_0 \) are non-empty. Let \( T : A \rightarrow B \) satisfy the following conditions:

(a) \( T \) is a generalized proximal contraction of first kind,

(b) \( T(A_0) \) is contained in \( B_0 \).

Then, there exists a unique element \( x \) in \( A \) such that

\[ d(x, Tx) = d(A, B) \quad (3.17) \]

and the sequence \( \{x_n\} \) converges to the best proximity point \( x \), where \( x_0 \) is any fixed element in \( A_0 \) and \( d(x_{n+1}, Tx_n) = d(A, B) \) for \( n \geq 0 \).

**Proof.** Let us select an element \( x_0 \) in \( A_0 \). On account of the fact \( T(A_0) \) is contained in \( B_0 \), it is guaranteed that there is an element \( x_1 \) in \( A_0 \) satisfying the condition that

\[ d(x_1, Tx_0) = d(A, B). \]

Further, since \( Tx_1 \) is a member of \( T(A_0) \) which is contained in \( B_0 \), it follows that there is an element \( x_2 \) in \( A_0 \) such that

\[ d(x_2, Tx_1) = d(A, B). \]

This process can be continued further. Having chosen \( x_n \) in \( A_0 \), there exists an element \( x_{n+1} \) in \( A_0 \) satisfying the condition that

\[ d(x_{n+1}, Tx_n) = d(A, B) \]

for every non-negative integer \( n \). In view of the fact that \( T \) is a generalized proximal contraction of first kind, we have

\[
\begin{align*}
\quad d(x_n, x_{n+1}) & \quad \leq \quad \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_{n+1}) + \gamma d(x_n, x_{n+1}) + \delta [d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})] \\
& \quad \leq \quad \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_{n+1}) + \gamma d(x_n, x_{n+1}) + \delta d(x_{n-1}, x_{n+1}) \\
& \quad \leq \quad \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_{n+1}) + \gamma d(x_n, x_{n+1}) + \delta s [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
& \quad < \quad (1 - \gamma - s \delta) d(x_n, x_{n+1}) \\
& \quad = \quad (\alpha + \beta + s \delta) d(x_{n-1}, x_n) \\
& \quad \leq \quad \frac{\alpha + \beta + s \delta}{1 - \gamma - s \delta} d(x_{n-1}, x_n) \\
& \quad \leq \quad \lambda d(x_{n-1}, x_n)
\end{align*}
\]

where \( \lambda = \frac{\alpha + \beta + s \delta}{1 - \gamma - s \delta} < 1 \). Repeating this process we obtain

\[ d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1). \quad (3.18) \]
For any $m \geq 1$, $p \geq 1$ it follows from (3.18)
\[
d(x_m, x_{m+p}) \leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})]
\]
\[
= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p})
\]
\[
\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})]
\]
\[
= sd(x_m, x_{m+1}) + s^2d(x_{m+1}, x_{m+2}) + s^2d(x_{m+2}, x_{m+p})
\]
\[
\leq sd(x_m, x_{m+1}) + s^2d(x_{m+1}, x_{m+2}) + s^3d(x_{m+2}, x_{m+3}) + \ldots
\]
\[
\qquad + s^{p-1}d(x_{m+p-2}, x_{m+p-1}) + s^{p-1}d(x_{m+p-1}, x_{m+p})
\]
\[
\leq \lambda^m d(x_1, x_0) + s^2\lambda^{m+1}d(x_1, x_0) + s^3\lambda^{m+2}d(x_1, x_0) + \ldots
\]
\[
\qquad + s^{p-1}\lambda^{m+p-2}d(x_1, x_0) + s^{p-1}\lambda^{m+p-1}d(x_1, x_0)
\]
\[
\leq \lambda^m (1 + \lambda + s\lambda^2 + \ldots + s^{p-2}\lambda^{p-2} + s^{p-1}\lambda^{p-1})d(x_1, x_0)
\]
\[
\leq \lambda^m (1 + \lambda + (s\lambda)^2 + \ldots + (s\lambda)^{p-2} + (s\lambda)^{p-1} + \ldots)d(x_1, x_0)
\]
\[
= \lambda^m \frac{1}{1 - s\lambda}d(x_1, x_0), \text{ where } s\lambda < 1
\]

Therefore, $\{x_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{x_n\}$ converges to some element $x$ in $A$. Therefore,
\[
d(x, B) \leq d(x, Tx_n) = \lim_{n \to \infty} d(x_n, 1, Tx_n) = d(x, B) \leq d(x, B).
\]
Therefore, $d(x, Tx_n) \to d(x, B)$. Since $B$ is approximately compact with respect to $A$, the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ converging to some element $y$ in $B$. So, it results that
\[
d(x, y) = \lim_{n \to \infty} d(x_{n_k+1}, Tx_{n_k}) = d(A, B).
\]
Hence $x$ must be a member of $A_0$. Because of the fact that $T(A_0)$ is contained in $B_0$
\[
d(u, Tx) = d(A, B)
\]
(3.20)
for some element $u$ in $A$. Since $T$ is a generalized proximal contraction of the first kind, and $d(u, Tx) = d(A, B)$ and $d(x_{n+1}, Tx_n) = d(A, B)$, it follows that
\[
d(u, x_{n+1}) \leq d(x, x_n) + \beta d(u, x) + \gamma d(x, x_{n+1}) + \delta[d(x, x_{n+1}) + d(x_n, u)].
\]
Letting $n \to \infty$,
\[
d(u, x) \leq (\beta + \delta)d(u, x)
\]
This implies $u = x$.
Thus, it follows from (3.20) that $d(x, Tx) = d(A, B)$. Let us assume that there is another best proximity point $x^*$ in $A$ so that
\[
d(x^*, Tx^*) = d(A, B).
\]
Since $T$ is a generalized proximal contraction of the first kind, it follows that
\[
d(x, x^*) \leq \alpha d(x, x^*) + \beta d(x, x^*) + \gamma d(x, x^*) + \delta[d(x, x^*) + d(x, x^*)]d(x, x^*) \leq (\alpha + 2\delta)d(x, x^*)
\]
(α + 2δ) < 1 implies x = x∗.

\[ \boxdot \]

**Corollary 3.1.** Let A and B be non-empty, closed subsets of a complete b-metric space \((X, d)\) with \(s \geq 1\) such that B is approximately compact with respect to A. Also suppose that \(A_0\) and \(B_0\) are non-empty. Let \(T : A \to B\) satisfy the following conditions:

(a) There exist a non-negative real number \(α\) with \(sα < 1\) such that, for all \(u_1, u_2, x_1, x_2\) in A, \(d(u_1, Tx_1) = d(A, B)\) and \(d(u_2, Tx_2) = d(A, B)\) imply the inequality that \(d(u_1, u_2) \leq αd(x_1, x_2)\).

(b) \(T(A_0)\) is contained in \(B_0\).

Then, there exists a unique element \(x\) in A such that

\[ d(x, Tx) = d(A, B) \tag{3.21} \]

and the sequence \(\{x_n\}\) converges to the best proximity point \(x\), where \(x_0\) is any fixed element in \(A_0\) and \(d(x_{n+1}, Tx_n) = d(A, B)\) for \(n \geq 0\).

**Corollary 3.2.** Let \(T\) be a self-mapping on a complete b-metric space \((X, d)\) with \(s \geq 1\). Further, let us assume that there exist non-negative real numbers \(α, β, γ, δ\) such that \(s(α + β + s(1 + γ + δ) < 1\) and

\[ d(Tx_1, Tx_2) \leq αd(x_1, x_2) + βd(x_1, Tx_1) + γd(x_2, Tx_2) + δd(x_1, Tx_2) + d(x_2, Tx_1) \tag{3.22} \]

for all \(x_1, x_2\) in the domain of the mapping \(T\). Then the mapping \(T\) has a unique fixed point.

The following main result is a best proximity point theorem for non-self generalized proximal contractions of the second kind in b-metric space.

**Theorem 3.3.** Let A and B be non-empty, closed subsets of a complete b-metric space \((X, d)\) with \(s \geq 1\) such that A is approximately compact with respect to B. Also suppose that \(A_0\) and \(B_0\) are non-empty. Let \(T : A \to B\) satisfy the following conditions:

(a) \(T\) is a continuous generalized proximal contraction of second kind,

(b) \(T(A_0)\) is contained in \(B_0\).

Then, there exists \(x\) in A such that

\[ d(x, Tx) = d(A, B) \tag{3.23} \]

**Proof.** Let us select an element \(x_0\) in \(A_0\). On account of the fact \(T(A_0)\) is contained in \(B_0\), it is guaranteed that there is an element \(x_1\) in \(A_0\) satisfying the condition that

\[ d(x_1, Tx_0) = d(A, B) \]

Further, since \(Tx_1\) is a member of \(T(A_0)\) which is contained in \(B_0\), it follows that there is an element \(x_2\) in \(A_0\) such that

\[ d(x_2, Tx_1) = d(A, B) \]

This process can be continued further. Having chosen \(x_n\) in \(A_0\), there exist an element \(x_{n+1}\) in \(A_0\) satisfying the condition that

\[ d(x_{n+1}, Tx_n) = d(A, B) \]

International Scientific Publications and Consulting Services
for every non-negative integer \( n \). In view of the fact that \( T \) is a generalized proximal contraction of second kind, we have that

\[
d(T_{x_n}, T_{x_{n+1}}) \leq \alpha d(T_{x_{n-1}}, T_{x_n}) + \beta d(T_{x_{n-1}}, T_{x_{n+1}}) + \gamma d(T_{x_{n}}, T_{x_{n+1}}) + \delta d(T_{x_{n-1}}, T_{x_{n}}) + \delta d(T_{x_{n}}, T_{x_{n+1}})
\]

\[
\leq \alpha d(T_{x_{n-1}}, T_{x_n}) + \beta d(T_{x_{n-1}}, T_{x_{n+1}}) + \gamma d(T_{x_{n}}, T_{x_{n+1}}) + \delta d(T_{x_{n}}, T_{x_{n+1}})
\]

\[
(1 - \gamma - s\delta) d(T_{x_{n-1}}, T_{x_n}) \leq (\alpha + \beta + s\delta) d(T_{x_{n-1}}, T_{x_n})
\]

\[
d(T_{x_n}, T_{x_{n+1}}) \leq \frac{\alpha + \beta + s\delta}{1 - \gamma - s\delta} d(T_{x_{n-1}}, T_{x_n})
\]

where \( \lambda = \frac{\alpha + \beta + s\delta}{1 - \gamma - s\delta} < 1 \).

Repeating this process we obtain

\[
d(T_{x_n}, T_{x_{n+1}}) \leq \lambda^n d(T_{x_0}, T_{x_1}). \tag{3.24}
\]

For any \( m \geq 1, \ p \geq 1 \) it follows from (3.24)

\[
d(T_{x_m}, T_{x_{m+p}}) \leq \sum_{k=0}^{p-1} d(T_{x_{m+k}}, T_{x_{m+k+1}}) + d(T_{x_{m+p}}, T_{x_{m+p-1}})
\]

\[
= s^k d(T_{x_{m+k}}, T_{x_{m+k+1}}) + s^k d(T_{x_{m+k+1}}, T_{x_{m+k+2}}) + d(T_{x_{m+p}}, T_{x_{m+p-1}})
\]

\[
\leq s^n d(T_{x_0}, T_{x_1}) + s^n d(T_{x_0}, T_{x_1}) + s^n d(T_{x_0}, T_{x_1}) + \ldots
\]

\[
= \lambda^m d(T_{x_0}, T_{x_1}) + \lambda^{m+1} d(T_{x_0}, T_{x_1}) + \lambda^{m+2} d(T_{x_0}, T_{x_1}) + \ldots
\]

\[
= \lambda^m d(T_{x_0}, T_{x_1}) + \lambda^{m+1} d(T_{x_0}, T_{x_1}) + \lambda^{m+2} d(T_{x_0}, T_{x_1}) + \ldots
\]

\[
= \lambda^m (1 + \lambda + \lambda^2 + \ldots + \lambda^{p-2}) + \lambda^{p-1} d(T_{x_0}, T_{x_1})
\]

\[
= \lambda^m (1 + \lambda + (s\lambda)^2 + \ldots + (s\lambda)^{p-2} + (s\lambda)^{p-1}) d(T_{x_0}, T_{x_1})
\]

\[
= \lambda^m \frac{1}{1 - s\lambda} d(T_{x_0}, T_{x_1}), \text{ where } s\lambda < 1.
\]

Hence

\[
\lim_{m \to \infty} d(T_{x_m}, T_{x_{m+p}}) = 0 \text{ for all } p > 0. \tag{3.25}
\]

Therefore, \( \{T_{x_n}\} \) is a Cauchy sequence. Because the space is complete, the sequence \( \{T_{x_n}\} \) converges to some element \( y \) in \( B \). Therefore,

\[
d(y, A) \leq d(y, x_{n+1}) = \lim_{m \to \infty} d(T_{x_m}, x_{n+1}) = d(A, B) \leq d(y, A).
\]
Therefore, \( d(y,x_{n+1}) \to d(y,A) \). Since \( A \) is approximately compact with respect to \( B \), the sequence \( \{x_n\} \) has a subsequence \( \{x_{n_k}\} \) converging to some element \( x \) in \( A \). Since \( T \) is a continuous mapping,

\[
d(x,Tx) = \lim_{n \to \infty} d(x_{n_k+1},Tx_{n_k}) = d(A,B).
\]

Corollary 3.3. Let \( A \) and \( B \) be non-empty, closed subsets of a complete \( b \)-metric space \( (X,d) \) with \( s \geq 1 \) such that \( B \) is approximately compact with respect to \( A \). Also suppose that \( A_0 \) and \( B_0 \) are non-empty. Let \( T : A \to B \) satisfy the following conditions:

(a) There exists a non-negative real number \( \alpha \) with \( \alpha < 1 \) such that, for all \( u_1,u_2,x_1,x_2 \) in \( A \), \( d(u_1,Tx_1) = d(A,B) \) and \( d(u_2,Tx_2) = d(A,B) \) imply the inequality that \( d(Tu_1,Tu_2) \leq \alpha d(Tx_1,Tx_2) \).

(b) \( T(A_0) \) is contained in \( B_0 \).

Then, there exists a unique element \( x \) in \( A \) such that

\[
d(x,Tx) = d(A,B)
\]

Theorem 3.4. Let \( A \) and \( B \) be non-empty, closed subsets of a complete \( b \)-metric space \( (X,d) \) with \( s \geq 1 \). Also suppose that \( A_0 \) and \( B_0 \) are non-empty. Let \( T : A \to B \) satisfy the following conditions:

(a) \( T \) is a generalized proximal contraction of the first kind as well as a generalized proximal contraction of the second kind.

(b) \( T(A_0) \) is contained in \( B_0 \).

Then, there exists a unique element \( x \) in \( A \) such that

\[
d(x,Tx) = d(A,B)
\]

and the sequence \( \{x_n\} \) converges to the best proximity point \( x \), where \( x_0 \) is any fixed element in \( A_0 \) and \( d(x_{n+1},Tx_n) = d(A,B) \) for \( n \geq 0 \).

Proof. Proceeding as in Theorem 3.2, it is possible to find a sequence \( \{x_n\} \) in \( A_0 \) such that

\[
d(x_{n+1},Tx_n) = d(A,B)
\]

for all non-negative integer \( n \). As in Theorem 3.2, it can be shown that the sequence \( \{x_n\} \) is a Cauchy sequence and hence converges to some element \( x \) in \( A \). Further, as in Theorem 3.3, it can be asserted that the sequence \( \{Tx_n\} \) is a Cauchy sequence and hence converges to some element \( y \) in \( B \). Therefore, it follows that

\[
d(x,y) = \lim_{n \to \infty} d(x_{n+1},Tx_n) = d(A,B).
\]

Hence, \( x \) becomes an element of \( A_0 \). Since \( T(A_0) \) is contained in \( B_0 \),

\[
d(u,Tx) = d(A,B)
\]

for some element \( u \) in \( A \). Since \( T \) is a generalized proximal contraction of the first kind, we have

\[
d(u,x_{n+1}) = \alpha d(x,x_n) + \beta d(u,x) + \gamma d(x_n,x_{n+1}) + \delta [d(x,x_{n+1}) + d(x,u)].
\]

Letting \( n \to \infty \), \( d(u,x) \leq (\beta + \delta) d(u,x) \), which implies that \( x \) and \( u \) must be identical. Thus, it follows that

\[
d(x,Tx) = d(u,Tx) = d(A,B).
\]
Acknowledgements

The authors are grateful to the Editor-in-Chief for his support and the referee(s) for giving useful suggestions.

References

http://dx.doi.org/10.1016/j.jmaa.2005.10.081

http://dx.doi.org/10.1016/j.na.2008.07.022

http://dx.doi.org/10.1016/j.topol.2009.01.017

http://dx.doi.org/10.1016/j.na.2007.10.014

http://dx.doi.org/10.1155/2009/197308


http://dx.doi.org/10.1080/01630563.2010.485713

http://dx.doi.org/10.1186/1687-1812-2012-42


http://dx.doi.org/10.1016/j.na.2009.01.173

http://dx.doi.org/10.1016/j.na.2010.06.008