Fixed Point Theorems for Nonself Asymptotically Nonexpansive type Mappings in CAT(0) Spaces

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Abstract
In this paper, we propose a three-step iteration scheme for nonself asymptotically nonexpansive type mappings in a complete CAT(0) metric space and establish necessary and sufficient conditions for convergence of this process to a fixed point of nonself asymptotically nonexpansive type mappings. We also establish a strong convergence result. These results generalize and unify many important results in the literature.

Keywords: CAT(0) space, nonself asymptotically nonexpansive type mapping, fixed points, strong convergence, iteration process.

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1 Introduction

Let C be a nonempty subset of metric space (X,d). A mapping T : C → C is said to be (i) Lipschitzian if d(Tx,Ty) ≤ kd(x,y) for all x,y ∈ C, k ≥ 0, (ii) nonexpansive if d(Tx,Ty) ≤ d(x,y) for all x,y ∈ C (iii) asymptotically nonexpansive if there exists a sequence {kₙ} in [1,∞) with limₙ→∞ kₙ = 1 such that d(Tⁿₓ,Tⁿᵧ) ≤ kₙd(x,y) for all x,y in C and n ∈ N, where N denotes the set of positive integers. Class of asymptotically nonexpansive mappings includes a class of nonexpansive mappings as a proper subclass [8], and both the mappings are Lipschitzian.

In 1974, Kirk [11] substantially weaken the assumption of asymptotic nonexpansiveness of T by replacing it with an assumption, which may hold even if none of the iterates of T is Lipschitzian. A mapping T : C → C is said to be asymptotically nonexpansive type if for each y ∈ C the following inequality holds:

\[ \limsup_{n \to \infty} \left( \sup_{x \in C} (d(T^n x, T^n y) - d(x,y)) \right) \leq 0. \]  

(1.1)

Every asymptotically nonexpansive mapping satisfies (1.1), but converse need not be true [11, p.345]. The concept of asymptotically nonexpansive type mappings is more general than that of asymptotically nonexpansive mappings. Iterative approximation of fixed points of nonexpansive, asymptotically nonexpansive and asymptotically nonexpansive type mappings have been studied by various authors in the setting of Hilbert spaces, Banach spaces and convex metric spaces, see [10, 21–26] and reference therein.

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When it is unique this geodesic segment is denoted by $\triangle x_1 x_2 x_3$. A geodesic triangle of its points.

2 Preliminaries

Let $(X,d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0,1] \subseteq \mathbb{R}$ to $X$ such that $c(0) = x$, $c(1) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,1]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique this geodesic segment is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle (x_1, x_2, x_3)$ in a geodesic metric space $(X,d)$ consists of three points $x_1, x_2, x_3$ in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle (x_1, x_2, x_3)$ in $(X,d)$ is a triangle $\overline{\triangle} (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2} (\hat{x}_i, \hat{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in $X$ and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\hat{x}, \hat{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2} (\hat{x}, \hat{y}).$$

If $x, y_1, y_2$ are points in a CAT(0) space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality [1, p.163].

CAT(0) spaces may be regarded as a metric version of Hilbert spaces. For example, in any Hilbert space $H$ we have the following extended version of parallelogram law:

$$\|z - (\alpha x + (1 - \alpha)y)\|^2 = \alpha \|z - x\|^2 + (1 - \alpha) \|z - y\|^2 - \alpha(1 - \alpha) \|x - y\|^2$$

for any $\alpha \in [0,1]$ and $x, y, z \in H$.

Any CAT(0) metric space $X$ has the inequality

$$d(z, \alpha x + (1 - \alpha)y)^2 \leq \alpha d(x, z)^2 + (1 - \alpha) d(z, y)^2 - \alpha(1 - \alpha) d(x, y)^2$$

(2.2)

for any $\alpha \in [0,1]$, $x, y \in X$.

If $\alpha = \frac{1}{2}$ then the inequality (2.2) becomes the (CN) inequality.

Lemma 2.1. [6] Let $(X,d)$ be a CAT(0) space, then:
(i) For \(x, y \in X\) and \(t \in [0, 1]\), there exists a unique point \(z \in [x, y]\) such that
\[
d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1-t)d(x, y).
\]  
(2.3)

We use the notation \((1-t)x \oplus ty\) for the unique point \(z\) satisfying (2.3).

(ii) For \(x, y, z \in X\) and \(t \in [0, 1]\), we have
\[
d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).
\]  
(2.4)

We shall denote with \(\text{Fix}(T)\) the set of fixed points of a mapping \(T\). A mapping \(P : X \to C\) is said to be retraction if \(C \subset X\) and \(P\) restricted to \(C\) is the identity, i.e. \(Px = x\) for any \(x \in C\). Clearly \(P^2 = P\), the set \(C\) is called a retract of \(X\).

**Lemma 2.2.** [1, Chapter II.2, Proposition 2.4] Let \(C\) be a convex subset of \(X\) which is complete in the induced metric. Then, for every \(x \in X\), there exists a unique point \(P(x) \in C\) such that \(d(x, P(x)) = d(x, C) := \inf\{d(x, y) : y \in C\}\). Moreover, the map \(x \to P(x)\) is a nonexpansive retract from \(X\) onto \(C\).

We now define nonself asymptotically nonexpansive type mapping in a CAT(0) space.

**Definition 2.1.** Let \(C\) be a nonempty subset of a CAT(0) space \(X\). A map \(T : C \to X\) is said to be asymptotically nonexpansive type, if for each \(y \in C\),
\[
\limsup_{n \to \infty} \sup_{x \in C} \left\{d(T^nx, T^ny) - d(x, y)\right\} \leq 0,
\]  
(2.5)

where \(P\) is the nonexpansive retraction of \(X\) onto \(C\).

Glowinski and Le Tallec [7] used three-step iterative schemes to find the approximate solutions of the elasto-visco-plasticity problem, liquid crystal theory and eigenvalue computation, and shown that the three-step iterative schemes gives better numerical results than the two-step and one-step iterative schemes. Noor [19] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces. Xu and Noor [27] studied three-step iterative scheme to approximate fixed points of asymptotically nonexpansive self mappings in Banach spaces. Motivated by the above facts, we now consider following three-step iteration for nonself asymptotically nonexpansive type mapping as follows:

\[
\begin{cases}
x_0 \in C, \\
x_{n+1} = P \left((1-\alpha_n)x_n \oplus \alpha_n T^ny_n\right) \\
y_n = P \left((1-\beta_n)x_n \oplus \beta_n T^ny_n\right) \\
z_n = P \left((1-\gamma_n)x_n \oplus \gamma_n T^ny_n\right)
\end{cases}
\]  
(2.6)

where \(\{\alpha_n\}, \{\beta_n\}\) and \(\{\gamma_n\}\) are sequences in \([\epsilon, 1-\epsilon]\) for some \(\epsilon \in (0, 1)\).

In the next section, we establish necessary and sufficient conditions for strong convergence of the sequence \(\{x_n\}\) given by (2.6) to a fixed point of nonself asymptotically nonexpansive type mapping.

### 3 Main results

**Lemma 3.1.** Let \(C\) be a nonempty convex subset of a CAT(0) space \(X\) and let \(T : C \to X\) be a mapping of asymptotically nonexpansive type with \(\text{Fix}(T) \neq \emptyset\) and \(\{x_n\}\) be sequence defined by (2.6). If \(x^* \in \text{Fix}(T)\), then
\[
d(x_{n+1}, x^*) \leq d(x_n, x^*) + 3 \sup_{x \in C} \left\{d(T^nx, x^*) - d(x, x^*)\right\}; \quad n \in \mathbb{N}.
\]

**Proof.** Let \(x^* \in \text{Fix}(T)\). Using (2.6) and (2.4), we have
Proof. We need following lemma to prove next result.

Similarly, we get

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Substituting (3.9) into (3.8), we get

Substituting (3.10) into (3.7), we get

This completes the proof.

We need following lemma to prove next result.

Lemma 3.2. [24] Suppose that \( \{a_n\} \) and \( \{b_n\} \) are two sequences of nonnegative numbers such that \( a_{n+1} \leq a_n + b_n \) for all \( n \geq 1 \). If \( \sum_{n=1}^{\infty} b_n \) converges, then \( \lim_{n \to \infty} a_n \) exists.

Lemma 3.3. Let \( C \) be a nonempty convex subset of a complete CAT(0) space \( X \) and let \( T : C \to X \) be a mapping of asymptotically nonexpansive type with \( \text{Fix}(T) \neq \emptyset \) and sequence \( \{x_n\} \) be defined by (2.6). Then

(i) \( \lim_{n \to \infty} d(x_n, x^*) \) exists for all \( x^* \in \text{Fix}(T) \);

(ii) \( \lim_{n \to \infty} d(x_n, \text{Fix}(T)) \) exists.

Proof. Since \( T \) is asymptotically nonexpansive type mapping, for each \( x^* \in \text{Fix}(T) \), we have

\[
\limsup_{n \to \infty} \left[ \sup_{x \in C} \left( d \left( T \left( P(T)^{n-1} x \right), T \left( P(T)^{n-1} x^* \right) \right) - d \left( x, x^* \right) \right) \right] \leq 0,
\]
we can choose \( n_0 \) such that \( n \geq n_0 \) imply
\[
\sup_{k \geq n} \left( \sup_{x \in C} \left\{ d \left( (T^n PT)^{k-1} x, x^* \right) \right\} \right) \leq \frac{1}{3n^2}.
\]

By Lemma 3.1, we have
\[
d(x_{m+n}, x^*) \leq d(x_m, x^*) + \frac{1}{m^2}.
\]

Therefore for \( n, m \geq n_0 \), we have
\[
d(x_{m+n}, x^*) \leq d(x_m, x^*) + \sum_{i=n}^{n+m-1} \frac{1}{i^2}.
\]

Taking infimum over all \( x^* \in \text{Fix}(T) \), we have
\[
d(x_{m+n}, \text{Fix}(T)) \leq d(x_m, \text{Fix}(T)) + \sum_{i=n}^{n+m-1} \frac{1}{i^2}.
\]

If follows from Lemma 3.2, (3.11) and (3.12) that
\[
\lim_{n \to \infty} d(x_n, x^*) \quad \text{and} \quad \lim_{n \to \infty} d(x_n, \text{Fix}(T))
\]
exists.

**Lemma 3.4.** Let \( C \) be a nonempty closed convex subset of a complete CAT(0) space \( X \), let \( T : C \to X \) be asymptotically nonexpansive type mapping with \( \text{Fix}(T) \neq \emptyset \), and the sequence \( \{x_n\} \) is defined by (2.6). If \( \lim_{n \to \infty} d(x_n, \text{Fix}(T)) = 0 \), then \( \{x_n\} \) is a Cauchy sequence.

**Proof.** Since \( \lim_{n \to \infty} d(x_n, \text{Fix}(T)) = 0 \), then for all \( \epsilon > 0 \) there exists \( k(\epsilon) \in \mathbb{N} \) such that for all \( n \geq k(\epsilon) \)
\[
d(x_n, \text{Fix}(T)) < \frac{\epsilon}{2},
\]
which implies that there exist \( x^* \in \text{Fix}(T) \) such that for all \( n \geq k(\epsilon) \)
\[
d(x_n, x^*) < \frac{\epsilon}{2}.
\]

Since the sequence \( \{d(x_n, x^*)\} \) is nonincreasing, we have for \( m, n \geq k(\epsilon) \) that
\[
d(x_n, x_m) = d(x_n, x^*) + d(x_m, x^*) = 2d(x_{k(\epsilon)}, x^*) < \epsilon,
\]
which shows that \( \{x_n\} \) is a Cauchy sequence.

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of a complete CAT(0) space \( X \), let \( T : C \to X \) be asymptotically nonexpansive type mapping with \( \text{Fix}(T) \) be a nonempty closed set. Then the sequence \( \{x_n\} \) defined by (2.6) converges strongly to a point in \( \text{Fix}(T) \) if and only if \( \liminf_{n \to \infty} d(x_n, \text{Fix}(T)) = 0 \).

**Proof.** Necessity is obvious. Thus, we will only prove the sufficiency. So assume that \( \liminf_{n \to \infty} d(x_n, \text{Fix}(T)) = 0 \), also \( \lim_{n \to \infty} d(x_n, \text{Fix}(T)) \) exists by Lemma 3.3 (ii). Hence we get \( \lim_{n \to \infty} d(x_n, \text{Fix}(T)) = 0 \). Using Lemma 3.4, we get that \( \{x_n\} \) is a Cauchy sequence in a closed subset \( C \) of a complete CAT(0) space and therefore converges to some \( q \in C \). Since \( \lim_{n \to \infty} d(x_n, \text{Fix}(T)) = 0 \) we get that \( d(q, \text{Fix}(T)) = 0 \), by closedness of \( \text{Fix}(T) \), we get that \( q \in \text{Fix}(T) \). This completes the proof.

A mapping \( T : C \to X \) is said to satisfy \textit{Condition-A*} if there exists a nondecreasing function \( f : [0, \infty) \to [0, \infty] \) with \( f(0) = 0 \) and \( f(r) > 0 \) for all \( r \in (0, \infty) \) and \( d(x, Tx) \geq f(d(x, \text{Fix}(T))) \), for all \( x \in C \).
If $T : C \rightarrow C$, then the Condition-A$^*$ reduces to the Condition-I of Senter and Dotson [20].

We now give strong convergence result employing Condition-A$^*$.

**Theorem 3.2.** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$, $T : C \rightarrow X$ be asymptotically nonexpansive type mapping with $\text{Fix}(T)$ be a nonempty closed set, and $\{x_n\}$ be a sequence defined by (2.6). If $\{x_n\}$ be approximate fixed point sequence for $T$, i.e. $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $T$ satisfy Condition-A$^*$, then $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof.** By Lemma 3.3, for any $x^* \in \text{Fix}(T)$, we see that

$$\lim_{n \rightarrow \infty} d(x_n, x^*) \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, \text{Fix}(T))$$

exist. Let $\lim_{n \rightarrow \infty} d(x_n, \text{Fix}(T)) = r$ for some $r \geq 0$.

Now in view of Theorem 3.1, to complete the proof we must show that $r = 0$.

Since $T$ satisfies Condition-A$^*$, so we have

$$d(x_n, Tx_n) \geq f(d(x_n, \text{Fix}(T))) \geq f(r).$$

Since $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, we get that $f(r) = 0$ and so $r = 0$.

This completes the proof. \qed

**Remark 3.1.** Any CAT($\kappa$) space is a CAT($\kappa'$) space for every $\kappa' > \kappa$ [1, page 165], therefore the results in this paper can be applied to any CAT($\kappa$) space with $\kappa < 0$.

**References**


http://dx.doi.org/10.1007/978-3-662-12494-9


http://dx.doi.org/10.1007/978-1-4612-1019-1


http://dx.doi.org/10.1007/BF02715544


http://dx.doi.org/10.1016/j.jmaa.2005.03.055


http://dx.doi.org/10.1016/j.camwa.2008.05.036


http://dx.doi.org/10.1137/1.9781611970838


http://dx.doi.org/10.1090/S0002-9939-1972-0298500-3


http://dx.doi.org/10.1007/BF02757136


http://dx.doi.org/10.1016/j.na.2007.04.011


http://dx.doi.org/10.1155/2010/367274

http://dx.doi.org/10.1155/2010/268780

http://dx.doi.org/10.1006/jmaa.2000.7042

http://dx.doi.org/10.1090/S0002-9939-1974-0346608-8

http://dx.doi.org/10.1006/jmaa.1994.1135

http://dx.doi.org/10.1016/0022-247X(91)90245-U

http://dx.doi.org/10.1017/S0004972700028884

http://dx.doi.org/10.1006/jmaa.1993.1309
http://dx.doi.org/10.1016/j.amc.2007.01.101


http://dx.doi.org/10.1006/jmaa.2001.7649