Weak* almost Dunford-Pettis operators in Banach lattices

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Abstract

We introduce and study the class of weak* almost Dunford-Pettis operators. As an application, we characterize Banach lattices with the weak DP* property. Also, we establish some sufficient conditions under which the class of order bounded weak* almost Dunford-Pettis operator coincide with that of almost limited. Finally, we derive some interesting results.

Keywords: Weak* Dunford-Pettis operator, Weak* almost Dunford-Pettis operator, Almost limited operator, Almost limited set, Weak Dp* property.

1 Introduction

Throughout this paper E and F will denote real Banach lattices. B_E is the closed unit ball of E and sol (A) denotes the solid hull of a subset A of a Banach lattice. We will use the term operator T : E → F between two Banach spaces to mean a bounded linear mapping. It is positive if T(x) ≥ 0 in F whenever x ≥ 0 in E. The objective of this paper is to study the class of weak* almost Dunford-Pettis operators. Also, we derive the following interesting consequences:

1. some characterizations of the class of weak* almost Dunford-Pettis operators;
2. some characterizations of the weak DP* property;
3. the coincidence of this class of operators with that of almost limited operators;
4. the domination property of the class of weak* almost Dunford-Pettis operators.

To state our results, we need to fix some notation and recall some definitions. A Riesz space (or a vector lattice) is an ordered vector space E with the additional property that for each pair of vectors x, y ∈ E the supremum and the infimum of the set \{x, y\} both exist in E. Following the classical notation, we shall write \( x \vee y := \sup \{x, y\} \) and \( x \wedge y := \inf \{x, y\} \). A Banach lattice E is a Banach space \((E, \|\|)\) such that E is a vector lattice and its norm satisfies the following property:

For each x, y ∈ E such that \( |x| \leq |y| \), we have \( \|x\| \leq \|y\| \).

If E is a Banach lattice, its topological dual \( E' \), endowed with the dual norm, is also a Banach lattice. A norm \( \|\| \) of a Banach lattice E is order continuous if for each generalized nets \((x_\alpha)\) such that \( (x_\alpha) \nexists 0 \) in E, \((x_\alpha)\) converges to 0 for the norm \( \|\| \) where the notation \( (x_\alpha) \nexists 0 \) means that the \( (x_\alpha) \) is decreasing, its infimum exists and \( \inf (x_\alpha) = 0 \). A Riesz space is said to be σ-Dedekind complete if every countable subset that is bounded above has a supremum (or, equivalently, whenever \( 0 \leq x_\alpha \uparrow \leq x \) implies the existence of \( \sup \{x_\alpha\} \)). The lattice operations in a Banach lattice E are weakly sequentially continuous if for every weakly null sequence \((x_\alpha)\) in E, \( |x_\alpha| \to 0 \) for σ(E, E'). The lattice operations in a Banach lattice \( E' \) are weakly sequentially continuous if for every weak* null sequence \((f_\alpha)\) in \( E' \), \( |f_\alpha| \to 0 \)

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for $\sigma(E',E)$. We refer the reader to [1] and [6] for unexplained terminologies on Banach lattice theory and positive operators.

2 A brief review

To be able to obtain the main results, a brief review of the literature is presented. As many Banach spaces do not have the Dunford-Pettis property, a weak notion is introduced, called the weak Dunford-Pettis property. A Banach space $E$ has the Dunford-Pettis property if every weakly compact operator defined on $E$ (and taking their values in a Banach space $F$) is Dunford-Pettis. Similarly, a Banach lattice $E$ has the weak Dunford-Pettis property if every weakly compact operator defined on $E$ is almost Dunford-Pettis, that is, the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence $(x_n)$ consisting of pairwise disjoint elements in $E$. It is obvious that if $E$ has the Dunford-Pettis property, then it has the weak Dunford-Pettis property. Let us recall from [1] that an operator $T$ from a Banach space $X$ into another $Y$ is called weak Dunford-Pettis if the sequence $(f_n(T(x_n)))$ converges to 0 whenever $(x_n)$ converges weakly to 0 in $X$ and $(f_n)$ converges weakly to 0 in $Y^*$. Alternatively, $T$ is weak Dunford-Pettis if $T$ maps relatively weakly compact sets of $X$ into Dunford-Pettis sets of $Y$ (see Theorem 5.99 of [1]). A norm bounded subset $A$ of a Banach lattice $E$ is said to be Dunford-Pettis if every weakly null sequence $(f_n)$ of $E^*$ (the topological dual of $E$) converges uniformly to zero on the set $A$, that is, $sup_{x\in A}\|f_n(x)\| \rightarrow 0$ (see Theorem 5.98 of [1]). A norm bounded subset $A$ of a Banach space $X$ is said to be limited if every weak* null sequence $(x_n)$ of $X^*$ converges uniformly on $A$, that is, $sup_{x\in A}\|f(x_n)\| \rightarrow 0$. Note that every relatively compact set is limited but the converse is not true in general. In fact, the set $\{e_n : n \in \mathbb{N}\}$ of unit coordinate vectors is a limited set in $l^\infty$ which is not relatively compact. Also we recall from [3] that a norm bounded subset $A$ of $E$ is said to be an almost limited set if every disjoint, weak* null sequence $(f_n)$ of $E^*$ converges uniformly to zero on $A$, that is, $sup_{x\in A}\|f_n(x)\| \rightarrow 0$. A Banach lattice $E$ has the dual positive Schur property, if every positive weak* null sequence in $E$ converges uniformly to zero on the set $A$, that is, $sup_{x\in A}\|f(x)\| \rightarrow 0$ (see Theorem 5.98 of [1]). A norm bounded subset $A$ of a Banach space $X$ is said to be almost limited in $F$.

3 Main results

Following Zi Li Chen [3] a norm bounded subset $A$ of $E$ is said to be an almost limited set if every disjoint, weak* null sequence $(f_n)$ of $E^*$ converges uniformly to zero on $A$, that is, $sup_{x\in A}\|f_n(x)\| \rightarrow 0$. Also by [3] an operator $T$ from a Banach space $X$ into a Banach lattice $F$ is said to be almost limited if the set $T(B_X)$ is almost limited in $F$.

**Theorem 3.1.** For an operator $T$ from a Banach space $X$ into a Banach lattice $F$, the following statements are equivalent:

1. $T$ is a weak* almost Dunford-Pettis operator.
(2) If $S$ is a weakly compact operator from an arbitrary Banach space $Z$ into $X$, then the operator product $T \circ S$ is almost limited.

(3) If $S$ is a weakly compact operator from $l^1$ into $X$, then the operator product $T \circ S$ is almost limited.

(4) For all weakly null sequence $(x_n) \subseteq X$, and for all disjoint weak* null sequence $(f_n)$ in $F^*$ it follows that $f_n(T(x_n)) \to 0$.

Proof. (1 $\to$ 2). Since $S$ is a weakly compact operator then $S(B_Z)$ is a weakly compact set. On the other hand, as $T$ is a weak* almost Dunford-Pettis operator, $T(S(B_Z))$ is an almost limited set and hence $T \circ S$ is almost limited.

(2 $\to$ 3). Obvious.

(3 $\to$ 4). Let $(x_n)$ in $X$ be a weakly null sequence and let $(f_n)$ in $F^*$ be a disjoint weak* null sequence. By Theorem 5.26 [1], the operator $S : l^1 \to X$ defined by $S((\lambda_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \lambda_i x_i$ for each $(\lambda_i)_{i=1}^{\infty} \in l^1$, is weakly compact. Thus, by our hypothesis $T \circ S$ is almost limited and hence $\|(T \circ S)(f_n)\| \to 0$ and the desired conclusion follows from the inequality

$$|f_n(T(x_n)) - f_n(T(S(e_n))))| \leq \sup_{(\lambda_i) \in B_1} |f_n(TS(\lambda_i))| = \|(T \circ S)(f_n)\|$$

for each $n$, where $(e_i)_{i=1}^{\infty}$ is the canonical basis of $l^1$.

(4 $\to$ 1). Let $W$ be a relatively weakly compact subset of $X$, and let $(f_n)$ be a disjoint weak* null sequence in $F^*$. If $(f_n)$ does not converge uniformly to zero on $T(W)$, then there exist a sequence $(x_n)$ of $W$, a subsequence of $(f_n)$ (which we shall denote by $(f_n)$ again), and some $\varepsilon > 0$ satisfying $|f_n(T(x_n))| > \varepsilon$ for all $n$. Since $W$ is weakly compact, we can assume that $(x_n) \to x$ weakly in $X$. Then $T(x_n) \to T(x)$ weakly in $F$ and so, by our hypothesis, we have

$$0 < \varepsilon < |f_n(T(x_n))| \leq |f_n(T(x_n) - x)| + |f_n(T(x))| \to 0,$$

which is impossible. Thus $(f_n)$ converges uniformly to zero on $T(W)$ and this shows that $T(W)$ is an almost limited set. This ends the proof of the Theorem.

Let us recall that, an operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is said to be order bounded if for each $z \in E^+$, the set $T([-z, z])$ is an order bounded set in $F$. Each order interval $[-z, z]$ of a $\sigma$-Dedekind complete Banach lattice $E$ is an almost limited set for each $z \in E^+$. In fact, if $(f_n)$ is a disjoint weak* null sequence in $E^*$, then by [7], $(|f_n|)$ is a weak* null sequence in $E^*$. Hence $\sup_{x \in [-z, z]} |f_n(x)| = |f_n(z)| \to 0$ for each $z \in E^+$. As a consequence, if $T : E \to F$ is an order bounded operator from a Banach lattice $E$ into a $\sigma$-Dedekind complete Banach lattice $F$, then $T([-z, z])$ is an almost limited set in $F$, and then $|f_n o T|(z) = \sup_{y \in T([-z, z])} |f_n(y)| \to 0$ for all disjoint weak* null sequence $(f_n)$ in $F^*$ and for each $z \in E^+$.

**Theorem 3.2.** Let $E$ and $F$ be two Banach lattices such that the lattice operations of $E$ are weak* sequentially continuous. Then for an order bounded operator from a Banach lattice $E$ into a Banach lattice $F$, and a norm bounded solid subset $A$ of $E$ the following statements are equivalent:

1. $T(A)$ is an almost limited set.

2. $f_n(T(x_n)) \to 0$ for each disjoint sequence $(x_n)$ in $A^+$ and for every disjoint weak* null sequence $(f_n)$ of $F^*$.

Proof. (1 $\to$ 2). It follows from that the inequality

$$|f_n(T(x_n))| \leq \sup_{y \in T(A)} |f_n(y)|.$$

(2 $\to$ 1). Suppose that the sequence $(f_n)$ is a disjoint weak* null sequence in $F^*$. Since $(T f_n)$ is a weak* null sequence in $E^*$ and $E^*$ has the weak* sequentially continuous lattice operations, then $(|T^* f_n|)$ is also a weak* null sequence in
$E'$. On the other hand, by (2), \((T^* f_n)(x_n) = f_n(T x_n) \to 0\), for every disjoint sequence \((x_n) \subseteq A^+\). Now by a particular case of [4], \(\sup_{x \in A} |T^* f_n(x)| \to 0\) and so \(\sup_{y \in T(A)} |f_n(y)| = \sup_{x \in A} |T^* f_n(x)| \to 0\). Then \(T(A)\) is an almost limited set.

For order bounded operators between two Banach lattices, we give a characterization of weak* almost Dunford-Pettis operators:

**Theorem 3.3.** Let \(T\) be an order bounded operator from a Banach lattice \(E\) into another \(F\) and \(E'\) has the weak* sequentially continuous lattice operations. Then the following assertions are equivalent:

1. \(T\) is a weak* almost Dunford-Pettis operator.

2. For all weakly null sequence \((x_n)\) in \(E\) consisting of pairwise disjoint terms, and for all weak* null sequence \((f_n)\) in \(F'\) consisting of pairwise disjoint terms it follows that \(f_n(T(x_n)) \to 0\).

**Proof.** (1 \(\to\) 2) Obvious.

(2 \(\to\) 1) Let \((x_n)\) be a weakly null sequence in \(E\), and let \((f_n)\) be a disjoint weak* null sequence in \(F'\). We have to prove that \(f_n(T(x_n)) \to 0\). Let \(A\) be the solid hull of the weak relatively compact subset \(\{x_n, n \in N\}\) of \(E\), by Theorem 4.34 of [1]. \((z_n) \to 0\) weak for each disjoint sequence \((z_n)\) in \(A^+\) and so, by our hypothesis, we have \(g_n(T(z_n)) \to 0\) for each disjoint weak* null sequence \((g_n)\) in \(F'\) and for each disjoint sequence \((z_n)\) in \(A^+\). Theorem 2.2, implies that \(T(A)\) is an almost limited set, and hence \(\sup_{y \in T(A)} |f_n(y)| \to 0\). So

\[ |f_n(T(x_n))| \leq \sup_{x \in A} |f_n(T x)| \leq \sup_{y \in T(A)} |f_n(y)| \to 0 \]

holds and the proof is finished.

Now for order bounded operators between two Banach lattices, we give another characterizations of weak* almost Dunford-Pettis operators.

**Theorem 3.4.** Let \(T\) be a order bounded operator from a Banach lattice \(E\) into a \(\sigma\)-Dedekind complete Banach lattice \(F\). Then the following assertions are equivalent:

1. \(T\) is weak* almost Dunford-Pettis.

2. \(f_n(T(x_n)) \to 0\) for every weakly null sequence \((x_n)\) in \(E^+\) and for every disjoint weak* null sequence \((f_n)\) of \(F'\).

3. \(f_n(T(x_n)) \to 0\) for every disjoint weakly null sequence \((x_n)\) in \(E\) and for all disjoint weak* null sequence \((f_n)\) of \(F'\).

4. \(f_n(T(x_n)) \to 0\) for every disjoint weakly null sequence \((x_n)\) in \(E^+\) and for all disjoint weak* null sequence \((f_n)\) of \(F'\).

5. If \(A\) is a solid relatively weakly compact subset of \(E\), then \(TA\) is an almost limited set of \(F\).

6. \(f_n(T(x_n)) \to 0\) for every weakly null sequence \((x_n)\) in \(E^+\) and for all disjoint weak* null sequence \((f_n)\) of \((F')^+\).

7. \(f_n(T(x_n)) \to 0\) for every disjoint weakly null sequence \((x_n)\) in \(E^+\) and for all disjoint weak* null sequence \((f_n)\) of \((F')^+\).

**Proof.** (1 \(\to\) 2) and (1 \(\to\) 3) follows from Theorem 2.1.

(2 \(\to\) 4) and (3 \(\to\) 4) are obvious.

(4 \(\to\) 5) Let \(A\) be a solid relatively weakly compact subset of \(E\) and let \((f_n)\) be a disjoint weak* null sequence in \(F'\). If \((z_n)\) is a disjoint sequence in \((A)^+\) then by Theorem 4.34 of [2] \(z_n \to 0\). Thus, by our hypothesis \(f_n(T(z_n)) \to 0\) for every disjoint sequence \((z_n)\) in \(A\) and every disjoint weak* null sequence \((f_n)\) of \(F'\). Now, by Theorem 2.5 of [3], \(T\)
A) is almost limited.
(5 \to 1), (2 \to 6) and (4 \to 7) are obvious.
(6 \to 2) and (7 \to 4) Let \((x_n)\) in \(E^+\) be a weakly null (resp. disjoint weakly null) sequence and let \((f_n)\) of \(F'\) be a disjoint weak* null sequence. Since \(F\) is a \(\sigma\)-Dedekind complete Banach lattice then by Proposition 1.4 of [7] \(|f_n(x_n)|\) is weak* null. So the sequences \((f_n^+)\), \((f_n^-)\) are weak* null. Finally, by (6) (resp. (7)), \(f_n(T(x_n)) = (f_n^+)(T(x_n)) - (f_n^-)(T(x_n)) \to 0\).

Now we obtain the following characterization of the weak DP* property which is a generalization of Theorem 3.2 of [3].

**Lemma 3.1.** For a \(\sigma\)-Dedekind complete Banach lattice \(E\) the following assertions are equivalent:

1. \(E\) has the weak DP* property.
2. Solid hull of every relatively weakly is almost limited.
3. For all disjoint weakly null sequence \((x_n)\) in \(E^+\) and for all disjoint weak* null sequence \((f_n)\) of \((E')^+\) it follows that \(f_n(x_n) \to 0\).
4. For all weakly null sequence \((x_n)\) in \(E^+\) and for all disjoint weak* null sequence \((f_n)\) of \((E')^+\) it follows that \(f_n(x_n) \to 0\).
5. For all disjoint weakly null sequence \((x_n)\) in \(E^+\) and for all disjoint weak* null sequence \((f_n)\) of \(E^+\) it follows that \(f_n(x_n) \to 0\).
6. For all disjoint weakly null sequence \((x_n)\) in \(E^+\) and for all disjoint weak* null sequence \((f_n)\) of \(E^+\) it follows that \(f_n(x_n) \to 0\).
7. For all weakly null sequence \((x_n)\) in \(E^+\) and for all disjoint weak* null sequence \((f_n)\) of \(E^+\) it follows that \(f_n(x_n) \to 0\).

**Proof.** The proof is the same in Theorem 3.2 of [3].

The next result characterizes pairs of Banach lattices \(E\) and \(F\) for which every order bounded weak* almost Dunford-Pettis operator \(T : E \to F\) is almost limited.

**Theorem 3.5.** Let \(E\) and \(F\) be two Banach lattices such that \(F\) is \(\sigma\)-Dedekind complete. Then each order bounded weak* almost Dunford-Pettis operator from \(E\) into \(F\) is almost limited if one of the following assertions is valid.

1. \(F\) has the dual positive Schur property;
2. \(E^+\) has an order continuous norm.

**Proof.** (1). In this case, every operator \(T : E \to F\) is almost limited. In fact for all disjoint weak* null sequence \((f_n)\) of \(F'\), by Proposition 1.4 of [7] \(|f_n(x_n)|\) is weak* null and by the dual positive Schur property of \(F\), \(||f_n|| \to 0\) and hence \(||Tf_n|| \to 0\), as desired.

(2). As the norm of \(E^+\) is order continuous then by Theorem 2.4.14 of [1] every norm bounded disjoint sequence \((x_n)\) in \(E^+\) is weakly null. Now, since \(T\) is an order bounded weak* almost Dunford-Pettis operator then by Theorem 3.1 for all disjoint weak* null sequence \((f_n)\) of \(F'\), \(f_n(T(x_n)) \to 0\) and by theorem 3.2 the proof is completed.

Note that from Theorem 3.4, it is easy to see that if \(F\) is a \(\sigma\)-Dedekind complete Banach lattice then every order bounded almost Dunford-Pettis operator \(T : E \to F\) is weak* almost Dunford-Pettis. But the converse is false in general. In fact, the identity operator \(T : l^1 \to l^1\) is weak* almost Dunford-Pettis operator but it fail to be almost Dunford-Pettis. The following result characterizes pairs of Banach lattices \(E, F\) for which every order bounded weak* almost Dunford-Pettis operator operator \(T : E \to F\) is almost Dunford-Pettis.
Theorem 3.6. Let $E$ and $F$ be two Banach lattices such that $F$ is $\sigma$-Dedekind complete. Then each order bounded weak* almost Dunford-Pettis operator from $E$ into $F$ is almost Dunford-Pettis if the following assertions are valid.

(1) $E$ has the positive Schur property;
(2) The norm of $F$ is order continuous.

Proof. (1) In this case, every operator $T : E \to F$ is almost Dunford-Pettis.

(2) Let $T : E \to F$ be an order bounded weak* almost Dunford-Pettis and let $(x_n)$ be a positive disjoint weakly null sequence in $E$. Let $f \in (F')^+$. By Theorem 1.23 of [2], for each $n$ there exists some $g_n \in [-f, f]$ with $f[Tx_n] = g_n(Tx_n)$. Since $T^*$ is an order bounded operator, there is some $h \in E^+$, $T^*[f, f] \subseteq [-h, h]$. So $f[Tx_n] = (T^*g_n)(x_n) \leq h(x_n)$ for all $n$. Since $(x_n)$ is a weakly null sequence then $h(x_n)$ and $f[Tx_n]$ are norm null. So $[Tx_n]$ is a weakly null sequence. Now let $(f_n)$ be a disjoint norm bounded sequence in $(F')^+$ as the norm of $F$ is order continuous, then by corollary 2.4.3 of [6] $(f_n)$ is weak* null and by our hypothesis $f_n(T(x_n)) \to 0$ and so by corollary 2.6 of [4], $\|Tx_n\| \to 0$.

As consequence of Theorem 3.4 we obtain the domination property for weak* almost Dunford-Pettis operators.

Corollary 3.1. Let $E$ and $F$ be two Banach lattices and $F$ is $\sigma$-Dedekind complete. If $S$ and $T$ are two positive operators from $E$ into $F$ such that $0 \leq S \leq T$. Then $S$ is a weak* almost Dunford-Pettis operator whenever $T$ is one.

Proof. Let $(x_n)$ be a disjoint weakly null sequence in $E^+$, and let $(f_n)$ be a disjoint weak* null sequence in $(F')^+$. By Theorem 3.4, we have to prove that $f_n(S(x_n)) \to 0$. Since $T$ is weak* almost Dunford-Pettis, then $f_n(T(x_n)) \to 0$. Now, by using the inequalities $f_n(S(x_n)) \leq f_n(T(x_n))$ for each $n$, we see that $f_n(S(x_n)) \to 0$.

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