Fuzzy stability of the monomial functional equation

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Abstract

By using the fixed point method, we prove the generalized Hyers-Ulam-Rassias stability of the monomial functional equation $\Delta^n f(y) = n! f(x)$ in fuzzy normed spaces.

Keywords: fuzzy normed spaces, monomial functional equation, fixed point alternative, Hyers-Ulam-Rassias stability.

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1 Introduction

One of the interesting questions in the theory of functional analysis concerning the stability problem of functional equations is as follows:

“When is it true that a mapping satisfying approximately a functional equation must be close to an exact solution of the given functional equation?”

The first stability problem was raised by S. M. Ulam \cite{Ulam1},\cite{Ulam2} during his talk at the University of Wisconsin in 1940. For very general functional equations, the concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Let $X$ and $Y$ be linear spaces and $Y^X$ be the vector space of all functions from $X$ to $Y$. Following \cite{Hyers}, for each $x \in X$, define $\Delta_x : Y^X \to Y^X$ by

$$\Delta_x f(y) = f(x+y) - f(y) \quad (f \in Y^X, y \in X).$$

Inductively, we define

$$\Delta_{x_1,...,x_n} f(y) = \Delta_{x_1,...,x_{n-1}} (\Delta_{x_n} f(y))$$

for each $y, x_1, ..., x_n \in X$ and all $f \in Y^X$. If $x_1 = ... = x_n = x$, we write

$$\Delta^n_x f(y) = \Delta_{x,...,x} f(y) \quad (n \text{ times})$$

By induction on $n$, it can be easily verified that

$$\Delta^n_x f(y) = \sum_{r=0}^{n} (-1)^{n-r} C^n_r f(rx + y) \quad (n \in \mathbb{N}, x, y \in X). \quad (1.1)$$

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The functional equation
\[ \Delta^\alpha f(y) - n! f(x) = 0 \]  
(1.2)
is called the monomial functional equation of degree \( n \), since the function \( f(x) = cx^n \) is a solution of this functional equation. Every solution of the monomial functional equation of degree \( n \) is said to be a monomial mapping of degree \( n \). In particular additive, quadratic, cubic and quartic functions are monomials of degree one, two, three and four respectively. The stability of monomial equations was initiated by D. H. Hyers in [12]. The problem has been recently considered by many authors, we refer, for example, to [4], [8]-[11], [14], [17] and [24].

In 2006, Z. Kaiser [14] proved the stability of monomial functional equation where the functions map a normed space over a field with valuation to a Banach space over a field with valuation and the control function is of the form \( \varepsilon(\|x\|^\alpha + \|y\|^\beta) \). In 2007, L. Cădariu and V. Radu proved the stability of the monomial functional equation
\[ \sum_{i=0}^{n} C_n^i (-1)^{n-i} f(ix + y) - n! f(x) = 0 \]  
(1.3)
in fuzzy normed spaces. Every solution of the monomial functional equation of degree \( n \) is said to be a monomial mapping of degree \( n \). In this case, \( f(x) = cx^n \) is a solution of this functional equation. Every solution of the monomial functional equation of degree \( n \) is said to be a monomial mapping of degree \( n \). In this paper, we use the fixed point method to prove the Hyers-Ulam-Rassias stability of monomial functional equation of an arbitrary degree in fuzzy normed spaces with the control function is of the form \( \varepsilon(\|x\|^\alpha + \|y\|^\beta) \).


In this paper, we use the fixed point method to prove the Hyers-Ulam-Rassias stability of monomial functional equation of an arbitrary degree in fuzzy normed spaces with the control function is of the form \( N'(\varphi(x,y), t) \).

2 Preliminaries and notations

The theory of fuzzy space has much progressed as developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view [1], [7], [16], [19] and [25]. Following Cheng and Mordeson [5], Bag and Samanta [1] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15] and investigated some properties of fuzzy normed spaces [2]. We use the definition of fuzzy normed spaces given in [1], [19] and [20] to investigate a fuzzy version of the Hyers-Ulam stability for the monomial functional equation (1.2) in the fuzzy normed vector space setting.

Definition 2.1. Let \( X \) be a real vector space. A function \( N : X \times \mathbb{R} \to [0, 1] \) (so-called fuzzy subset) is said a fuzzy norm on \( X \) if
\begin{enumerate}
  \item [(N1)] \( N(x,t) = 0 \) for \( t \leq 0 \);
  \item [(N2)] \( x = 0 \) if and only if \( N(x,t) = 1 \) for all \( t > 0 \);
  \item [(N3)] \( N(cx,t) = N(x, \frac{t}{|c|}) \) if \( c \neq 0 \);
  \item [(N4)] \( N(x+y, s+t) \geq \min \{ N(x,s), N(y,t) \} \);
  \item [(N5)] \( N(x,.) \) is a non-decreasing function of \( \mathbb{R} \) and \( \lim_{t \to +\infty} N(x,t) = 1 \);
  \item [(N6)] For \( x \neq 0 \), \( N(x,.) \) is a left continuous function on \( \mathbb{R} \).
\end{enumerate}

for all \( x, y \in X \) and all \( s, t \in \mathbb{R} \). The pair \( (X, N) \) is called a fuzzy normed vector space (briefly, FNS).

Definition 2.2. Let \( (X, N) \) be an FNS. A sequence \( \{x_n\} \) in \( X \) is said to be convergent if there exists an \( x \in X \) such that
\[ \lim_{t \to +\infty} N(x_n - x, t) = 1 \]  
for all \( t > 0 \). In this case, \( x \) is called the limit of the sequence \( \{x_n\} \) in \( X \) and we denote it by
\[ N - \lim_{t \to +\infty} x_n = x. \]

Definition 2.3. Let \( (X, N) \) be an FNS. A sequence \( \{x_n\} \) in \( X \) is called Cauchy if for each \( \varepsilon > 0 \) and each \( t > 0 \) there exists an \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \) and all \( p > 0 \), we have \( N(x_n + p - x_n) > 1 - \varepsilon \).

It is well-known that every convergent sequence in an FNS is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space. We say that a mapping \( f : X \to Y \) between FNS \( X \) and \( Y \) is continuous at a point \( x_0 \in X \) if for each sequence \( \{x_n\} \) converging to \( x_0 \in X \), then the sequence \( \{f(x_n)\} \) converges to \( f(x_0) \). If \( f : X \to Y \) is continuous at each \( x \in X \), then \( f : X \to Y \) is
said to be continuous on $X$ (see [2]).

**Definition 2.4.** Let $X$ be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then $(X, d)$ is called a generalized metric space. $(X, d)$ is called complete if every $d$-Cauchy sequence in $X$ is $d$-convergent.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

**Definition 2.5.** Let $(X, d)$ be a generalized complete metric space. A mapping $J : X \rightarrow X$ satisfies a Lipschitz condition with a Lipschitz constant $L > 0$ if

$$d(J(x), J(y)) \leq Ld(x, y) \quad (x, y \in X).$$

If $L < 1$, then $J$ is called a strictly contractive operator.

**Theorem 2.1.** [3], [6] Suppose we are given a complete generalized metric space $(X, d)$ and a strictly contractive mapping $J : X \rightarrow X$, with the Lipschitz constant $L < 1$. If there exists a nonnegative integer $k$ such that $d(J^kx, J^{k+1}x) < \infty$ for some $x \in X$, then the following are true:

(1) the sequence $J^kx$ converges to a fixed point $x^* \in X$;
(2) $x^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^kx, y) < \infty\}$;
(3) $d(y, x^*) \leq \frac{1}{1-L}d(y, J^kx)$ for all $y \in Y$.

In 1996, Isac and Th. M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

### 3 Main Results

Throughout this section, we assume that $X$ is a linear space, $(Y, N)$ is a fuzzy Banach space and $(Z, N')$ is a fuzzy normed space.

From a result of A. Gilányi [8], Z. Kaiser [14] proved the following lemma.

**Lemma 3.1.** For every mapping $f : X \rightarrow Y$ and $n, k \in \mathbb{N}$, there correspond positive integers $N_x(k, n), x_1, x_2, \ldots, x_{n(k-1)}$ such that

$$f(kx) - k^n f(x) = \frac{1}{n!} \sum_{i=0}^{n(k-1)} N_x(k, n) f_i(x) - G(x)$$

for all $x \in X$, where $F_i : X \rightarrow Y$ and $G : X \rightarrow Y$ are defined by

$$F_i(x) = \Delta_n^k f(ix) - n! f(x)$$

and

$$G(x) = \Delta_n^k f(0) - n! f(kx)$$

for all $x \in X$ and $i = 0, 1, \ldots, n(k-1)$.

**Theorem 3.1.** Assume that a mapping $f : X \rightarrow Y$ satisfies the inequality

$$N'(\Delta_n^k f(x) - n! f(x), t) \geq N'(\varphi(x, y), t)$$

where $\varphi : X^2 \rightarrow Z$ is a mapping which there is a constant $0 < L < 1$ satisfying

$$N'(\varphi(x, y), t) \geq N'(k^n L \varphi \left(\frac{x}{k}, \frac{y}{k}\right), t)$$

(3.5)
for all \( x, y \in X \), all \( t > 0 \) and some \( k \geq 2 \). Define

\[
\psi_k(x, t) = \min \left\{ N' \left( \varphi(kx, 0), n!t \right), N' \left( \varphi(x, ix), \frac{n!t}{\xi^i} \right) : i = 0, 1, \ldots, n(k-1) \right\}
\]

for each \( k \geq 2 \). Then the limit

\[
M(x) := N - \lim_{s \to +\infty} \frac{f(k^s x)}{k^s t}
\]

exists for each \( x \in X \) and defines a monomial mapping \( M : X \to Y \) of degree \( n \) such that

\[
N(f(x) - M(x), t) \geq \min \left\{ N' \left( \varphi(kx, 0), n!t \right), N' \left( \varphi(x, ix), \frac{n!t}{\xi^i} \right) : i = 0, 1, \ldots, n(k-1) \right\}
\]

for all \( x \in X \), and all \( t > 0 \).

**Proof.** Form Lemma 3.1 and (3.4), we obtain

\[
N(f(kx) - k^n f(x), t) = N \left( \sum_{i=0}^{n(k-1)} \xi_i F_i(x) - G(x), n!t \right) \]

\[
\geq \min \left\{ N(G(x), n!t), N(\xi_1 F_1(x) + G(x), n!t) : i = 0, 1, \ldots, n(k-1) \right\}
\]

So, we get

\[
N(f(kx) - k^n f(x), t) \geq \psi_k(x, t)
\]

(3.8)

Let us consider the set

\[ S := \{ g : X \to Y \} \]

and introduce the generalized metric on \( S \):

\[
d(g, h) = \inf \{ \mu \in [0, +\infty) : N(g(x) - h(x), \mu t) \geq \psi_k(x, t), \ \forall x \in X, \forall t > 0 \}
\]

where, as usual, \( \inf \emptyset = +\infty \). It is easy to show that \( (S, d) \) is complete (see the proof of [18], Lemma 2.1). Now we consider the linear mapping \( J : S \to S \) such that

\[
J(g(x)) := \frac{f(kx)}{k^n}
\]

for all \( x \in X \).

Let \( g, h \in S \) be given such that \( d(g, h) = \varepsilon \). Then

\[
N(g(x) - h(x), \varepsilon t) \geq \psi_k(x, t)
\]
for all \( x \in X \) and all \( t > 0 \). Hence
\[
N(J(g(x)) - J(h(x)), L\varepsilon t) = \frac{1}{k^n}(g(kx) - h(kx), L\varepsilon t)
\]
\[
\geq \min \left\{ N'(\varphi(k^2 x, 0), n!k^n t), N'(\varphi(kx, kx), \frac{n!k^n t}{\varepsilon t}) \right\}_{i=0,1,\ldots,n(k-1)}
\]
\[
\geq \min \left\{ N'\left(\frac{\varphi(kx, 0), n!Lk^n t}{Lk^n t}\right), N'\left(\varphi(x, ix), \frac{n!Lk^n t}{n!Lk^n t}\right) \right\}_{i=0,1,\ldots,n(k-1)}
\]
\[
\geq \min \left\{ N'(\varphi(kx, 0), n!t), N'\left(\varphi(x, ix), \frac{n!t}{\varepsilon t} \right) \right\}_{i=0,1,\ldots,n(k-1)}
\]
\[
= \psi_k(x, t)
\]
for all \( x \in X \) and all \( t > 0 \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq L\varepsilon \). This means that
\[
d(Jg, Jh) \leq Ld(g, h)
\]
for all \( g, h \in S \). It follows from (3.8) that
\[
N\left(\frac{f(kx)}{k^n} - f(x), \frac{t}{k^n}\right) \geq \psi_k(x, t)
\]
for all \( x \in X \) and all \( t > 0 \). So
\[
d(f, Jf) \leq \frac{1}{k^n} < \infty.
\]
Therefore, by Theorem 2.1, there exists a mapping \( M : X \rightarrow Y \) satisfying the following:

1. \( M \) is fixed point of the operator \( J \), that is,
\[
M(kx) = k^n M(x)
\]
for all \( x \in X \).

The mapping \( M \) is a unique fixed point of \( J \) in the set
\[
A = \{ g \in S : d(f, g) \leq \infty \};
\]
This implies that \( M \) is a unique mapping satisfying (3.9) such that there exists a \( \mu \in (0, \infty) \) satisfying
\[
N(f(x) - M(x), \mu t) \geq \psi_k(x, t)
\]
for all \( x \in X \) and all \( t > 0 \).

2. We find that \( d(J^s f, M) \rightarrow 0 \) as \( s \rightarrow \infty \). This implies the equality
\[
N - \lim_{s \rightarrow +\infty} \frac{f(k^s x)}{k^m} = M(x)
\]
for all \( x \in X \);

3. We have
\[
d(f, M) \leq \frac{1}{k^n - k^m} \frac{d(f, Jf)}{1 - E}
\]
for all \( x \in X \);
Thus we can get
\[
N \left( f(x) - M(x), \frac{t}{k^n - k^n L} \right) \geq \min \left\{ N' \left( \phi(kx, 0), n! t \right), N' \left( \phi(x, ix), \frac{n! t}{\xi_0} \right) \right\}
\]

Then
\[
N \left( f(x) - M(x), t \right) \geq \min \left\{ N' \left( \phi(kx, 0), n! k^n (1 - L) t \right), N' \left( \phi(x, ix), \frac{n! k^n (1 - L) t}{\xi_0} \right) \right\}
\]

which means that the inequality (3.7) holds true, that is,
\[
N \left( f(x) - M(x), t \right) \geq \min \left\{ N' \left( \phi(kx, k^2 y), t \right) \right\}
\]

for all \( x \in X \) and all \( t > 0 \).

In addition, it follows from the condition (3.5) that
\[
N \left( \Delta_n^k f(ky), \frac{t}{k^n} \right) \geq N' \left( \phi(kx, y), t \right)
\]

for all \( x, y \in X \) and all \( t > 0 \). So
\[
N \left( \Delta_n^k f(ky), \frac{t}{k^n} \right) \geq N' \left( \phi(kx, y), k^{n-1} t \right)
\]

for all \( x, y \in X \) and all \( t > 0 \).

Since
\[
\lim_{x \to +\infty} N' \left( \phi(x), k^{n-1} t \right) = 1
\]

for all \( x, y \in X \) and all \( t > 0 \),
\[
N \left( \Delta_n^k f(y) - n! f(x), t \right) \geq N' \left( \phi(x), k^{n-1} t \right)
\]

for all \( x, y \in X \) and all \( t > 0 \). Thus
\[
\Delta_n^k f(y) - n! f(x) = 0
\]

for all \( x, y \in X \). It means that \( M : X \to Y \) is a monomial function of degree \( n \), as desired.

We obtain the following corollaries concerning the stability for approximate mappings controlled by a sum of powers of norms then by a product of powers of norms.

**Corollary 3.1.** Suppose that \( (X, \|\cdot\|) \) is a normed vector space, \( (\mathbb{R}, \mathbb{R}^+) \) a fuzzy normed space and \( (Y, N) \) is a fuzzy Banach space. Let \( \theta \geq 0 \) and \( \alpha > 0 \) be real numbers with \( \alpha < n \) and let \( f : X \to Y \) be a mapping satisfying
\[
N \left( \Delta_n^k f(y) - n! f(x), t \right) \geq N' \left( \theta (\|x\|^p + \|y\|^p), t \right)
\]

for all \( x, y \in X \) and all \( t > 0 \). Then the limit
\[
M(x) := N - \lim_{x \to +\infty} f(kx)
\]

exists for each \( x \in X \) and defines a monomial mapping \( M : X \to Y \) of degree \( n \) such that
\[
N \left( f(x) - M(x), t \right) \geq \min \left\{ N' \left( \|x\|^p, n! t \right), N' \left( \frac{(1 + i^p)\|x\|^p}{k^n - k^p}, \frac{n! t}{\xi_0} \right) \right\}
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.1 by taking

\[
\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)
\]

for all \( x, y \in X \). Then we can choose \( L = k^{p-n} \) and we get the desired result. \( \square \)

**Corollary 3.2.** Suppose that \((X, \|\cdot\|)\) is a normed vector space, \((X, N)\) a fuzzy normed space and \((Y, N)\) is a fuzzy Banach space. Let \( \theta \geq 0, p > 0 \) and \( q > 0 \) be real numbers with \( p + q < n \) and let \( f : X \to Y \) be a mapping satisfying

\[
N(\Delta^n f (y) - n! f(x), t) \geq N'(\theta(\|x\|^p, \|y\|^p), t)
\]  

(3.13)

for all \( x, y \in X \) and all \( t > 0 \). Then the limit

\[
M(x) := N - \lim_{x \to +\infty} \frac{f(k^n x)}{k^n}
\]

(3.14)

exists for each \( x \in X \) and defines a monomial mapping \( M : X \to Y \) of degree \( n \) such that

\[
N(f(x) - M(x), t) = \min \left\{ 1, N'(\left(\frac{t^n \|x\|^{p+q} + n! t}{k^n - k^{p+q} \cdot \frac{t^n}{k^n}}\right)_{i=0,1,\ldots,n(k-1)} \right\}
\]

(3.15)

for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.1 by taking

\[
\varphi(x,y) := \theta(\|x\|^p, \|y\|^p)
\]

for all \( x, y \in X \). Then we can choose \( L = k^{(p+q)-n} \) and we get the desired result. \( \square \)

By the similar method of the proof of Theorem 3.1, we can prove the following theorem and corollaries.

**Theorem 3.2.** Assume that a mapping \( f : X \to Y \) satisfies the inequality (3.4) where \( \varphi : X^2 \to Z \) is a mapping which there is a constant \( 0 < L < 1 \) satisfying

\[
N'(\varphi(x,y), t) \geq N'\left(\frac{L}{k^n} \varphi(kx, ky), t\right)
\]

(3.16)

for all \( x, y \in X \), all \( t > 0 \) and some \( k \geq 2 \). Then the limit

\[
M(x) := N - \lim_{x \to +\infty} k^{n} f\left(\frac{x}{k^n}\right)
\]

(3.17)

exists for each \( x \in X \) and defines a monomial mapping \( M : X \to Y \) of degree \( n \) such that

\[
N(f(x) - M(x), t) = \min \left\{ N'(\frac{L \varphi(0,0)}{k^n(1-L)}, n! t), N'(\frac{L \varphi(x, x)}{k^n(1-L)} \cdot \frac{n! t}{t^n})_{i=0,1,\ldots,n(k-1)} \right\}
\]

(3.18)

for all \( x \in X \), and all \( t > 0 \).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 3.1. Consider the linear mapping \( J : S \to S \) such that

\[
J(g(x)) := k^n f\left(\frac{x}{k^n}\right)
\]
for all $x \in X$. It follows from (3.8) that
\[ N \left( f(x) - k^n f \left( \frac{x}{k} \right), \frac{Lt}{k^n} \right) \geq \Psi_t(x, t) \]
for all $x \in X$ and all $t > 0$. So
\[ d(f, Jf) \leq \frac{L}{k^n} < \infty. \]
The rest of the proof is similar to the proof of Theorem 3.1.

**Corollary 3.3.** Suppose that $(X, \|\cdot\|)$ is a normed vector space, $(\mathbb{R}, N')$ a fuzzy normed space and $(Y, N)$ is a fuzzy Banach space. Let $\theta \geq 0$ and $p > 0$ be real numbers with $p > n$ and let $f : X \to Y$ be a mapping satisfying
\[ N(\Delta^n_x f(y) - n! f(x), t) \geq N'(\theta(\|x\|^p + \|y\|^p), t) \] (3.19)
for all $x, y \in X$ and all $t > 0$. Then the limit
\[ M(x) := N - \lim_{x \to +\infty} k^n f \left( \frac{x}{k} \right) \] (3.20)
exists for each $x \in X$ and defines a monomial mapping $M : X \to Y$ of degree $n$ such that
\[ N(f(x) - M(x), t) \geq \min \left\{ N' \left( \frac{\|x\|^p}{1 - k^n - p}, n!t \right), N' \left( \frac{(1 + ip')\|x\|^p + |x|^q}{k^n - p}, \frac{n!t}{\xi} \right) \right\}_{i=0,1,\ldots,n(k-1)} \] (3.21)
for all $x \in X$ and all $t > 0$.

**Proof.** The proof follows from Theorem 3.2 by taking
\[ \varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \]
for all $x, y \in X$. Then we can choose $L = k^{n-p}$ and we get the desired result.

**Corollary 3.4.** Suppose that $(X, \|\cdot\|)$ is a normed vector space, $(\mathbb{R}, N')$ a fuzzy normed space and $(Y, N)$ is a fuzzy Banach space. Let $\theta \geq 0$, $p > 0$ and $q > 0$ be real numbers with $p + q > n$ and let $f : X \to Y$ be a mapping satisfying
\[ N(\Delta^n_x f(y) - n! f(x), t) \geq N'(\theta(\|x\|^p, \|y\|^q), t) \] (3.22)
for all $x, y \in X$ and all $t > 0$. Then the limit
\[ M(x) := N - \lim_{x \to +\infty} k^n f \left( \frac{x}{k} \right) \] (3.23)
exists for each $x \in X$ and defines a monomial mapping $M : X \to Y$ of degree $n$ such that
\[ N(f(x) - M(x), t) \geq \min \left\{ 1, N' \left( \frac{i^n p^q |x|^{p+q} + n!t}{k^n q - p}, \frac{n!t}{\xi} \right) \right\}_{i=0,1,\ldots,n(k-1)} \] (3.24)
for all $x \in X$ and all $t > 0$.

**Proof.** The proof follows from Theorem 3.2 by taking
\[ \varphi(x, y) := \theta(\|x\|^p, \|y\|^q) \]
for all $x, y \in X$. Then we can choose $L = k^{n-(p+q)}$ and we get the desired result.
We obtain the following theorems and corollaries concerning the stability for approximate mappings controlled by a fraction function.

**Theorem 3.3.** Let \( \varphi : X^2 \rightarrow [0,\infty) \) be a function such that there exists an constant \( 0 < L < 1 \) with

\[
\varphi(x,y) \leq n! L \varphi \left( \frac{x}{L}, \frac{y}{L} \right)
\]  

(3.25)

for all \( x,y \in X \), for all \( t > 0 \) and for some integers \( k \geq 2 \). Let \( f : X \rightarrow Y \) be a mapping satisfying

\[
N \left( \Delta^n f(y) - n! f(x), t \right) \geq \frac{t}{t + \varphi(x,y)}
\]  

for all \( x,y \in X \) and all \( t > 0 \). Then the limit

\[
M(x) := N - \lim_{s \rightarrow +\infty} \frac{f(k^s x)}{k^{ns}}
\]  

(3.27)

exists for each \( x \in X \) and defines a monomial function \( M : X \rightarrow Y \) of degree \( n \) such that

\[
N(f(x) - M(x), t) \geq \min \left\{ \left( \frac{n! k^n (1 - L)t}{n! k^n (L) + \varphi(kx,0)} \right), \left( \frac{n! k^n (1 - L)t}{n! k^n (L) + \varphi(kx,0)} \right) \right\}_{i=0,1,...,n(k-1)}
\]  

(3.28)

for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.1 by taking

\[
N(\varphi(x,y), t) := \frac{t}{t + \varphi(x,y)}
\]

for all \( x,y \in X \) and all \( t > 0 \).

**Corollary 3.5.** Suppose that \( (X, \| - \|) \) is a normed vector space and \( (Y,N) \) is a fuzzy Banach space. Let \( \theta \geq 0 \) and \( p > 0 \) be real numbers with \( p < n \) and let \( f : X \rightarrow Y \) be a mapping satisfying

\[
N \left( \Delta^n f(y) - n! f(x), t \right) \geq \frac{t}{t + \varphi(x,y)}
\]  

(3.29)

for all \( x,y \in X \) and all \( t > 0 \). Then the limit

\[
M(x) := N - \lim_{s \rightarrow +\infty} \frac{f(k^s x)}{k^{ns}}
\]  

(3.30)

exists for each \( x \in X \) and defines a monomial mapping \( M : X \rightarrow Y \) of degree \( n \) such that

\[
N(f(x) - M(x), t) \geq \min \left\{ \left( \frac{n! (k^n - k^p)t}{n! (k^n - k^p)t + \varphi(kx,0)} \right), \left( \frac{n! (k^n - k^p)t}{n! (k^n - k^p)t + \varphi(kx,0)} \right) \right\}_{i=0,1,...,n(k-1)}
\]  

(3.31)

for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.3 by taking

\[
\varphi(x,y) := \theta(\| x \|^p + \| y \|^p)
\]

for all \( x,y \in X \). Then we can choose \( L = k^{p-n} \) and we get the desired result.
Corollary 3.6. Suppose that \((X, \|\cdot\|)\) is a normed vector space and \((Y, N)\) is a fuzzy Banach space. Let \(\theta \geq 0, p > 0\) and \(q > 0\) be real numbers with \(p + q < n\) and let \(f : X \to Y\) be a mapping satisfying
\[
N(\Delta^n f(y) - n!f(x), t) \geq \frac{t}{t + \theta(\|x\|^p, \|y\|^q)}
\]  \hspace{1cm} (3.32)
for all \(x, y \in X\) and all \(t > 0\). Then the limit
\[
M(x) := N - \lim_{s \to +\infty} \frac{f(k^s x)}{k^{ns}}
\]  \hspace{1cm} (3.33)
exists for each \(x \in X\) and defines a monomial mapping \(M : X \to Y\) of degree \(n\) such that
\[
N(f(x) - M(x), t) \geq \min \left\{ 1, \frac{n!(k^n - k^{(p+q)}t)}{n!(k^n - k^{(p+q)}t) + \theta \|x\|^p + \|y\|^q} \right\}
\]  \hspace{1cm} (3.34)
for all \(x \in X\) and all \(t > 0\).

Proof. The proof follows from Theorem 3.3 by taking
\[
\phi(x, y) := \theta(\|x\|^p, \|y\|^q)
\]
for all \(x, y \in X\). Then we can choose \(L = k^{(p+q)-n}\) and we get the desired result. \(\square\)

Theorem 3.4. Let \(\phi : X^2 \to [0, \infty)\) be a function such that there exists an constant \(0 < L < 1\) with
\[
\phi(x, y) \leq \frac{L}{k^n} \phi(kx, ky)
\]  \hspace{1cm} (3.35)
for all \(x, y \in X\), for all \(t > 0\) and for some integers \(k \geq 2\). Let \(f : X \to Y\) be a mapping satisfying
\[
N(\Delta^n f(y) - n!f(x), t) \geq \frac{t}{t + \phi(x, y)}
\]  \hspace{1cm} (3.36)
for all \(x, y \in X\) and all \(t > 0\). Then the limit
\[
M(x) := N - \lim_{s \to +\infty} k^{ns} f \left( \frac{x}{k^i} \right)
\]  \hspace{1cm} (3.37)
exists for each \(x \in X\) and defines a monomial function \(M : X \to Y\) of degree \(n\) such that
\[
N(f(x) - M(x), t) \geq \min \left\{ \left( \frac{n!k^n(1-L)t}{n!k^n(1-L)t + L\phi(kx, 0)} \right), \left( \frac{n!k^n(1-L)t}{n!k^n(1-L)t + L\phi(x, 0)} \right) \right\}
\]  \hspace{1cm} (3.38)
for all \(x \in X\) and all \(t > 0\).

Proof. The proof follows from Theorem 3.2 by taking
\[
N'(\phi(x, y), t) := \frac{t}{t + \phi(x, y)}
\]
for all \(x, y \in X\) and all \(t > 0\). \(\square\)

Corollary 3.7. Suppose that \((X, \|\cdot\|)\) is a normed vector space and \((Y, N)\) is a fuzzy Banach space. Let \(\theta \geq 0\) and
\( p > 0 \) be real numbers with \( p > n \) and let \( f : X \rightarrow Y \) be a mapping satisfying
\[
N(\Delta^n f(y) - n!f(x), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.39}
\]
for all \( x, y \in X \) and all \( t > 0 \). Then the limit
\[
M(x) := N - \lim_{s \to +\infty} f(k^n x) \tag{3.40}
\]
exists for each \( x \in X \) and defines a monomial mapping \( M : X \rightarrow Y \) of degree \( n \) such that
\[
N(f(x) - M(x), t) \geq \frac{f(k^n x)}{k^{ns}} \min \left\{ \left( \frac{n!(1 - k^n - p)t}{n!(1 - k^n - p)t + \theta\|x\|^p} \right), \left( \frac{n!(k^n - k^n)t}{n!(k^n - k^n)t + \xi \theta(1 - p)^n\|x\|^p} \right) \right\} \tag{3.41}
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.4 by taking
\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]
for all \( x, y \in X \). Then we can choose \( L = k^n - p \) and we get the desired result. \( \square \)

**Corollary 3.8.** Suppose that \((X, \| \cdot \|)\) is a normed vector space and \((Y, N)\) is a fuzzy Banach space. Let \( \theta \geq 0 \), \( p > 0 \) and \( q > 0 \) be real numbers with \( p + q > n \) and let \( f : X \rightarrow Y \) be a mapping satisfying
\[
N(\Delta^n f(y) - n!f(x), t) \geq \frac{t}{t + \theta(\|x\|^p, \|y\|^q)} \tag{3.42}
\]
for all \( x, y \in X \) and all \( t > 0 \). Then the limit
\[
M(x) := N - \lim_{s \to +\infty} k^{ns} f \left( \frac{x}{k^n} \right) \tag{3.43}
\]
exists for each \( x \in X \) and defines a monomial mapping \( M : X \rightarrow Y \) of degree \( n \) such that
\[
N(f(x) - M(x), t) \geq \min \left\{ 1, \left( \frac{n!(k^n + q)t}{n!(k^n + q)t + \theta\|x\|^p, \|y\|^q} \right) \right\} \tag{3.44}
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.4 by taking
\[
\varphi(x, y) := \theta(\|x\|^p, \|y\|^q)
\]
for all \( x, y \in X \). Then we can choose \( L = k^{n-(p+q)} \) and we get the desired result. \( \square \)

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