\(\varepsilon\)-Mixed equilibrium problems and fixed points in Hilbert spaces

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Abstract
We define \(\varepsilon\)-mixed equilibrium point in Hilbert spaces. We find a common element of the set of fixed points of finitely many non-expansive mappings and the set of solutions of an \(\varepsilon\)-mixed equilibrium problem (AMEP) in Hilbert spaces. Also we obtain some theorems about mixed equilibrium problems and fixed points.

Keywords: \(\varepsilon\)-Mixed equilibrium point, \(\varepsilon\)-Fixed point, Hilbert spaces.

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1 Introduction
The fact that equilibrium problems covers several problems in economics, physics, engineering, chemistry, biology etc. From the mathematical modeling point of view equilibrium can be described by different types of theorem such as fixed point theorems, optimization problems. Equilibrium systems can be studied from several points of view: number of solutions; properties of the solution set; and the numerical approximation of solutions. Now, we will present several particular classes of \(\varepsilon\)-mixed equilibrium systems and the relations between them. Some methods have been proposed to solve the equilibrium problems, mixed equilibrium problems and approximate equilibrium problems; see, for instance, [4], [5], [6]. Recently, L.-C. Ceng, S. Al-Homidan, Q. H. Ansari, and J.-C. Yao [1] introduce a method for mixed equilibrium problems and fixed point problems. Throughout the paper, Let \(H\) be a real Hilbert space and \(C\) be a nonempty, closed, bounded and convex subset of \(H\).

**condition 1.1.** The following condition appears implicitly in [3].
We assume that the map \(F : C \times C \rightarrow \mathbb{R}\) satisfies the following conditions:

(i) \(F(u, u) = 0\), \(\forall u \in C\)

(ii) \(F\) is monotone, that is, \(F(u, v) + F(v, u) \leq 0\), \(\forall u, v \in C\)

(iii) For all \(u, v, w \in C\),
\[
\lim_{t \to 0} F(wt + (1 - t)u, v) \leq F(u, v)
\]

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(iv) For each fixed $u \in C$, the function $y \mapsto F(u, v)$ is convex and lower semicontinuous.

**Definition 1.1.** [5] Let $F : C \times C \rightarrow R$ be a bifunction and $f : C \rightarrow R$ be a real-valued function. We say that $u^* \in C$ is a mixed equilibrium point of $(F, f)$ if there exists an $u^* \in C$ such that

$$F(u^*, v) + f(v) - f(u^*) \geq 0 \quad \forall v \in C;$$

The set of such $u^* \in C$ is denoted by $MEP(F, f)$; that is,

$$MEP(F, f) = \{u^* \in C : F(u^*, v) + f(v) - f(u^*) \geq 0, \quad \forall v \in C\}.$$

In particular, if $f = 0$, this problem reduces to the equilibrium problem (EP), which is to find $u^* \in C$ such that $F(u^*, v) \geq 0$, $\forall v \in C$.

In the following we will present a known lemma which is needed in the proof of some results (see [3]).

**Lemma 2.1.** [3] Let $F : C \times C \rightarrow R$ be a map satisfies condition 1.1 and $f : C \rightarrow R$ be a real-valued function. Let $r$ be a positive parameter, and let $u \in C$. Then there exists a $v \in C$ such that

$$F(v, w) + f(w) - f(v) + \frac{(v - w, w - u)}{r} \geq 0 \quad \forall w \in C.$$

### 2 Main results

In this section we consider $\varepsilon$–mixed equilibrium problems and fixed points in Hilbert spaces.

**Condition 2.1.** [6] Assume that the the map $F : C \times C \rightarrow R$ for $\varepsilon > 0$ satisfies the following conditions:

(a) $F(u, u) = \varepsilon \quad \forall u \in C$

(b) $F$ is $\varepsilon$–monotone, that is, $F(u, v) + F(v, u) \leq \varepsilon \quad \forall u, v \in C$

(c) For all $u, v, w \in C$,

$$\lim_{t \to 0} F(wt + (1-t)u, v) \leq F(u, v)$$

(d) For each fixed $u \in C$, the function $y \mapsto F(u, v)$ is convex and lower semicontinuous.

**Definition 2.1.** Let $F : C \times C \rightarrow R$ be a bifunction, $f : C \rightarrow R$ be a real-valued function and $\varepsilon > 0$. We say that $u^* \in C$ is an approximate mixed equilibrium point (or $\varepsilon$–mixed equilibrium point) of $(F, f)$ if there exists an $u^* \in C$ such that

$$F(u^*, v) + f(v) - f(u^*) \geq \varepsilon, \quad \forall v \in C$$

In this paper, the set of such an $u^* \in C$ is denoted by $AMEP(F, f)$; that is,

$$AMEP(F, f) = \{x^* \in C : F(u^*, v) + f(v) - f(u^*) \geq \varepsilon, \quad \forall v \in C\},$$

and we have

$$AFix(T) = \{u \in X : d(Tu, u) \leq \varepsilon \quad for \ some \ \varepsilon > 0\},$$

$$Fix(T) = \{u \in X : Tu = u\}.$$

**Lemma 2.1.** Let $F : C \times C \rightarrow R$ be a map satisfies Condition 2.1 and $f : C \rightarrow R$ be a real-valued function. Let $r$ be a positive parameter, $u \in C$ and $\varepsilon > 0$. Then there exists a $v \in C$ such that

$$F(v, w) + f(w) - f(v) + \frac{(v - w, w - u)}{r} \geq \varepsilon \quad \forall w \in C.$$
Proof. Let \( F_1 : C \times C \rightarrow \mathbb{R} \) is a bifunction, \( f_1 : C \rightarrow \mathbb{R} \) be a real-valued functions as \( F_1(v,w) + f_1(w) - f_1(v) = F(v,w) + f(w) - f(v) - \varepsilon \) that satisfies condition 1.1 by lemma 1.1, there exists \( v \in C \) such that
\[
F_1(v,w) + f_1(w) - f_1(v) + \frac{(v - w, w - u)}{r} \geq 0 \quad \forall w \in C.
\]
Thus
\[
F(v,w) + f(w) - f(v) + \frac{(v - w, w - u)}{r} \geq \varepsilon \quad \forall w \in C.
\]

\[\square\]

**Theorem 2.1.** Let \( F : C \times C \rightarrow \mathbb{R} \) be a map satisfies condition 2.1 and \( f : C \rightarrow \mathbb{R} \) be a real-valued function. For \( r > 0, u \in C, \varepsilon > 0 \) defined \( T^\varepsilon_r : H \rightarrow C \) such that:
\[
T^\varepsilon_r(u) = \{ v \in C \mid F(v,w) + f(w) - f(v) + \frac{(v - w, w - u)}{r} \geq \varepsilon \ \forall w \in C \}.
\]

Then:

(a) \( \text{dom } T^\varepsilon_r = H. \)

(b) For each \( u \in H, T^\varepsilon_r(u) \neq \emptyset. \)

(c) \( T^\varepsilon_r \) is single-valued.

(d) \( T^\varepsilon_r \) is firmly nonexpansive, that is,
\[
\|T^\varepsilon_r(u) - T^\varepsilon_r(v)\|^2 \leq \langle T^\varepsilon_r(u) - T^\varepsilon_r(v), u - v \rangle \quad \forall u, v \in H
\]

(e) \( \text{Fix}(T^\varepsilon_r) = \text{AMEP}(F, f) \)

(f) \( \text{AMEP}(F, f) \) is nonempty, closed and convex.

**Proof.** [4], Corollary 1; asserts that for every \( u \in H \) there exists a point \( v \in K \) such that
\[
F(v,w) + f(w) - f(v) + \frac{(v - w, w - u)}{r} \geq \varepsilon \ \forall w \in C.
\]

(b) For \( u \in H \) and \( r > 0 \), let \( v_1, v_2 \in T^\varepsilon_r(u) \). Then,
\[
F(v_1, v_2) + f(v_2) - f(v_1) + \frac{(v_1 - v_2, v_2 - u)}{r} \geq \varepsilon,
\]
and
\[
F(v_2, v_2) + f(v_2) - f(v_2) + \frac{(v_2 - v_2, v_2 - u)}{r} \geq \varepsilon.
\]

Then
\[
F(v_1, v_2) + f(v_2) - f(v_1) + \frac{(v_1 - v_2, v_2 - u)}{r} \geq \varepsilon
\]
and
\[
F(v_2, v_1) + f(v_1) - f(v_2) + \frac{(v_2 - v_1, v_1 - u)}{r} \geq \varepsilon.
\]
Since $F$ is $\varepsilon$–monotone,
\[
\varepsilon \geq F(v_1, v_2) + F(v_2, v_1) \\
= F(v_1, v_2) + f(v_2) - f(v_1) + F(v_2, v_1) + f(v_1) - f(v_2) \\
\geq 1/r(v_1 - v_2, v_2 - u) + 1/r(v_2, v_1 - u) + 2\varepsilon \\
\geq 1/r(v_2 - v_1, v_1 - v_2) + \varepsilon.
\]
Now, since $\varepsilon > 0$ and $r > 0$,
\[\|v_2 - v_1\|^2 \geq 0.\]
So, $v_1 = v_2$.

(c) Now we claim that $T_2^\varepsilon$ is firmly nonexpansive. Indeed, for $u, v \in H$,
\[
F(T_2^\varepsilon(u), T_2^\varepsilon(v)) + f(T_2^\varepsilon(v)) - f(T_2^\varepsilon(u)) + 1/r(T_2^\varepsilon(v) - T_2^\varepsilon(u), T_2^\varepsilon(u) - u) \geq \varepsilon
\]
and
\[
F(T_2^\varepsilon(v), T_2^\varepsilon(u)) + f(T_2^\varepsilon(u)) - f(T_2^\varepsilon(v)) + 1/r(T_2^\varepsilon(u) - T_2^\varepsilon(v), T_2^\varepsilon(v) - v) \geq \varepsilon
\]
Adding the two inequalities, we have
\[
F(T_2^\varepsilon(u), T_2^\varepsilon(v)) + F(T_2^\varepsilon(v), T_2^\varepsilon(u)) + 1/r(T_2^\varepsilon(v) - T_2^\varepsilon(u), T_2^\varepsilon(u) - T_2^\varepsilon(v) - u + v) \\
\geq 2\varepsilon
\]
with (A2), we have
\[
1/r(T_2^\varepsilon(v) - T_2^\varepsilon(u), T_2^\varepsilon(u) - T_2^\varepsilon(v) - (u - v)) \geq \varepsilon
\]
Now since $\varepsilon > 0$ and $r > 0$, then
\[
(T_2^\varepsilon(v) - T_2^\varepsilon(u), T_2^\varepsilon(u) - T_2^\varepsilon(v) - (u - v)) \geq 0
\]
so
\[
\|T_2^\varepsilon(u) - T_2^\varepsilon(v)\|^2 \leq (T_2^\varepsilon(u) - T_2^\varepsilon(v), u - v).
\]
(d) Take $u \in C$. Then
\[
u \in \text{Fix}(T_2^\varepsilon) \iff u = T_2^\varepsilon(u) \\
\iff F(u, w) + f(w) - f(u) + 1/r(u - w - u) \\
\geq \varepsilon \quad \forall w \in C \\
\iff F(u, w) + f(w) - f(u) \geq \varepsilon \quad \forall w \in C \\
\iff u \in \text{AMEP}(F, f).
\]
(e) At last, we claim that $\text{AMEP}(F, f)$ is closed and convex. Since $T_2^\varepsilon$ is firmly nonexpansive, $T_2^\varepsilon$ is also nonexpansive and since the fixed point set of a nonexpansive operator is closed and convex hence by [7] $\text{AMEP}(F, f)$ is closed and convex.

\[\square\]

**Lemma 2.2.** Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction and $f : C \longrightarrow \mathbb{R}$ be a real-valued function. Let $T : C \longrightarrow C$ be a nonlinear onto mapping and satisfying
\[d(u, Tu) + F(Tu, u) + f(u) - f(Tu) < \varepsilon \quad \forall u \in C.
\]
If $\text{MEP}(F, f) \neq \emptyset$, then $\text{AFix}(T) \neq \emptyset$. 

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Proof. If \( v \in \text{MEP}(F, f) \), then
\[
0 \leq F(v, u) + f(u) - f(v) \forall u \in C
\]

Since \( v \in C \), there exists \( u \in C \) such that \( Tu = v \). Therefore \( F(Tu, u) + f(u) - f(Tu) \geq 0 \) and so
\[
d(u - Tu) + F(Tu, u) + f(u) - f(Tu) < \varepsilon.
\]

It follows that \( d(u, Tu) < \varepsilon \) and \( u \in A\text{Fix}(T) \).

\[\square\]

**Lemma 2.3.** Let \( F : C \times C \rightarrow R \) be a map satisfies condition 2.1 and \( f : C \rightarrow R \) be a real-valued function. For \( u \in C \), define a mapping \( T : C \rightarrow C \) such that is contraction. If \( F(Tu, v) + f(v) - f(Tu) \geq \varepsilon \) for all \( v \in C \) then there exists \( u_0 \in \text{AMEP}(F, f) \).

Proof. Since \( C \) is a nonempty closed, bounded and convex subset of \( H \) and \( T \) is continuous, then by Browders Theorem there exists \( u_0 \in C \) such that \( Tu_0 = u_0 \). Since \( F(Tu_0, v) + f(v) - f(Tu_0) \geq \varepsilon \forall v \in C \), thus \( F(u_0, v) + f(v) - f(u_0) \geq \varepsilon \forall v \in C \). So there exists \( u_0 \in C \) such that \( u_0 \in \text{AMEP}(T) \).

\[\square\]

In the following we will present a known theorem which is needed in the proof of some results (see [2]).

**Lemma 2.4.** Let \( F_1, F_1 : C \times C \rightarrow R \) be a bifunction, \( f, f_1 : C \rightarrow R \) be a real-valued functions to relation \( F_1(v, w) + f_1(w) = F(v, w) + f(w) - f(v) - \varepsilon \) that satisfies condition 1.1 and \( S : C \rightarrow C \) be a nonexpensive mapping. Then \( \text{Fix}(S) \cap \text{AMEP}(F) \neq \emptyset \), if and only if \( \text{Fix}(S) \cap \text{AMEP}(F_1) \neq \emptyset \).

**Theorem 2.2.** [2] Let \( F : C \times C \rightarrow R \) be a map satisfies condition 1.1. \( f : C \rightarrow R \) be a real-valued function and \( S : C \rightarrow C \) be a nonexpensive mapping such that \( \text{Fix}(S) \cap \text{AMEP}(F, f) \neq \emptyset \). Let \( \{u_n\} \) and \( \{v_n\} \) be sequences generated initially by an arbitrary element \( u_1 \in H \) and then by
\[
\begin{cases}
F(v_n, w) + f(w) - f(v_n) + 1/r_n(w - v_n, v_n - u_n) \geq 0 & \forall w \in C, \\
u_{n+1} = \alpha_n v_n + (1 - \alpha_n) S v_n & \forall n \geq 1.
\end{cases}
\]

Where \( \{\alpha_n\} \) and \( \{r_n\} \) satisfy the following conditions:

(i) \( \{\alpha_n\} \subset [\alpha, \beta] \) for some \( \alpha, \beta \in (0, 1) \);

(ii) \( \{r_n\} \subset (0, \infty) \) and \( \lim inf_n r_n > 0 \). Then, the sequences \( \{u_n\} \) and \( \{v_n\} \) converge weakly to an element of \( \text{Fix}(S) \cap \text{AMEP}(F, f) \neq \emptyset \).

**Theorem 2.3.** Let \( F : C \times C \rightarrow R \) be a map satisfies condition 2.1, \( f : C \rightarrow R \) be a real-valued function and \( S : C \rightarrow C \) be a nonexpensive mapping such that \( \text{Fix}(S) \cap \text{AMEP}(F, f) \neq \emptyset \). Let \( \{u_n\} \) and \( \{v_n\} \) be sequences generated initially by an arbitrary element \( u_1 \in H \) and then by
\[
\begin{cases}
F(v_n, w) + f(w) - f(v_n) + 1/r_n(w - v_n, v_n - u_n) \geq \varepsilon & \forall w \in C, \\
u_{n+1} = \alpha_n v_n + (1 - \alpha_n) S v_n & \forall n \geq 1.
\end{cases}
\]

Where \( \{\alpha_n\} \) and \( \{r_n\} \) satisfy the following conditions:

(i) \( \{\alpha_n\} \subset [\alpha, \beta] \) for some \( \alpha, \beta \in (0, 1) \);

(ii) \( \{r_n\} \subset (0, \infty) \) and \( \lim inf_n r_n > 0 \).

Then, the sequences \( \{u_n\} \) and \( \{v_n\} \) converge weakly to an element of \( \text{Fix}(S) \cap \text{AMEP}(F, f) \).

Proof. In first consider bifunction \( F_1, F_1 : C \times C \rightarrow R \) to relation \( F_1(v, w) + f_1(w) = F(v, w) + f(w) - f(v) - \varepsilon \) that satisfies Conditions 1.1. Since by \( \text{Fix}(S) \cap \text{AMEP}(F, f) \neq \emptyset \), Lemma 2.4 implies that \( \text{Fix}(S) \cap \text{AMEP}(F, f_1) \neq \emptyset \) and holds conditions Theorem 2.2, therefore the sequences \( \{u_n\} \) and \( \{v_n\} \) converge weakly to an element of \( z \in \text{Fix}(S) \cap \text{AMEP}(F_1, f_1) \). Hence the sequences \( \{u_n\} \) and \( \{v_n\} \) converge weakly to \( z \in \text{Fix}(S) \cap \text{AMEP}(F, f) \).

\[\square\]
References

http://dx.doi.org/10.1016/j.cam.2008.03.032

http://dx.doi.org/10.1155/JIA/2006/65983


http://dx.doi.org/10.1016/j.cam.2007.02.022

http://dx.doi.org/10.1155/2013/659493