Journal Nonlinear Analysis and Application 2014 (2014) 1-14

ISPACS
International Scientific Publications and Consulting Services

Available online at www.ispacs.com/jnaa
Volume 2014, Year 2014 Article ID jnaa-00184, 14 Pages
doi:10.5899/2014/jnaa-00184
Research Article

JNAA
Journal of Nonlinear Analysis and Application

Stabilization and Riesz basis property for an overhead crane model with feedback in velocity and rotating velocity

Touré K. Augustin[1], Mensah E. Patrice[2], Taha M. Mathurin[2*]

(1) *Institut National Polytechnique Houphouët-Boigny de Yamoussoukro, BP 1093 Yamoussoukro, Côte d'Ivoire*

(2) *Institut National Polytechnique Houphouët-Boigny de Yamoussoukro, BP 2444 Yamoussoukro, Côte d'Ivoire*

Abstract

This paper studies a variant of an overhead crane model's problem, with a control force in velocity and rotating velocity on the platform. We obtain under certain conditions the well-posedness and the strong stabilization of the closed-loop system. We then analyze the spectrum of the system. Using a method due to Shkalikov, we prove the existence of a sequence of generalized eigenvectors of the system, which forms a Riesz basis for the state energy Hilbert space.

Keywords: Hyperbolic boundary value problem, strong stability, asymptotic behavior, semigroup theory.

1 Introduction

We study an overhead crane model consisting of a cable carrying a load. The cable is linked at its top end to a platform moving along a rail by means of a feedback control force in velocity and rotating velocity on the platform. The equations of motion for this system are given by

$$
\begin{aligned}
y_{tt} - y_{xx} &= 0, \quad 0 < x < 1, \quad t \geq 0, && (1.1)\\
-y_x(0,t) + my_{tt}(0,t) &= -\beta y_t(0,t) + \gamma y_{xt}(0,t), \quad t \geq 0, && (1.2)\\
y_x(1,t) + My_{tt}(1,t) &= 0, \quad t \geq 0, && (1.3)\\
y(x,0) = y_0(x),\ y_t(x,0) &= y_1(x),\ 0 < x < 1, && (1.4)
\end{aligned}
$$

where β, γ are two given non negative constants; $y(x,t)$ stands for transversal deviation at the point of the cable whose curvilinear abscissa is x at time t (as shown on the figure below); a subscript letter denotes the partial derivation with respect to that variable; m is the mass of the wagon and M the conveyed mass. Many authors have studied the stabilization of this simplified model with several control feedbacks [16], [6]. The goal of this work is to establish conditions on the feedback parameters β and γ, to get the strong stability and the Riesz basis property of the system $(1.1)-(1.4)$. The strong stability is mainly due to La Salle's principle [4] and to obtain the Riesz basis property one uses a method due to Shkalikov [17]. The content of this paper is as follows. In the next section we convert

*Corresponding author. Email address: malodja@yahoo.fr, Tel:+22507658264

the system in terms of an evolution equation for which well-posedness is obtained. In the third section we prove the strong stability. Then, we study the spectrum of the system and prove the Riesz basis property for the case considered for a suitable choice of the feedback parameters β and γ.

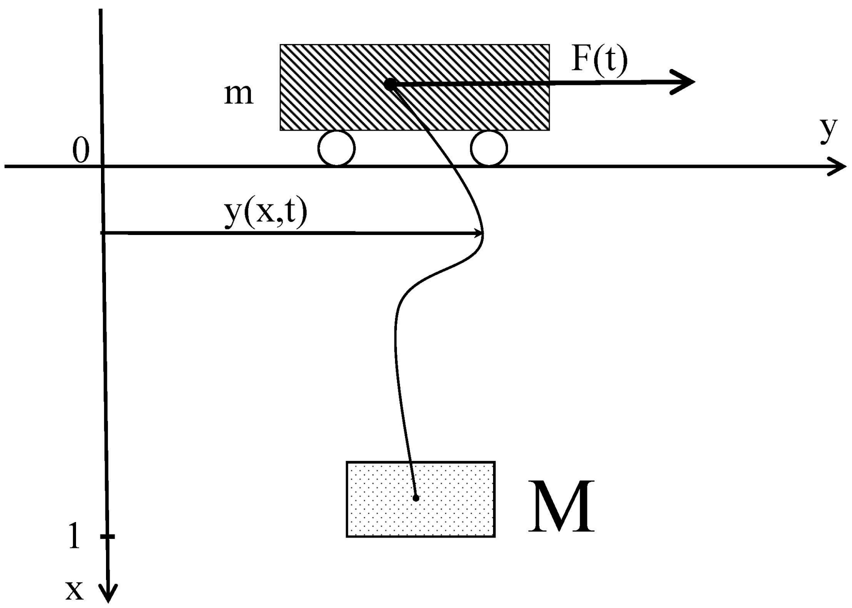

Figure 1: A model of an overhead crane

2 Well-Posedness of the system

Let us introduce the following space:

$$\mathscr{H} = H^1(0,1) \times L^2(0,1) \times \mathbb{R}^2,$$

where the spaces $L^2(0,1)$ and $H^k(0,1)$ are defined as :

$$L^2(0,1) = \left\{ y : [0,1] \to \mathbb{R} \,/\, \int_0^1 y^2 dx < \infty \right\}, \tag{2.5}$$

$$H^k(0,1) = \left\{ y : [0,1] \to \mathbb{R} \,/\, y, y^{(1)}, \ldots, y^{(k)} \in L^2(0,1) \right\}. \tag{2.6}$$

In $\mathscr{H}$ we define the following inner-product:

$$\begin{aligned} < U, \widetilde{U} >_{\mathscr{H}} &= \int_0^1 (y_x \widetilde{y}_x + z\widetilde{z}) dx + M u \widetilde{u} + \frac{v \widetilde{v}}{(m - \beta\gamma)} \\ &+ \varepsilon \left(\int_0^1 z dx + M u + v + \beta y(0,t) \right) \times \\ &\left(\int_0^1 \widetilde{z} dx + M \widetilde{u} + \widetilde{v} + \beta \widetilde{y}(0,t) \right), \end{aligned} \tag{2.7}$$

where $U=(y,z,u,v)^T\in\mathscr{H}$, $\widetilde{U}=(\widetilde{y},\widetilde{z},\widetilde{u},\widetilde{v})^T\in\mathscr{H}$, the superscript T stands for the transpose and ε is a non negative real. The energy norm is then defined as follows:

$$\begin{aligned}\|(y,z,u,v)^T\|^2 &= \int_0^1 (y_x^2+z^2)dx+Mu^2+\frac{v^2}{(m-\beta\gamma)}\\ &+\varepsilon\left(\int_0^1 zdx+Mu+v+\beta y(0,t)\right)^2.\end{aligned} \tag{2.8}$$

Next, we define the unbounded operator $A:D(A)\subset\mathscr{H}\to\mathscr{H}$ as follows:

$$A\begin{pmatrix} y\\ z\\ u\\ v\end{pmatrix}=\begin{pmatrix} z\\ y_{xx}\\ -\frac{1}{M}y_x(1)\\ y_x(0)-\beta y_t(0)\end{pmatrix}, \tag{2.9}$$

where the domain $D(A)$ of operator A is defined as

$$\begin{aligned}D(A) &= \Big\{(y,z,u,v)^T\,/\,y\in H^2(0,1),\,z\in H^1(0,1),\,u=z(1),\\ &\quad v=mz(0)-\gamma y_x(0)\Big\}.\end{aligned} \tag{2.10}$$

With the previous notations, the set of equations $(1.1)-(1.4)$ can be formally written as

$$\dot{U}=AU \text{ with initial data } U(0)\in\mathscr{H}, \tag{2.11}$$

where $U=(y,z,u,v)^T$ and $y=y(.,t),\ z=y_t,\ u=y_t(1,t),\ v=my_t(0,t)-\gamma y_x(0,t)$. For this reason the system $(1.1)-(1.4)$ will be interpreted in terms of equation (2.11).

Theorem 2.1. *Suppose that $m>\beta\gamma$. The operator A, defined by* (2.9) *and* (2.10)*, generates a $\mathscr{C}_0$ semigroup of contractions on $\mathscr{H}$. (For the terminology on the semigroup theory, the reader is referred to [11]).*

Proof. We apply the Lumer-Phillips theorem, see, ([11],p.14). First, we show that the operator A is dissipative. For any $U=(y,z,u,v)^T\in D(A)$, we get:

$$<AU,U>_{\mathscr{H}}=-\beta z^2(0)-\frac{\gamma}{m-\beta\gamma}\left[y_x(0)-\beta z(0)\right]^2\le 0. \tag{2.12}$$

It follows from (2.12) that the operator A is dissipative.
Next, we show that it is m-dissipative. It suffices to prove that operator $(I-A):\mathscr{D}(\mathscr{A})\subset\mathscr{H}\to\mathscr{H}$ is onto; that is for any given $W=(f,g,\widetilde{u},\widetilde{v})^T\in\mathscr{H}$, we have to find $U=(y,z,u,v)^T\in D(A)$ so that

$$(I-A)U=W, \tag{2.13}$$

which is equivalent to the following set of equations:

$$y-z = f, \tag{2.14}$$
$$z-y_{xx} = g, \tag{2.15}$$
$$u+\frac{1}{M}y_x(1) = \widetilde{u}, \tag{2.16}$$
$$v-y_x(0)+\beta y_t(0) = \widetilde{v}. \tag{2.17}$$

Eliminating $z = y - f$, one obtains

$$
\begin{aligned}
y - y_{xx} &= g + f, && (2.18)\\
y_x(1) &= M(\widetilde{u} + f(1) - y(1)), && (2.19)\\
y_x(0) &= \frac{m+\beta}{\gamma+1}\left(y(0) - f(0) - \frac{\widetilde{v}}{m+\beta}\right). && (2.20)
\end{aligned}
$$

Multiplying (2.18) by $w \in H^1(0,1)$ and integrate by parts on $[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1 \big(yw(x) + y_x w_x(x)\big)dx + \\
&My(1)w(1) + \frac{m+\beta}{\gamma+1}y(0)w(0) = \int_0^1 (y+f)w(x)dx + M\widetilde{u}w(1) \\
&\qquad + \frac{m+\beta}{\gamma+1}\widetilde{v}(0)w(0). \qquad (2.21)
\end{aligned}
$$

This is the weak formulation of $(2.18) - (2.20)$. The left-hand side of (2.21) is a continuous coercive bilinear form of y and w, which will be denoted by a. Moreover the right-hand side of (2.21) is a continuous linear form on $H^1(0,1)$ denoted by L. Using the well-known Lax-Milgram theorem, see e.g.[19], there exists a unique $y \in H^1(0,1)$ so that :

$$a(y,w) = L(w) \quad \text{for each } w \in H^1(0,1), \qquad (2.22)$$

since

$$y_{xx} = y - g - f \in L^2(0,1), \qquad (2.23)$$

we get $y \in H^2(0,1)$ and $z = y - f \in H^1(0,1)$ hence $U = (y,z,u,v) \in D(A)$. Finally, since A is m-dissipative, $D(A)$ is dense in $\mathscr{H}$. This shows that operator $I - A$ is onto and the proof of theorem 2.1 follows from the Lumer-Phillips theorem [11]. □

Remark 2.1. *It follows from Theorem 2.1 that the problem* (2.11) *has:*
a unique strong solution $U \in \mathscr{C}^1(\mathbb{R}_+,\mathscr{H}) \cap \mathscr{C}^0(\mathbb{R}_+,D(A))$ *if* $U_0 \in D(A)$*;*
a unique weak solution $U \in \mathscr{C}^0(\mathbb{R}_+,\mathscr{H})$ *if* $U_0 \in \mathscr{H}$.

Finally we can conclude that the problem defined by the system $(1.1) - (1.4)$ is well posed. Next, using the invariance principle of La Salle [6],we prove the strong stabilization of the semigroup generated by A.

3 Strong stabilization of the system

Let us define the energy function of the system $(1.1) - (1.4)$:

$$E(t) = \frac{1}{2}\int_0^1 \left(y_x^2 + y_t^2\right)dx + \frac{M}{2}y_t^2(1,t) + \frac{1}{2(m-\beta\gamma)}\left[my_t(0,t) - \gamma y_x(0,t)\right]^2. \qquad (3.24)$$

To obtain the strong stabilization of the system $(1.1) - (1.4)$ interpreted in terms of the evolutive problem (2.11), we need the two decisive lemmas below.

Lemma 3.1. *Let* φ *be a solution of the following system:*

$$
\begin{aligned}
\varphi_x(x) + \varphi(x) &= 1, && (3.25)\\
\varphi(1) &= -M. && (3.26)
\end{aligned}
$$

Then the function φ verifies the properties below:

$$\begin{aligned} \varphi &\leq -M, && (3.27)\\ \varphi(1) &= -M, && (3.28)\\ \varphi_x(x) &\geq 1. && (3.29)\end{aligned}$$

Proof. The proof is immediate. □

Lemma 3.2. *Let y be a strong solution of $(1.1)-(1.4)$ such that:*

$$y_t(0,t) = y_x(0,t) = 0,$$

then $y(x,t)$ is a constant function.

Proof. We follow the idea of [16]. Multiplying both sides of (1.1) by $\varphi(x)y_x(x,t)$ and integrate respect to the variables x and t we obtain:

$$\begin{aligned} I &= \int_0^1\int_0^T y_{tt}(x,t)\varphi(x)y_x(x,t)dxdt \\ &= \int_0^1 \Big[y_t(x,t)\varphi(x)y_x(x,t)\Big]_0^T dx - \frac{1}{2}\int_0^T \Big[\varphi y_t^2(x,t)\Big]_0^1 dt \\ &\quad +\frac{1}{2}\int_0^T\int_0^1 \varphi_x(x)y_t^2(x,t)dxdt. \end{aligned} \tag{3.30}$$

$$\begin{aligned} J &= \int_0^T\int_0^1 y_{xx}(x,t)\varphi(x)y_x(x,t)dtdx && (3.31)\\ &= \frac{1}{2}\int_0^T \Big[\varphi(x)y_x^2(x,t)\Big]_0^1 dt - \frac{1}{2}\int_0^1\int_0^T \varphi_x(x)y_x^2(x,t)dtdx. && (3.32)\end{aligned}$$

From $I=J$, $\varphi(1)=-M$, $y_t(0,t)=0$ and $y_x(0,t)=0$, we deduce that:

$$\begin{aligned} &\int_0^T \Big[\frac{1}{2}\int_0^1 \Big(\varphi_x(x)y_t^2(x,t)+ \\ &\varphi_x(x)y_x^2(x,t)\Big)dx + \frac{M}{2}y_t^2(1,t)\Big]dt \quad = \quad -\int_0^1 \big[y_t(x,t)\varphi(x)y_x(1,t)\big] \\ &\qquad -\frac{M}{2}\int_0^T y_x^2(1,t)dt. \end{aligned} \tag{3.33}$$

Using the assumptions of the lemma and the definition of the energy function $E(t)$ we get:

$$\frac{dE(t)}{dt} = -\beta y_t^2(0,t) - \frac{\gamma}{m-\beta\gamma}\Big[y_x(0,t)-\beta y_t(0,t)\Big]^2 = 0, \tag{3.34}$$

and using the properties of the function φ obtained in Lemma 3.1 we get

$$\int_0^T E(t)dt = TE(0) \leq -\int_0^1 \Big[y_t\varphi y_x\Big]_0^T dx \leq C\Big[E(0)+E(T)\Big] \leq 2CE(0). \tag{3.35}$$

Hence we get

$$(T-2C)E(0) \leq 0, \quad \forall T \in \mathbb{R},$$

thus $E(t)=E(0)=0$. Finally $y_t=0$, $y_x=0$ so that $y(x,t)$ is a constant. This completely proves the lemma. □

Theorem 3.1. *Let $U(t)$ be a weak solution of the problem (2.11) with the initial data $U_0=(y_0,y_1,u_0,v_0)\in\mathscr{H}$. Then*

$U(t)$ converge to $(C,0,0,0)$, where:

$$C = y_0(0) + \frac{1}{\beta}\int_0^1 y_t(x)dx + \frac{Mu_0 + v_0}{\beta}. \tag{3.36}$$

Proof. Since by virtue of Theorem 2.1 the operator A generates a semigroup of contractions, its domain is dense in the state energy Hilbert space. Hence we suppose that $U_0 = (y_0, y_1, u_0, v_0) \in D(A)$. Thanks to works of Cazenave and Haraux [4], the set

$$\omega(U_0) = \left\{ W \in \mathscr{H} / \exists (t_n) \to \infty \text{ such that } U(t_n) \to W \in \mathscr{H} \right\},$$

is non empty included in $D(A)$. Let $U_1 \in \omega(U_0)$ and $\widetilde{U}(t)$ be the strong solution of the problem (2.11) with initial data U_1. From the invariant principle of La Salle, the energy function $\widetilde{E}(t)$ associated with $\widetilde{U}(t)$ is a constant. Hence we get:

$$\frac{d\widetilde{E}(t)}{dt} = -\beta \widetilde{y}_t^2(0,t) - \frac{\gamma}{m-\beta\gamma}\left[\widetilde{y}_x(0,t) - \beta \widetilde{y}_t(0,t)\right]^2 = 0, \tag{3.37}$$

which leads to $\widetilde{y}_t(0,t) = \widetilde{y}_x(0,t) = 0$. Using Lemma 3.2, we get $\widetilde{U}(t) = U_1 = (C,0,0,0)$ so that the elements of $\omega(U_0)$ are constant functions. Let us now prove that $\omega(U_0)$ contains only one element. Choose $U = (C,0,0,0) \in \omega(U_0)$ and a sequence (t_n) which tends to infinity such as $U(t_n)$ converges to $(C,0,0,0)$ in $\mathscr{H}$. Consider now the function F defined as follows:

$$F(t) = \int_0^1 y_t(x,t)dx + Mu(t) + v(t) + \beta y(0,t). \tag{3.38}$$

Since this function is constant respect to the variable t we get

$$F(t) = F(0) = \int_0^1 y_1 dx + Mu_0 + v_0 + \beta y_0(0). \tag{3.39}$$

With (3.38) in mind, when t_n tends to $+\infty$ we get

$$C = y_0(0) + \frac{1}{\beta}\int_0^1 y_1(x)dx + \frac{Mu_0 + v_0}{\beta}. \tag{3.40}$$

□

4 Spectral analysis and Riesz basis property

In this section we suppose that: $\gamma \geq 0$ and $\gamma \neq m$. The main result of this section is the Riesz basis property for the system $(1.1)-(1.4)$. Our approach consists of proving that there is a sequence of generalized eigenvectors of operator A which forms a Riesz basis of the energy space. The study of the spectral problem associated with the evolutive system $(1.1)-(1.4)$ reveals that the spectral parameter appears in boundary conditions. For this kind of problems the classical theorem of Bari seems very difficult to apply [8]. Let us recall that the basic idea of Bari's theorem is that if $\left\{\phi_n\right\}_1^\infty$ is a Riesz basis for a Hilbert space $\mathscr{H}$ and another ω-linearly independent sequence basis $\left\{\psi_n\right\}_1^\infty$ of $\mathscr{H}$ satisfying $\sum_{n=1}^\infty \|\psi_n - \varphi_n\|^2 < \infty$, then $\left\{\psi_n\right\}_1^\infty$ also forms a Riesz basis itself. Here we use a method due to Shkalikov [17]. The basic idea of this method is to build with operator A a new operator called Shkalikov's linearized operator which verifies the Riesz basis property and then deduces the same property for the operator A. Here we have to work in complexified Hilbert space $\mathscr{H}$. For convenience we do not change the notation for this space. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and let $U = (y,z,u,v)^T \in D(A)$ be a corresponding eigenvector. To find U we have to solve the following equation $AU = \lambda U$ and hence the following set of equations

$$\begin{aligned} z &= \lambda y, && (4.41)\\ y_{xx} &= \lambda z, && (4.42)\\ -\frac{1}{M} y_x(1) &= \lambda z(1), && (4.43)\\ y_x(0) - \beta z(0) &= \lambda\Big(mz(0) - \gamma y_x(0)\Big). && (4.44) \end{aligned}$$

By eliminating z, we get

$$\begin{aligned} y_{xx} - \lambda^2 y &= 0, && (4.45)\\ y_x(1) + M\lambda^2 y(1) &= 0, && (4.46)\\ m\lambda^2 y(0) + \beta\lambda y(0) - (1+\gamma\lambda) y_x(0) &= 0. && (4.47) \end{aligned}$$

The orders of the boundary conditions are respectively $k_1 = 2$, $k_2 = 2$, the global order is then $k_1 + k_2 = 4$.
When λ is a nonzero eigenvalue, the Shkalikov's characteristic polynomial (see [17],p.1314) associated with (4.45) is

$$\omega^2 - 1 = 0, \tag{4.48}$$

which zeros are

$$\omega_1 = 1 \text{ and } \omega_2 = -1. \tag{4.49}$$

The solutions of (4.45) can be found as

$$y(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}. \tag{4.50}$$

Upon substituting (4.50) in the boundary conditions we obtain the following matrix equation

$$\begin{pmatrix} \Big[(1+\gamma\lambda)\lambda - (m\lambda^2 + \beta\lambda)\Big] & -\Big[(1+\gamma\lambda)\lambda + (m\lambda^2 + \beta\lambda)\Big] \\ (\lambda + M\lambda^2)e^{\lambda} & (-\lambda + M\lambda^2)e^{-\lambda} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.51}$$

A necessary and sufficient condition for this matrix equation to have nontrivial solutions for c_1 and c_2 is that the following characteristic determinant

$$\Delta(\lambda) = \begin{vmatrix} \Big[(1+\gamma\lambda)\lambda - (m\lambda^2 + \beta\lambda)\Big] & -\Big[(1+\gamma\lambda)\lambda + (m\lambda^2 + \beta\lambda)\Big] \\ (\lambda + M\lambda^2)e^{\lambda} & (-\lambda + M\lambda^2)e^{-\lambda} \end{vmatrix}, \tag{4.52}$$

vanishes; in other words

$$\begin{aligned} \Delta(\lambda) &= \Big\{[(\gamma+m)M\lambda^4 + (M\beta + m + M + \gamma)\lambda^3 + (1+\beta)\lambda^2]e^{\lambda} \\ &\quad + [(\gamma-m)M\lambda^4 + (-M\beta + m + M - \gamma)\lambda^3 + (-1+\beta)\lambda^2]e^{-\lambda}\Big\} \\ &= 0. && (4.53) \end{aligned}$$

It can easily seen that for the eigenvalues of large modulus $|\lambda|$ the dominant terms of the expression in bracket are $(\gamma+m)$ and $(\gamma-m)$ which are nonzero if $\gamma \geq 0$ and $\gamma \neq m$.
In this case, according to the theory of Shkalikov, we say that the boundary conditions of (4.46) – (4.47) are regular.
Our next task is to prove that the eigenvalues of operator A, with sufficiently large modulus are algebraically simple and isolated.
Since operator A is m-dissipative the eigenvalues λ of A are all in the left half complex plane, and hence verify $\Re e(\lambda) \leq 0$.

Lemma 4.1. *Consider the system given by* (2.12) *where* $\gamma > 0$ *and* $\gamma \neq m$.

1. *If $\beta > 0$ then zero is a geometrically simple eigenvalue of A and algebraically simple.*
2. *If $\beta = 0$ then zero is a geometrically simple eigenvalue of A and its algebraically multiplicity is 2.*
3. *If $\beta \geq 0$ the eigenvalues of A with sufficiently large modulus are algebraically simple..*

Proof.

1. Suppose $\beta > 0$.
Let $\lambda = 0$, the solutions of the Cauchy's problem $(4.45)-(4.47)$ are nonzero complex constant functions. Hence $\lambda = 0$ is an eigenvalue of A and the corresponding eigenvectors subspace is generated by the vector $(1,0,0,0)^T$ of $\mathscr{H}$. Thus the geometrical multiplicity of $\lambda = 0$ is 1. Let us now prove that its algebraic multiplicity is 1. The algebraic multiplicity of $\lambda = 0$ is greater than 1 if and only if there exists $W = (y,z,u,v)^T \in \ker A^2 \setminus \ker A$, which is equivalent to :
$$A(AW) = 0 \quad \text{and} \quad AW \neq 0. \tag{4.54}$$
Hence the vector AW belongs to *Ker A*, by normalizing W, we may suppose that
$$AW = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.55}$$
The later is equivalent to the following set of equations:
$$\begin{aligned} z &= 1, && (4.56) \\ y_{xx} &= 0, && (4.57) \\ -\frac{1}{M} y_x(1) &= 0, && (4.58) \\ y_x(0) - \beta z(0) &= 0, && (4.59) \end{aligned}$$
so that $y_x(0) = 0$, and since $\beta > 0$ we deduce from (4.59) that $z(0) = 0$ which leads to a contradiction.

2. Suppose $\beta = 0$. We use the previous notations. As above, $\lambda = 0$ is a geometrically simple eigenvalue of A.
On the other hand we get $z = 1$, and from (4.59) we deduce that y is a constant. Hence W belongs to the vector subspace of $\mathscr{H}$ generated by the vectors $(1,0,0,0)^T$ and $(0,1,1,1)^T$ so that the dimension of $\ker A^2$ is 2.
Next we prove that $\ker A^3 \setminus \ker A^2 = \{0\}$.
Consider $W = (y,z,u,v)^T \in \ker A^3 \setminus \ker A^2$. Then we have
$$A^2(AW) = 0 \quad \text{with} \quad AW \neq 0. \tag{4.60}$$
Hence $AW \in \ker A^2$ and we have
$$\begin{aligned} z &= C_1, && (4.61) \\ y_{xx} &= C_2, && (4.62) \\ -\frac{1}{M} y_x(1) &= C_2, && (4.63) \\ y_x(0) &= C_2, && (4.64) \end{aligned}$$

where $C_2 \neq 0$, so that

$$\begin{aligned} z &= C_1, && (4.65)\\ y_x &= C_2 x + C_3, && (4.66)\\ -\frac{1}{M}(C_2 + C_3) &= C_2, && (4.67)\\ C_3 &= C_2, && (4.68)\end{aligned}$$

which leads to a contradiction. Finally when $\beta = 0$ the algebraically multiplicity of $\lambda = 0$ is 2.

3. Using (4.52), we deduce by a straightforward computation that the nonzero eigenvalues of A are exactly the roots of the following equation

$$e^{2\lambda} = \left(\frac{\lambda(m-\gamma) - 1 + \beta}{\lambda(m+\gamma) + 1 + \beta}\right)\left(\frac{\lambda M - 1}{\lambda M + 1}\right). \tag{4.69}$$

If λ is a nonzero eigenvalue of A with algebraic multiplicity greater than 1, by differentiating (4.69) with respect to λ, we obtain

$$\begin{aligned} & e^{2\lambda}\Big[2(\lambda(m+\gamma)+1+\beta)(1+\lambda M) \\ & +(m+\gamma)(1+\lambda M) + M(\lambda(m+\gamma)+1+\beta)\Big] = M\Big(\lambda(m-\gamma)-1+\beta\Big) \\ & \qquad +(m-\gamma)(\lambda M - 1). \end{aligned} \tag{4.70}$$

Now, combining (4.69) and (4.70) we get

$$1 + \frac{1}{1-\lambda^2 M^2} + \frac{m - \gamma\beta}{(\lambda(m+\gamma)+1+\beta)(\lambda(\gamma - m) + 1 - \beta)} = 0. \tag{4.71}$$

This equation shows that there are at most four nonzero eigenvalues of A which could not be algebraically simple.
Hence The eigenvalues of A with sufficiently large modulus are algebraically simple.

□

Remark 4.1. *Since the characteristic equation* (4.69) *can not be explicitly solved one needs asymptotic methods to investigate the behavior of the eigenvalues. For these methods the reader can be referred to [3] or [9].*

Here we recall the following important result due to R. E. LANGER [9], which can be also found in [3].

Proposition 4.1. *Let $f(s)$ a function of the form*

$$f(s) = \sum_{k=0}^{n} a_k (1 + o(1))\, e^{\alpha_k s}, \quad |s| \to +\infty, \quad s = \sigma + i\tau,$$

*where a_k are nonzero complex numbers and α_k being real numbers such that
$\alpha_0 < \alpha_1 < \ldots < \alpha_n$.
Then the zeros of $f(s)$ all lie in a strip $a \le \sigma \le b$ and can asymptotically represented by those of the following comparison function $f^*(s) = \sum_{k=0}^{n} a_k e^{\alpha_k s}$. In particular, there exists $R > 0$, such that the number $N(T_1, T_2)$ of the zeros of $f(s)$ in any rectangle $a < \sigma \le b$, $T_1 < \sigma < T_2$ with $T_1 > R$ is limited by the following relation*

$$\left| N(T_1, T_2) - \frac{T_2 - T_1}{2\pi}(\alpha_n - \alpha_0) \right| \le n.$$

Note that $h = o(1)$ when $|s| \to +\infty$ means here that $h(s) \to 0$ when $|s| \to +\infty$.

Remark 4.2. *In particular the multiplicity of any zero of $f(s)$ is not greater than n.*

Lemma 4.2. *Consider the system given by* (2.12) *where $\gamma \neq m$ and $\gamma \geq 0$. Suppose $\beta \geq 0$. Then the following holds.*

1. *Each nonzero eigenvalue λ of A is such that $\Re e(\lambda) < 0$.*
2. *The eigenvalues of A with sufficiently large modulus are isolated.*

Proof.

1. Consider λ a nonzero eigenvalue of operator A and $y = (y,z,u,v)^T \in D(A)$ a corresponding eigenvector. Since A is a dissipative operator,
   $\Re e(\lambda) \leq 0$. Multiplying both sides of (4.45) by the conjugate function $\bar{y}$ of y and integrating by parts with respect to x we get

$$\int_0^1 y_{xx}\bar{y}dx - \int_0^1 \lambda^2 y\bar{y}dx = \left[y_x\bar{y}\right]_0^1 - \int_0^1 y_x\bar{y}_x dx - \lambda^2 \int_0^1 y\bar{y}dx = 0. \tag{4.72}$$

   Now using the boundary conditions (4.46) and (4.47) we have

$$\lambda^2 M|y(1)|^2 + \int_0^1 |y_x|^2 dx + \lambda^2 \int_0^1 |y|^2 dx + \frac{\lambda(\lambda m + \beta)}{1+\gamma\lambda}|y(0)|^2 = 0. \tag{4.73}$$

   If $\Re e(\lambda) = 0$ then $\lambda = i\tau$ with $\tau \in \mathbb{R}$ and equation (4.73) becomes

$$-\tau^2 M|y(1)|^2 + \int_0^1 |y_x|^2 dx - \tau^2 \int_0^1 |y|^2 dx + \frac{(-\tau^2 m + i\tau\beta)}{1+i\tau\gamma}|y(0)|^2 = 0. \tag{4.74}$$

   Hence the complex number $\frac{(-\tau^2 m+i\tau\beta)}{1+i\tau\gamma}|y(0)|^2$ is a real. So that isolating its imaginary part we get

$$(\tau^3\gamma m + \tau\beta)|y(0)|^2 = 0. \tag{4.75}$$

   Since $\tau \neq 0$, we get

$$(\tau^2\gamma m + \beta)|y(0)|^2 = 0, \tag{4.76}$$

   and using the fact that $\tau^2\gamma m \neq 0$ we obtain $y(0) = 0$.
   From (4.47) we deduce that $y_x(0) = 0$. Hence y is a solution of the following Cauchy's problem

$$\left\{ \begin{array}{lcl} y_{xx} - \lambda^2 y & = & 0, \\ y(0) & = & 0, \\ y_x(0) & = & 0, \end{array} \right. \tag{4.77}$$

   whose unique solution is $y = 0$. Hence we obtain $U = (y,z,u,v)^T = 0$ which leads to a contradiction.
   Therefore we get $\Re e(\lambda) < 0$ for each nonzero eigenvalue of A.

2. Let λ be an eigenvalue of A. Equation (4.53) shows that λ verifies the following equation

$$\begin{aligned} &\Big\{[(\gamma+m)M\lambda^4 + (M\beta + m + M + \gamma)\lambda^3 + (1+\beta)\lambda^2]e^{2\lambda} \\ &+[(\gamma-m)M\lambda^4 + (-M\beta + m + M - \gamma)\lambda^3 + (-1+\beta)\lambda^2]\Big\} \quad = \quad 0. \end{aligned} \tag{4.78}$$

   In this equation, the left hand expression is called an exponential sum, see [9] for more details. Next using the proposition 4.1 we deduce that for $\gamma \neq m$, the zeros of the above equation are asymptotically those of the following equation:

$$(\gamma+m)e^{2\lambda} + (\gamma - m) = 0. \tag{4.79}$$

The zeros of the above equation take the form

$$\lambda_n' = \begin{cases} \dfrac{1}{2}\ln\left|\dfrac{\gamma-m}{\gamma+m}\right| + in\pi, & n\in\mathbb{Z}, \quad \text{if} \quad m>\gamma, \\ \dfrac{1}{2}\ln\left|\dfrac{\gamma-m}{m+\gamma}\right| + i\left(n+\dfrac{1}{2}\right)\pi, & n\in\mathbb{Z}, \quad \text{if} \quad m<\gamma. \end{cases} \tag{4.80}$$

From (4.80), we deduce that the eigenvalues of A with sufficiently large modulus are isolated.

□

Finally we can conclude that for $\beta \geq 0$ and $\gamma \neq m$, the eigenvalues operator A are asymptotically algebraically simple and isolated and since the boundary conditions of the spectral problem (4.45)-(4.47) are also regular we say according to the theory of Shkalikov in [17], that they are strongly regular.

Theorem 4.1. *Consider the system given by* (2.11) *where $\beta \geq 0$ and $\gamma \neq m$, then there exists a fundamental system of generalized eigenvectors of the operator A which forms a Riesz basis in $\mathscr{H} = H^1(0,1)\times L^2(0,1)\times\mathbb{R}^2$.*

Proof. Following the notations of Shkalikov in [17], for integer $r \geq 0$ we set

$$\mathscr{W}_2^r = W_2^{1+r}(0,1)\oplus W_2^r(0,1), \tag{4.81}$$

where $W_2^k(0,1)$ is the Sobolev space of smooth functions on the segment $[0,1]$, having $k-1$ absolutely continuous derivatives and $k-th$ derivative from $L^2(0,1)$ with the norm $\|f\|_{W^k} = \left\|f^{(k)}\right\|_{L^2(0,1)} + \|f\|_{L^2(0,1)}$.
We rewrite (4.45) in the form

$$l(y,\lambda) = -y'' + \lambda^2 y = 0. \tag{4.82}$$

Now we consider the operator H defined as follows

$$\tilde{v} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in \mathscr{W}_2^r \longmapsto H\tilde{v} = \begin{bmatrix} v_1 \\ v_0'' \end{bmatrix} \in \mathscr{W}_2^r, \tag{4.83}$$

where $v_0 = y$, $v_1 = \lambda y$. We also define $H^i(\tilde{v}) \in \mathscr{W}_2^{r-i}$ where H^i is the $i-th$ power of H. Now we normalize the boundary conditions (4.46) and (4.47) according to Shkalikov's method [17]. First we rewrite them as follows

$$U_1(y,\lambda) = -m\lambda^2 y(0) - \beta\lambda y(0) + (1+\lambda\gamma)y'(0) = 0, \tag{4.84}$$
$$U_2(y,\lambda) = y'(1) + M\lambda^2 y(1) = 0, \tag{4.85}$$

and we do the following transformations:

$$\lambda^i y^{(k)}(x) = (H^i\tilde{v})_0^{(k)}(x) \quad \text{if} \quad i+k<n+r,$$

$$\lambda^i y^{(k)}(x) = \lambda^{i+k-n-r+1}(H^{n+r-k-1}\tilde{v})_0^{(k)}(x) \quad \text{if} \quad i+k\geq n+r,$$

where $x=0$ or $x=1$, n being the number of boundary conditions and the subscript index means that one takes the first component of the associated vector. In our case we have $n=2$. We rewrite the above boundary conditions as follows:

$$U_i(\tilde{v},\lambda) = \sum_{k=0}^{\nu_i(r)} \lambda^k U_i^k(\tilde{v}), \quad 1\leq i\leq n,$$

where for any $1\leq i\leq n$, $U_i^k(\tilde{v})$ is a linear form of the variable $\tilde{v}$ which does not depends on λ. We set

$$N_r = \nu_1(r) + \nu_2(r) + \cdots + \nu_q(r),$$

where the numbers v_i are those who appear above. Here we then take $r=0$, $\lambda^2 y(0)$ is replaced by $\lambda v_1(0)$ and the term $\lambda^2 y(1)$ by $\lambda v_1(1)$ and $\lambda y(0)$ is replaced by v_1. Hence we obtain :

$$U_1(y,\lambda) = -\lambda v_1(0)+\frac{1}{m}(-\beta v_1(0)+v_0'(0)+\lambda\gamma v_0'(0))=0, \tag{4.86}$$

$$U_2(y,\lambda) = \frac{1}{M}v_0'(1)+\lambda v_1(1)=0, \tag{4.87}$$

and we get : $v_1(0)=1$, $v_2(0)=1$ so that $N_0=2$. We shall denote $\mathscr{W}^r_{2,U}$ the Shkalikov space defined as follows

$$\mathscr{W}^r_{2,U} = \Big\{ \widetilde{v}\in\mathscr{W}^r_2, U_j(H^k(\widetilde{v}))=0, \quad 1\le j\le n \quad \text{for} \quad 0\le k\le n+r-2 \text{ and all boundary conditions of order } \le n+r-k-2 \Big\}. \tag{4.88}$$

Following the theory of Shkalikov $\mathscr{W}^r_{2,U}$ is a closed subspace of finite codimension in $\mathscr{W}^r_2$. In our case, since $n=2$ for $r=0$ and $r=1$ we have

$$\mathscr{W}^0_{2,U} = \left\{ \widetilde{v}=\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}\in W^1_2(0,1)\oplus L^2(0,1)\right\}. \tag{4.89}$$

$$\mathscr{W}^1_{2,U} = \left\{ \widetilde{v}=\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}\in W^1_2(0,1)\oplus W^2_2(0,1)\right\}. \tag{4.90}$$

We define the Shkalikov's operator as follows

$$H_0\begin{pmatrix} v_0\\ v_1\\ z_1\\ z_2\end{pmatrix} = \begin{bmatrix} v_1 \\ v_0'' \\ \frac{1}{m}\Big(-\beta v_1(0)+v_0'(0)-\lambda\gamma v_0'(0)\Big) \\ -\frac{1}{M}v_0'(1)\end{bmatrix}. \tag{4.91}$$

From the Corollary 3.2 of the Theorem 3.1 of Shkalikov [17], there is a set of generalized eigenvectors of the operator H_0 which forms a Riesz basis of the Hilbert space $\mathscr{W}^0_{2,U}\oplus\mathbb{C}^2 = H^1(0,1)\oplus L^2(0,1)\oplus\mathbb{C}^2$. Now we built now a Riesz basis for operator A. We have

$$D(H_0) = \left\{\begin{pmatrix} u\\ v\\ z_1\\ z_2\end{pmatrix}\in\mathscr{W}^1_{2,U}\oplus\mathbb{C}^2/z_1=v(0), z_2=v(1)\right\}. \tag{4.92}$$

Next we prove that the spectral problem associated with operator H_0 is equivalent to the one defined by A.
First, suppose that

$$H_0 U = \lambda U \quad \text{where} \quad U=(u,v,z_1,z_2)^T\in D(H_0),$$

is an eigenvector associated with $\lambda\in Sp(H_0)$.

Then we obtain

$$v = \lambda u, \tag{4.93}$$
$$u'' = \lambda v, \tag{4.94}$$
$$\frac{1}{m}\Big(-\beta v(0)+v'(0)-\lambda\gamma v'(0)\Big) = \lambda z_1 = \lambda v(0), \tag{4.95}$$
$$-\frac{1}{M}u'(1) = \lambda z_2 = \lambda v(1), \tag{4.96}$$
$$U \in D(H_0), \tag{4.97}$$

by substitution we have

$$u''-\lambda^2 u = 0, \tag{4.98}$$
$$-\lambda v(0)+\frac{1}{m}(-\beta v(0)+v'(0)+\lambda\gamma v'(0)) = 0, \tag{4.99}$$
$$\lambda v(1)+\frac{1}{M}u'(1) = 0. \tag{4.100}$$

Thus λ is an eigenvalue of A associated with the spectral problem (4.45) – (4.47).
Next let λ be an eigenvalue of A associated with the spectral problem (4.45) – (4.47), we easily deduce that λ is an eigenvalue of H_0.
Since we know from the previous study of H_0 that there is a set of generalized eigenvectors of the operator H_0 which forms a Riesz basis of the Hilbert space $\mathscr{W}^0_{2,U}\oplus\mathbb{C}^2 = H^1(0,1)\oplus L^2(0,1)\oplus\mathbb{C}^2$, we deduce that there is also a set of generalized eigenvectors of the operator A which forms a Riesz basis of the Hilbert space $\mathscr{H}$.
This completely proves the theorem. □

References

[1] G. D. Birkhoff, Boundary value and expansion problems, Transactions of this Society, 9 (1917) 373.

[2] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contraction dans les espaces de Hilbert, North-Holland, (1972).

[3] O. J. V. Castañón, Sur la répartition des zéros de certaines fonctions méromorphes liées à la fonction zêta de Riemann, Thèse de doctorat, Université Bordeaux, 1 (2008).

[4] T. Cazenave, A. Haraux, Introduction aux Problèmes d'évolution semi-linéaires, Ellipses, (1990).

[5] S. Cox, E. Zuazua, The rate at which Energy decays in a string damped at one end, Indiana University Mathematics Journal, 44.2 (1995). 545-573. (1972).

[6] F. Conrad, A. Mifdal, Strong stability of a model of an overhead crane Control and Cybernetics, 27 (3) (1998) 363-394.

[7] P. Grabowski, Spectral approach to well-posedness and stability analysis of hybrid feedback systems, Journal of Mathematical Systems Estimation and Control, 6 (1996).

[8] B. Z. Guo, Riesz basis approach to the stabilization of a flexible beam with a tip mass, SIAM J. Control Optim, 39 (2001) 1736-1747.
http://dx.doi.org/10.1137/S0363012999354880

[9] R. E. Langer, On the zeros of exponential sums and integrals, Bull. Amer. Math. Soc, 37 (4) (1931) 213-239.
http://dx.doi.org/10.1090/S0002-9904-1931-05133-8

[10] A. Mifdal, Eťude de la stabilité forte et uniforme d'un système hybride. Application à un modèle de pont roulant, Thèse de doctorat de l'Université Nancy, 1 (1997).

[11] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations Springer-Verlag, New York, (1983).

[12] B. P. Rao, Decay Estimates of solutions for a Hybrid System of of flexible Structure, European Journal of applied Mathematics, 4 (1993) 303-319.
http://dx.doi.org/10.1017/S0956792500001133

[13] D. L. Russell, Decay rates for weakly damped systems in Hilbert space obtained with control-theoric methods, J. Differential Equations, 19 (1975) 344-370.
http://dx.doi.org/10.1016/0022-0396(75)90009-1

[14] D. L. Russell, Nonharmonic Fourier series in control theory of distributed systems, J. Math. Anal. Appl, 18 (1967) 542-559.
http://dx.doi.org/10.1016/0022-247X(67)90045-5

[15] P. Rideau, Contrôle d'un Assemblage de Poutres Flexibles par des Capteurs Actionneurs Ponctuels; Etude du Spectre du système, Thèse, Ecole Nationale Suprieure des Mines de Paris, Sophia-Antipolis, France, (1985).

[16] F. Z. Saouri, Stabilisation de quelques systmes élastiques. Analyse spectrale et comportement asymptotique, Thèse de doctorat, Université Henry Poincaré I, (2000).

[17] A. Shkalikov, Boundary problem for ordinary differenial operators with parameter in boundary conditions, Journal of Soviet Mathematics, 33 (1986) 1311-1342.
http://dx.doi.org/10.1007/BF01084754

[18] J. D. Tamarkin, Some general problems in the theory of ordinary linear differential equations, Mathematische Zeitschrift, 27 (1927) 24.

[19] H. Tanabe, Equations of Evolution, Pitman, London, (1979).