Some results of best simultaneous approximation in normed spaces

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Abstract
In this paper we prove some results of best simultaneous approximation on normed spaces and then we see some applications.

Keywords: Best approximation, Best simultaneous approximation, Uniformly convex, Approximatively compact, Normed space.

1 Introduction

The problem of best approximation and best simultaneous approximation in normed spaces has been studied by several authors (for example see [14] and [11]). Also several results of best simultaneous approximation in the context of normed linear space were obtained by Goel, et al (9] and [10]). Moreover to see the extensive basic information of best approximation in inner product spaces we refer the readers to [5]. On the other hand these subjects have many applications in the other areas (see [3] and [12]). In this paper first, we prove our results for best simultaneous approximation in some subsets of a normed space, such as convex and finite dimensional sets. Also we see a notable result in uniformly convex Banach spaces. At the second part the notion of approximatively compact spaces has been considered and some properties of these spaces has been proved. The last theorem states a relation between the best and best simultaneous approximation under some conditions. But first of all let us to introduce the preliminaries.

Definition 1.1. ([5]) Let \((X, ||\cdot||)\) be a normed space, \(F\) be a bounded set in \(X\) and \(K\) be any subset of \(X\). An element \(x^* \in K\) is said to be a best approximation to \(F\), if

\[
d(F,K) = \sup_{f \in F} || f - x^* ||,
\]

where

\[
d(F,K) = \inf_{k \in K} \sup_{f \in F} || f - k ||.
\]

If we put \(F = \{x\}\) then

\[
dist(x,K) = \inf_{k \in K} || x - k ||,
\]

is called the distance of \(x\) and \(K\).

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**Definition 1.2.** ([14]) Let \((X, \|\|)\) be a normed space. Then \(G\) is a proximinal subset of \(X\) if for each \(x \in X\), the following set
\[
\alpha_G(x) = \{g \in G : \|g - x\| = \text{dist}(x, G)\},
\]
is non empty. A set \(G\) is said to be approximatively compact if for each \(x \in X\) and each sequence \(\{g_n\}\) in \(G\) with \(\lim_{n \to \infty} \|g_n - x\| = \text{dist}(x, G)\), there exists a subsequence \(\{g_{n_k}\}\) converging to an element of \(G\). An element \(x\) is said to be orthogonal to another element \(y \in X\) and we write \(x \perp y\) if for each scalar \(\lambda\), \(\|x\| \leq \|x - \lambda y\|\). Also we define \(M^\perp\) as
\[
M^\perp = \{x \in X : x \perp g, \forall g \in G\}. 
\]
Orthogonality in \(X\) is said to be homogeneous if \(x_1 \in M^\perp\) implies that for each scalar \(\lambda\), \(\lambda x_1 \in M^\perp\).

**Definition 1.3.** ([14]) A normed space \((X, \|\|)\) is said to be uniformly convex if for both given sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that for each \(n \in \mathbb{N}\), \(\|x_n\| \leq 1\), \(\|y_n\| \leq 1\), and
\[
\lim_{n \to \infty} \frac{\|x_n + y_n\|}{2} = 1,
\]
implies that \(\lim \|x_n - y_n\| = 0\). Also \(X\) is called strictly convex if for each \(x, y \in X\) such that \(\|x\| = \|y\| = 1\) and \(x \neq y\), implies that \(\|\frac{x + y}{2}\| < 1\).

Let \(X\) be a vector space over a field \(\mathbb{F}\). We remind that a subset \(C\) of \(X\) is called convex if for each \(x, y \in C\) and \(0 \leq t \leq 1\) we have
\[
tx + (1-t)y \in C.
\]

2 Main section

**Lemma 2.1.** Let \((X, \|\|)\) be a normed space.

1. Suppose that \(K \subseteq X\) and \(F\) be bounded subset of \(X\). The map \(\phi(x) = \sup_{f \in F} \|f - x\|\) is a continuous function on \(X\).

2. If \(K\) is a finite dimensional subspace of \(X\). Then there exists a best simultaneous approximation \(b \in K\) to any given compact subset \(F\) of \(X\).

**Proof.**

1. For any \(f \in F\) and \(x, y \in X\) we have
\[
\sup_{f \in F} \|f - x\| \leq \sup_{f \in F} (\|f - x\| + \|x - y\|).
\]
Now suppose that \(\|x - y\| < \varepsilon\). Then \(\phi(x) \leq \phi(y) + \varepsilon\). By the same way we have \(\phi(y) \leq \phi(x) + \varepsilon\). Then we have \(\|\phi(x) - \phi(y)\| < \varepsilon\) and so the proof is complete.

2. From compactness of \(F\) we imply that there exists \(M > 0\) such that
\[
\sup_{f \in F} \|f\| = M.
\]
Now as a subset of \(K\), suppose that \(S = B^M(0)\). Then
\[
\inf_{k \in S} \sup_{f \in F} \|f - k\| = \inf_{k \in K} \sup_{f \in F} \|f - k\| \\
\leq M.
\]
But \(S\) is compact and \(\phi\) is continuous. Then \(\phi\) attains its maximum over \(S\), for some \(b \in K\) which will be the best simultaneous to \(F\).
Theorem 2.1. Let $(X, \|\cdot\|)$ be a normed space. Suppose that $K$ be a convex subset of $X$ and $F \subseteq X$. If $k_1, k_2 \in K$ be two best simultaneous approximations to $F$ then for $0 \leq \lambda \leq 1$, $k^* = \lambda k_1 + (1 - \lambda)k_2$ is also a best simultaneous approximation to $F$.

Proof. First note that by definition of convexity, we have

$$d(F, K) = \inf_{k \in K} \sup_{f \in F} \|f - k\| \leq \sup_{f \in F} \|f - k^*\|.$$ 

On the other hand we see that

$$\sup_{f \in F} \|f - k^*\| = \sup_{f \in F} \|f - (\lambda k_1 + (1 - \lambda)k_2)\| = \sup_{f \in F} \|\lambda(f - k_1) + (1 - \lambda)(f - k_2)\| \leq \lambda \sup_{f \in F} \|(f - k_1)\| + (1 - \lambda)\sup_{f \in F} \|(f - k_2)\| = \lambda d(F, K) + (1 - \lambda)d(F, K) = d(F, K).$$

So we have

$$d(F, K) = \sup_{f \in F} \|f - k^*\|,$$

and hence the proof is complete. \hfill \square

Theorem 2.2. Let $(X, \|\cdot\|)$ be a strictly convex space and $K$ be a finite dimensional subspace of $X$. Then for any compact subset $F$ of $X$, there exists a unique best simultaneous approximation of $K$ to $F$.

Proof. The existence solution follows from lemma 2.1. Now we investigate the uniqueness case. Let $k_1$ and $k_2$ be two distinct best simultaneous approximations to $F$. Then we have

$$\inf_{k \in K} \sup_{f \in F} \|f - k\| = \sup_{f \in F} \|f - k_1\| = \sup_{f \in F} \|f - k_2\| = d. \tag{2.1}$$

According to pervious theorem $\frac{k_1 + k_2}{2}$ is also a best simultaneous approximation.i.e.

$$\sup_{f \in F} \|f - \frac{k_1 + k_2}{2}\| = d.$$

But $F$ is compact and hence there exists $f_0 \in F$ such that

$$\sup_{f \in F} \|f - \frac{k_1 + k_2}{2}\| = \|f_0 - \frac{k_1 + k_2}{2}\| = d. \tag{2.2}$$

From (2.1) we have $\|f_0 - k_1\| \leq d$ and $\|f_0 - k_2\| \leq d$ and hence by strict convexity

$$\|f_0 - k_1 + f_0 - k_2\| \leq 2d,$$

which implies that

$$\|f_0 - \frac{k_1 + k_2}{2}\| < d.$$

This is a contradiction to (2.2) and hence the proof is complete. \hfill \square

In the sequel we wish to prove an interesting theorem on uniformly convex Banach spaces.

Theorem 2.3. Suppose that $K$ be a closed and convex subset of a uniformly convex Banach spaces $X$. Then for any compact subset $F$ of $X$, there exists a unique best approximation to $F$ from elements of $K$. 

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Proof. Since $K$ is convex without lose of generality we may assume that
\[ d = \inf_{k \in K} \sup_{f \in F} \|f - k\|. \]

Suppose that \( \{k_n\} \) be a sequence in $K$ such that
\[ \lim_{n \to \infty} \sup_{f \in F} \|f - k_n\| = d, \]
and
\[ d_m = \sup_{f \in F} \|f - k_m\|, \quad m \geq 1. \]

We see that $d_m \geq d$ and hence
\[ \frac{\|f - k_m\|}{d_m} \leq 1. \quad (2.3) \]

Take
\[ \frac{1}{2} \frac{k_m}{d_m} + \frac{k_n}{d_n} = \frac{(d_m d_n + k_md_m)(d_m + d_n)}{(d_m + d_n)2d_md_n}. \]

If we assume that $y_{m,n} = \frac{d_k + d_n}{d_n + d_n}$, by convexity of $K$, $y_{m,n} \in K$ and hence we have
\[ \sup_{f \in F} \|f - y_{m,n}\| \geq d, \]
and
\[ \sup_{f \in F} = \|f - y_{m,n}\| = \left( \frac{d_m + d_n}{2d_md_n} \right) \frac{d_m + d_n}{d_m + d_n}. \]

Since $F$ is a compact set of $X$, there exists $f \in F$ such that
\[ \frac{\|f - k_m\|}{d_m} - \frac{\|f - k_n\|}{d_n} \geq d \frac{d_m + d_n}{d_m d_n}. \]

By (2.3) and uniform convexity of $X$, for a given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for each $m, n \geq n_0$ we have
\[ \frac{\|f - k_m\|}{d_m} - \frac{\|f - k_n\|}{d_n} < \varepsilon. \quad (2.4) \]

Let $m, n \to \infty$. By (2.4) and using the fact that $d_m \to d$, we see that $\{k_n\}$ is a Cauchy sequence. Suppose that $k_n \to k$. Then $k \in K$ as $K$ is closed and a simple calculation shows that $k$ is a best simultaneous approximation. If $k_1$ and $k_2$ be two best approximations, then we have
\[ \lim_{n \to \infty} k_n = k_1 \text{ and } \lim_{m \to \infty} k_m = k_2. \]

Hence
\[ \lim_{n \to \infty} \sup_{f \in F} \|f - k_n\| = d = \lim_{m \to \infty} \sup_{f \in F} \|f - k_m\|. \]

This implies that $\sup_{f \in F} \|f - k_1\| = \sup_{f \in F} \|f - k_2\|$ and hence $k_1 = k_2$. \qed
3 Approximatively compact sets and best approximations

**Lemma 3.1.** Let \((X, \|\cdot\|)\) be a normed space. Then for each \(x \in X\), \(\alpha_G(x)\) is a closed subset of \(X\).

**Proof.** If \(\alpha_G(x)\) were not closed, there would exist a sequence \(\{x_n\}\) in \(\alpha_G(x)\) such that \(x_n \to x\) and \(x \notin \alpha_G(x)\). Then \(\text{dist}(x, \alpha_G(x)) \leq \|x_n - x\| \to 0\), so that \(\text{dist}(x, \alpha_G(x)) = 0\). But \(\|x - y\| > 0\), for each \(y \in \alpha_G(x)\), since \(x \notin \alpha_G(x)\). This contradicts the proximinality of \(\alpha_G(x)\).

**Theorem 3.1.** Let \((X, \langle ., . \rangle)\) be an inner product space with induced norm \(\|\cdot\|\). If \(G\) is a subset of \(X\) such that for each \(x \in X\), \(\alpha_G(x)\) is convex, then \(\alpha_G(x)\) is compact.

**Proof.** Fix any \(x \in X\) and suppose that \(\{y_n\}\) be a sequence in \(\alpha_G(x)\) such that
\[
\|y_n - x\| \to \text{dist}(x, \alpha_G(x)).
\] (3.5)

By parallelogram law we have
\[
\|y_n - y_m\|^2 = \|x - y_m\|^2 + \|x - y_n\|^2 - 2\|x - (y_n + y_m)/2\|^2.
\]

On the other hand \((y_n + y_m)/2 \in \alpha_G(x)\). Hence we have
\[
\|y_n - y_m\|^2 \leq 2\|x - y_m\|^2 + \|x - y_n\|^2 - 4\text{dist}(x, \alpha_G(x))^2.
\] (3.6)

By (3.5) and (3.6), we see that \(\{y_n\}\) is a Cauchy sequence in \(\alpha_G(x)\). Hence by lemma 3.1, \(\{y_n\}\) converges to some point \(y \in \alpha_G(x)\) and so the proof is complete.

The above theorem has the following corollaries.

**Corollary 3.1.** If \(G\) is an approximatively compact set of the normed space \((X, \|\cdot\|)\), then the set \(\alpha_G(x)\) is compact.

**Proof.** Let \(\{y_n\}\) be a sequence in \(\alpha_G(x)\). This means that
\[
\|x - y_n\| = \text{dist}(x, G), \ \forall n \in \mathbb{N}.
\]

Hence \(\lim_{n \to \infty} \|x - y_n\| = \text{dist}(x, G)\). Since \(G\) is approximatively compact, \(\{y_n\}\) has a subsequence \(\{y_{n_k}\}\) converging to an element \(y \in \alpha_G(x)\) and by lemma 3.1 \(y \in \alpha_G(x)\). Hence \(\alpha_G(x)\) is compact.

**Corollary 3.2.** Let \((X, \langle ., . \rangle)\) be an inner product space. Then each complete convex subset of \(X\) is approximatively compact.

In the following theorem we see an interesting relation between the best and best simultaneous approximation.

**Theorem 3.2.** Let \(M\) be a subspace of a normed space \((X, \|\cdot\|)\). Then for each compact set \(K\) of \(X\) such that \(K \subseteq M^\perp\), \(K\) has a best simultaneous approximation such \(k\), to \(M\). Moreover if the orthogonality in \(X\) is homogenous and \(x_1, x_2, \ldots, x_n \in M^\perp\) are linearly dependent, then \(k\) is a best approximation for their average \(\frac{x_1 + x_2 + \ldots + x_n}{n}\).

**Proof.** Since \(K\) is compact and \(\|\cdot\|\) is a continuous map, then for some \(x_0 \in K\), we may assume that \(\sup_{x \in K} \|x\| = \|x_0\|\). So for each scalar \(\alpha\), we have
\[
\sup_{x \in K} \|x\| = \|x_0\| \\
\leq \|x_0 - \alpha g\|, \ \forall g \in M \\
\leq \sup_{x \in K} \|x - \alpha g\|, \ \forall g \in M.
\]
This implies that
\[
\sup_{x \in K} \parallel x \parallel \leq \inf_{g \in M} \sup_{x \in K} \parallel x - \alpha g \parallel \leq \sup_{x \in K} \parallel x \parallel.
\]

Hence
\[
\sup_{x \in K} \parallel x \parallel = \inf_{g \in M} \sup_{x \in K} \parallel x - \alpha g \parallel.
\]

This shows that \( k = 0 \) is a best simultaneous approximation to \( K \) and the proof of first part is complete. Now suppose that \( x_1, x_2, \ldots, x_n \in M^\perp \) are linearly dependent. If \( x_1 \) and \( x_2 \) in \( M^\perp \) are linearly dependent then for some scalar \( \lambda \) we have
\[
\frac{x_1 + x_2}{2} = \frac{x_1 + \lambda x_2}{2} = (1 + \lambda) x_1.
\]

A simple calculation shows that \( k = 0 \) is a best approximation for each \( x \in M^\perp \) and as orthogonality in \( X \) is homogeneous, \( x_1 \in M^\perp \) implies that \( \frac{x_1 + x_2}{2} \in M^\perp \). So \( k = 0 \) is a beat approximation to \( \frac{1 + \lambda}{2} x_1 \) i.e. to \( \frac{x_1 + x_2}{2} \). Now by induction the proof is complete.

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