Growth of Solutions to System of Nonlinear Wave Equations with Degenerate Damping and Strong Sources

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Abstract

M. A. Rammaha and Sawanya Sakuntasathien, [Global existence and blow up of solutions to systems of nonlinear wave equations with degenerate damping and source terms, Nonlinear Analysis 72(2010)2658-2683], introduced and studied the concept of existence and nonexistence of solutions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n=1,2,3$. In the present work we will prove that the solutions of system of nonlinear wave equations with degenerate damping and source terms supplemented with the initial and Dirichlet boundary conditions grows exponentially in a bounded domain $\Omega \subset \mathbb{R}^n$, $n > 0$, provided that the initial data are large enough, with positive initial energy and the strong nonlinear functions $f_1$ and $f_2$ satisfying appropriate conditions.

Keywords: Exponential growth, Degenerate damping, Strong nonlinear source, Positive initial energy

1 Introduction

In this work we consider the following system of nonlinear wave equations with degenerate damping and strong nonlinear source terms:

$$
\begin{align*}
    u_{tt} - \Delta u + (a_1 |u|^k + a_2 |v|^l) |u|^{m-1} u_t &= f_1(u,v), \\
    v_{tt} - \Delta v + (a_3 |v|^\theta + a_4 |u|^\rho) |v|^{r-1} v_t &= f_2(u,v),
\end{align*}
$$

where $m,r > 0$, $k,l,\theta,\rho \geq 1$ and the two functions $f_1(u,v)$ and $f_2(u,v)$ given by

$$
\begin{align*}
    f_1(u,v) &= a_5 |u+v|^{2(\rho+1)}(u+v) + a_6 |u|^{\rho+2} |v|^\rho, \\
    f_2(u,v) &= a_7 |u+v|^{2(\rho+1)}(u+v) + a_8 |u|^{\rho+2} |v|^\rho, \quad \rho > -1
\end{align*}
$$

In (1.1), $u = u(t,x), v = v(t,x)$ where $x \in \Omega$ is a bounded domain of $\mathbb{R}^n$, $(n \geq 1)$ with a smooth boundary $\partial \Omega$ and $t > 0$, $a_i > 0, i = 1,2,\ldots$. System (1.1) is supplemented with the following initial conditions

$$
(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), x \in \Omega
$$

and boundary conditions

$$
u(x) = v(x) = 0, x \in \partial \Omega.\quad (1.4)$$

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Some special cases of the single wave equations with nonlinear damping and nonlinear source terms in the form:

\[ u_{tt} - \Delta u + a|u|^m u_t = b|u|^{p-1}u \quad (1.5) \]

arise in quantum field theory which describe the motion of charged mesons in an electromagnetic field. Equation (1.5) together with initial and boundary conditions of Dirichlet type, has been extensively studied and results concerning existence, blow up and asymptotic behavior of smooth, as well as weak solutions have been established by several authors over the past decades. The study of single wave equation with the presence of different mechanisms of dissipation, damping and for more general forms of nonlinearities has been extensively studied and results concerning existence, nonexistence and asymptotic behavior of solutions have been established by several authors and many results appeared in the literature over the past decades. See ([1], [3], [9], [10], [12], [14], [16], [21], [24], [26]) and references therein.

Concerning the system of wave equations, Milla Miranda and Medeiros [17] considered the following system

\[ \begin{align*}
    u_{tt} - \Delta u + |v|^{p+2} |u|^p &= f_1(x), \\
    v_{tt} - \Delta v + |u|^{p+2} |v|^p &= f_2(x),
\end{align*} \quad (1.6) \]

in \( \Omega \times (0, T) \). Using the method of potential well, the authors determined the existence of weak solutions of system (1.6).

In [2] Agre and Rammaha studied the following system:

\[ \begin{align*}
    u_{tt} - \Delta u + |u|^{m-1} u_t &= f_1(u, v), \\
    v_{tt} - \Delta v + |v|^{l-1} v_t &= f_2(u, v),
\end{align*} \quad (1.7) \]

in \( \Omega \times (0, T) \) with initial and boundary conditions and the nonlinear functions \( f_1 \) and \( f_2 \) satisfying appropriate conditions. They proved under some restrictions on the parameters and the initial data many results on the existence of a weak solution. They also showed that any weak solution with negative initial energy blows up in finite time using the same techniques as in [9].

In [21], author considered the same problem treated in [2], and he improved the blow up result for a large class of initial data in which the initial energy can take positive values.

In the work [16], authors considered the nonlinear viscoelastic system:

\[ \begin{align*}
    u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(x, s) ds + |u|^{m-1} u_t &= f_1(u, v), \\
    v_{tt} - \Delta v + \int_0^t h(t-s) \Delta v(x, s) ds + |v|^{l-1} v_t &= f_2(u, v),
\end{align*} \quad (1.8) \]

where

\[ \begin{align*}
    f_1(u, v) &= a|u+v|^{2(p+1)}(u+v) + b|u|^p u|v|^{p+2}, \\
    f_2(u, v) &= a|u+v|^{2(p+1)}(u+v) + b|u|^{p+2}|v|^p v.
\end{align*} \quad (1.9) \]

and they prove a global nonexistence theorem for certain solutions with positive initial energy, the main tool of the proof is a method used in [21]. The second author considered the more general system

\[ \begin{align*}
    |u|^{\gamma} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(x, s) ds + |u|^{m-1} u_t &= f_1(u, v), \\
    |v|^{\gamma} v_{tt} - \Delta v - \Delta u_{tt} + \int_0^t h(t-s) \Delta v(x, s) ds + |v|^{l-1} v_t &= f_2(u, v),
\end{align*} \quad (1.10) \]

in [22] with the same function \( f_1, f_2 \) given in (1.9). He proved that the energy associated to the system (1.10) is unbounded and it grows up as an exponential function as time goes to infinity, provided that the initial data are large enough. The key ingredient in his proof is a method used in Vitillaro [26] and developed in [21] for a system of wave equations.
Concerning the study of decay of solutions of systems of evolution equations, let us mention the work in [23], where they treated the nonlinear viscoelastic system in (1.8) and under some restrictions on the nonlinearity of the damping and the source terms, they prove that, for certain class of relaxation functions and for some restrictions on the initial data, the rate of decay of the total energy depends on those of the relaxation functions.

Recently, in [18] M. A. Rammaha and Sawanya Sakuntasathien focus on the global well-posedness of the system of nonlinear wave equations in a bounded domain \( \Omega \subset \mathbb{R}^n \), \( n = 1, 2, 3 \), with Dirichlet boundary conditions. The nonlinearities \( f_1(u,v) \) and \( f_2(u,v) \) act as a strong source in the system.

In section 3 (Theorem 3.1), we will state and prove our main result, where we prove that, under some restrictions on the initial data and (with positive initial energy) for some conditions on the functions \( f \), the solution of problem (1.1)-(1.4) grows exponentially on the initial data and (with positive initial energy) for some conditions on the functions \( f_1 \) and \( f_2 \), the solution of problem (1.1)-(1.4) grows exponentially i.e

\[
\lim_{t \to \infty} \left[ \|u(t)\|_{2(\rho+2)} + \|v(t)\|_{2(\rho+2)} \right] \to \infty.
\]

**2 Preliminaries and notations**

The constants \( c_i, i = 0, 1, 2, \ldots \) used throughout this paper are positive generic constants, which may be different in various occurrences.

We introduce the "modified" energy functional \( E(t) \) associated to our system:

\[
2E(t) = \|u\|_2^2 + \|v\|_2^2 + J(u,v) - 2 \int_\Omega F(u,v) \, dx. \tag{2.12}
\]

where

\[
J(u,v) = \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \tag{2.13}
\]

Let us point out that the integral \( \int_\Omega F(u,v) \, dx \) in (2.13) makes sense because \( H_0^1(\Omega) \subset L^2(\rho+2)(\Omega) \), for

\[
\begin{align*}
-1 &< \rho & \text{if } n = 1, 2, \\
-1 &< \rho \leq \frac{4-n}{n-2} & \text{if } n \geq 3. \tag{2.14}
\end{align*}
\]

There exists a function \( F(u,v) \) such that

\[
F(u,v) = \frac{1}{2(\rho+2)} [a_1f_1(u,v) + vf_2(u,v)]
\]

\[
= \frac{1}{2(\rho+2)} [a_5|u+v|^{2(\rho+2)} + 2a_6|uv|^{\rho+2}] \geq 0, \tag{2.15}
\]

where

\[
\frac{\partial F}{\partial u} = f_1(u,v), \quad \frac{\partial F}{\partial v} = f_2(u,v). \tag{2.16}
\]

The following technical lemmas will play an important role in the sequel.
The following lemma is very useful to prove our main result for a large class of initial data. It is similar to that in [21].

**Lemma 3.1.** There exist two positive constants $c_1$ and $c_2$ such that

\[
\frac{c_2}{2(p+2)} (|u|^{2(p+2)} + |v|^{2(p+2)}) \leq F(u,v) \leq \frac{c_1}{2(p+2)} (|u|^{2(p+2)} + |v|^{2(p+2)}).
\]  

(1.6)

**Lemma 2.2.** Suppose that (2.14) holds. Then there exists $\eta > 0$ such that for any $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ the inequality

\[
2(p+2) \int_{\Omega} F(u,v)dx \leq \eta (J(u,v))^{p+2}
\]

holds.

Direct computations, using Minkowski, Hölder’s and Young’s inequalities and the embedding $H^1_0(\Omega) \hookrightarrow L^{2(p+2)}(\Omega)$ yields the proof of this previous lemma 2.2.

It is not hard to see this lemma.

**Lemma 2.3.** Suppose that (2.14) holds. Let $(u,v)$ be the solution of the system (1.1)-(1.4) then the energy functional is a non-increasing function, that is for all $t \geq 0$,

\[
E'(t) = - \int_{\Omega} \left( |u(t)|^{p} + |v(t)|^{r} \right) |u_t(t)|^{m+1} dx - \int_{\Omega} \left( |v(t)|^{\theta} + |u(t)|^{\phi} \right) |v_t(t)|^{r+1} dx
\]

(2.18)

3 Main section

Our main result reads as follows:

**Theorem 3.1.** Suppose that (2.14) holds. Assume further that

\[
\rho > \max \left( \frac{k+m-3}{2}, \frac{l+m-3}{2}, \frac{\theta r-3}{2}, \frac{\rho r-3}{2} \right).
\]

(3.19)

Then any solution of problem (1.1)-(1.4) with initial data satisfying

\[
\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 > \alpha_1^2, \quad \text{and} \quad E(0) < E_1
\]

(3.20)

grows exponentially, where the constants $\alpha_1$ and $E_1$ are defined in (3.21).

We take $a_i = 1, i = 1, 2, \ldots$ for convenience. Let us now introduce the following:

\[
B = \eta^{\frac{1}{2(p+2)}}, \quad \alpha_1 = B^{\frac{(p+2)}{2(p+2)}}, \quad E_1 = \left( \frac{1}{2} - \frac{1}{2(p+2)} \right) \alpha_1^2,
\]

(3.21)

where $\eta$ is the optimal constant given in (2.17).

The following lemma is very useful to prove our main result for a large class of initial data. It is similar to that in [21].

**Lemma 2.1.** [21] There exist two positive constants $c_1$ and $c_2$ such that

\[
\frac{c_2}{2(p+2)} (|u|^{2(p+2)} + |v|^{2(p+2)}) \leq F(u,v) \leq \frac{c_1}{2(p+2)} (|u|^{2(p+2)} + |v|^{2(p+2)}).
\]
Proof. (of theorem 3.1)
We based on the technic used in [22]. We set
\[ H(t) = E_1 - E(t). \] (3.25)

By using the definition of \( H(t) \), we get
\[ H'(t) = -E'(t) \]
\[ = \int \left( |u(t)|^k + |v(t)|^l \right) |u_t(t)|^{m+1} \, dx \]
\[ + \int \left( |v(t)|^p + |u(t)|^p \right) |v_t(t)|^{l+1} \, dx. \]
\[ \geq 0, \forall t \geq 0. \]

Consequently, since \( E'(t) \) is absolutely continuous,
\[ H(0) = E_1 - E(0) > 0. \]

Then,
\[ 0 < H(0) \leq H(t) = E_1 - \frac{1}{2} \left( \|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 \right) \]
\[ + \frac{1}{2(p + 2)} \left( \|u + v\|^{2(p+2)} + 2 \|uv\|^{p+2} \right). \] (3.26)

From (2.12) and (3.23), we obtain, for all \( t \geq 0 \),
\[ E_1 - \frac{1}{2} \|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{2(p + 2)} \left[ \|u + v\|^{2(p+2)} + 2 \|uv\|^{p+2} \right] \]
\[ < E_1 - \frac{1}{2} \alpha_1^2 + \frac{1}{2(p + 2)} \left[ \|u + v\|^{2(p+2)} + 2 \|uv\|^{p+2} \right] \]
\[ = - \frac{1}{2(p + 2)} \alpha_1^2 + \frac{1}{2(p + 2)} \left[ \|u + v\|^{2(p+2)} + 2 \|uv\|^{p+2} \right] \]
\[ < \frac{c_0}{2(p + 2)} \left[ \|u + v\|^{2(p+2)} + 2 \|uv\|^{p+2} \right]. \]

Hence,
\[ 0 < H(0) \leq H(t) < \frac{c_1}{2(p + 2)} \left[ \|u\|^{2(p+2)} + \|v\|^{2(p+2)} \right], \forall t > 0. \] (3.27)

Then we define the functional
\[ M(t) = \frac{1}{2} \int_{\Omega} (u^2 + v^2)(x,t) \, dx, \]
we introduce
\[ L(t) = H(t) + \varepsilon M'(t), \] (3.28)
for \( \varepsilon \) small to be chosen later to get small perturbation of \( E(t) \) and we will show that \( L(t) \) grows exponentially. Our goal is to show that \( L(t) \) satisfies a differential inequality of the form
\[ \frac{dL(t)}{dt} \geq \xi L(t). \] (3.29)

By taking a derivative of (3.28) and by equations (1.1), we obtain
\[ L'(t) = H'(t) + \varepsilon \left( \|u_2\|_{\mathbb{L}^2}^2 + \|v_1\|_{\mathbb{L}^2}^2 \right) - \varepsilon \left( \|u \|_{\mathbb{L}^2}^2 + \|v\|_{\mathbb{L}^2}^2 \right) - \varepsilon \int_{\Omega} u \left( |u(t)|^k + |v(t)|^l \right) |u_t|^{m-1} u_t \, dx \\
- \varepsilon \int_{\Omega} v \left( |v(t)|^\theta + |u(t)|^\rho \right) |v_t|^{r-1} v_t \, dx \\
- \varepsilon \int_{\Omega} (uf_1(u,v) + vf_2(u,v)) \, dx. \] (3.30)

Adding and substituting \(2E(t)\), using the definition of \(H(t)\) lead to

\[ L'(t) = H'(t) + 2\varepsilon \left( \|u_2\|_{\mathbb{L}^2}^2 + \|v_1\|_{\mathbb{L}^2}^2 \right) \\
- \varepsilon \int_{\Omega} u \left( |u(t)|^k + |v(t)|^l \right) |u_t|^{m-1} u_t \, dx \\
- \varepsilon \int_{\Omega} v \left( |v(t)|^\theta + |u(t)|^\rho \right) |v_t|^{r-1} v_t \, dx \\
+ \varepsilon \left( 1 - \frac{1}{(\rho + 2)} \right) \left( \|u + v\|_{\mathbb{L}^{2(\rho + 2)}}^2 + 2\|uv\|_{\mathbb{L}^{\rho + 2}}^{\rho + 2} \right) + 2\varepsilon H(t) - 2\varepsilon E_1. \] (3.31)

Then using (3.24) we obtain

\[ L'(t) \geq H'(t) + 2\varepsilon \left( \|u_2\|_{\mathbb{L}^2}^2 + \|v_1\|_{\mathbb{L}^2}^2 \right) \\
+ \varepsilon c_3 \left( \|u + v\|_{\mathbb{L}^{2(\rho + 2)}}^2 + 2\|uv\|_{\mathbb{L}^{\rho + 2}}^{\rho + 2} \right) + 2\varepsilon H(t), \\
- \varepsilon \int_{\Omega} u \left( |u(t)|^k + |v(t)|^l \right) |u_t|^{m-1} u_t \, dx \\
- \varepsilon \int_{\Omega} v \left( |v(t)|^\theta + |u(t)|^\rho \right) |v_t|^{r-1} v_t \, dx \] (3.32)

where \(c_3 = 1 - \frac{1}{\rho + 2} - 2E_1 (B\alpha_2)^{-2(\rho + 2)} > 0\), since \(\alpha_2 > B \frac{2(\rho + 2)}{\rho + 1}\). In order to estimate the last two terms in (3.32) we use the following Young's inequality:

\[ XY \leq \frac{\delta_\alpha X^\alpha}{\alpha} + \frac{\delta_\beta Y^\beta}{\beta}, \]

where \(X, Y > 0\), \(\delta > 0\), and \(\alpha, \beta \in \mathbb{R}^*\) such that \(\frac{1}{\alpha} + \frac{1}{\beta} = 1\). Consequently, we get for all \(\delta_1 > 0\)

\[ a |u_t|^{m-1} u_t \leq \frac{\delta^{m+1}}{m+1} |u|^{m+1} + \frac{m}{m+1} \delta_1^{-(m+1)/m} |u_t|^{m+1} \]

and therefore,

\[ \int_{\Omega} u \left( |u(t)|^k + |v(t)|^l \right) |u_t|^{m-1} u_t \, dx \]

\[ \leq \frac{\delta^{m+1}}{m+1} \int_{\Omega} \left( |u(t)|^k + |v(t)|^l \right) |u|^{m+1} \, dx \\
+ \frac{m}{m+1} \delta_1^{-(m+1)/m} \int_{\Omega} \left( |u(t)|^k + |v(t)|^l \right) |u_t|^{m+1} \, dx \] (3.33)

Similarly, for all \(\delta_2 > 0\)

\[ a |v_t|^{r-1} v_t \leq \frac{\delta^{r+1}}{r+1} |v|^{r+1} + \frac{r}{r+1} \delta_2^{-(r+1)/r} |v_t|^{r+1}, \]
which gives

\[
\int_{\Omega} \left( |v(t)|^\theta + |u(t)|^\rho \right) |v| |v_t|^{r-1} \, dx \\
\leq \frac{\delta^{r-1}}{r-1} \int_{\Omega} \left( |v(t)|^\theta + |u(t)|^\rho \right) |v|^{r+1} \, dx \\
+ \frac{1}{r-1} \delta_2^{-(r+1)/r} \int_{\Omega} \left( |v(t)|^\theta + |u(t)|^\rho \right) |v_t|^{r+1} \, dx
\]

(3.34)

Inserting the estimates (3.33), (3.34) into (3.32), we obtain

\[
L'(t) \geq H'(t) + 2\epsilon \left( \|u_t\|_{2}^2 + \|v_t\|_{2}^2 \right) \\
+ \epsilon c_1 \left( \|u + v\|_{2(\theta + 2)}^{2(\rho + 2)} + \|uv\|_{\rho + 2}^{2(\rho + 2)} \right) + 2\epsilon H(t) \\
- \epsilon \frac{m+1}{m+1+1} \int_{\Omega} \left( |u(t)|^k + |v(t)|^l \right) |u|^{m+1} \, dx \\
- \epsilon \frac{r+1}{r+1} \int_{\Omega} \left( |v(t)|^\theta + |u(t)|^\rho \right) |v|^{r+1} \, dx
\]

(3.35)

Using Young’s inequality, to get

\[
\int_{\Omega} \left( |u(t)|^k + |v(t)|^l \right) |u|^{m+1} \, dx = \|u\|_{k+m+1}^{k+m+1} + \int_{\Omega} |v|^{l} |u|^{m+1} \, dx \\
\leq \|u\|_{k+m+1}^{k+m+1} + \frac{1}{r+m+1} \|v\|_{r+m+1}^{l+m+1} \|u\|_{r+m+1}^{l+m+1} + \frac{m+1}{r+m+1} \|v\|_{r+m+1}^{l+m+1}
\]

(3.36)

and

\[
\int_{\Omega} \left( |v(t)|^\theta + |u(t)|^\rho \right) |v|^{r+1} \, dx = \|v\|_{\theta+r+1}^{\theta+r+1} + \int_{\Omega} |u|^{\rho} |v|^{r+1} \, dx \\
\leq \|v\|_{\theta+r+1}^{\theta+r+1} + \frac{\rho}{\rho+r+1} \|u\|_{\theta+r+1}^{\theta+r+1} \|v\|_{r+r+1}^{\theta+r+1} + \frac{\rho}{\rho+r+1} \|v\|_{r+r+1}^{\theta+r+1}
\]

(3.37)

By using lemma 2.1, (3.35) becomes

\[
L'(t) \geq H'(t) + 2\epsilon \left( \|u_t\|_{2}^2 + \|v_t\|_{2}^2 \right) \\
+ \epsilon c_1 \left( \|u\|_{2(\theta + 2)}^{2(\rho + 2)} + \|uv\|_{\rho + 2}^{2(\rho + 2)} \right) + 2\epsilon H(t) \\
- \epsilon \frac{m+1}{m+1+1} \left( \|u\|_{k+m+1}^{k+m+1} + \frac{1}{r+m+1} \|v\|_{r+m+1}^{l+m+1} \right) \\
- \epsilon \frac{r+1}{r+1} \left( \|v\|_{\theta+r+1}^{\theta+r+1} + \frac{\rho}{\rho+r+1} \|u\|_{\theta+r+1}^{\theta+r+1} \right)
\]

(3.38)
Since (3.19) holds, we obtain by using the algebraic inequality
\[ z^{\nu} \leq (z + 1) \leq \left(1 + \frac{1}{a}\right) (z + a), \forall z \geq 0, \nu \leq 1, a \geq 0. \] (3.38)
we have, for all \( t \geq 0, \)
\[
\begin{align*}
\|u\|_{l^{k+m+1}} & \leq c_1 \|u\|_{l^{k+m+1}} \leq d \left( \|u\|_{l^{2(\rho+2)}}^{2(\rho+2)} + H(t) \right) \leq d \left( \|u\|_{l^{2(\rho+2)}}^{2(\rho+2)} + H(t) \right), \\
\|u\|_{l^{t+m+1}}^{\rho+r+1} & \leq c_2 \|u\|_{l^{t+m+1}}^{\rho+r+1} \leq d \left( \|u\|_{l^{2(\rho+2)}}^{2(\rho+2)} + H(t) \right), \\
\|v\|_{l^{t+m+1}}^{\rho+r+1} & \leq c_3 \|v\|_{l^{t+m+1}}^{\rho+r+1} \leq d \left( \|v\|_{l^{2(\rho+2)}}^{2(\rho+2)} + H(t) \right), \end{align*}
\] (3.39)
where \( d = 1 + 1/H(0). \) Choosing \( \delta_1, \delta_2, \gamma_1 \) and \( \gamma_2 \) such that
\[
M_1 = \frac{m}{r^{m+1}} \delta_1^{-(m+1)/m},
\]
\[
M_2 = \frac{r}{r+1} \delta_2^{-(r+1)/r},
\]
and
\[
M_3 = \frac{\delta_1^{m+1}}{m+1} \left( 1 - \frac{m+1}{r^{m+1}} \gamma_1^{-(l+m+1)/(m+1)} \right) - \left( \frac{r^{l+1}}{r^{m+1}} \frac{l}{r+1} \gamma_2^{(r+l+1)/(r+1)} \right),
\]
and
\[
M_4 = \frac{\delta_2^{l+1}}{l+1} \left( 1 - \frac{l+1}{r^{l+1}} \gamma_2^{-(l+1)/(l+1)} \right) - \left( \frac{r^{m+1}}{m+1} \frac{r}{r+1} \gamma_1^{(l+m+1)/l} \right).
\]
and
\[
M_5 = \frac{\delta_1^{m+1}}{m+1} \left( 1 + \frac{m+1}{r^{m+1}} \gamma_1^{-(l+m+1)/(m+1)} \right) + \left( \frac{r^{l+1}}{r^{m+1}} \frac{l}{r+1} \gamma_2^{(r+l+1)/(r+1)} \right) + \left( \frac{l}{r+1} \frac{r^{l+1}}{r^{m+1}} \gamma_2^{(r+l+1)/(r+1)} \right).
\]
This implies
\[
L'(t) \geq H'(t) + 2 \varepsilon \left( \|u_t\|_2^2 + \|v_t\|_2^2 \right) + (1 - \varepsilon M_1) \int_{\Omega} \left( |u(t)|^k + |v(t)|^l \right) |u_t|^x dx \]
\[
+ (1 - \varepsilon M_2) \int_{\Omega} \left( |v(t)|^\theta + |u(t)|^\rho \right) |v_t|^{x+1} dx \]
\[
+ \varepsilon (c_4 - \varepsilon d M_3) \|u\|_{l^{2(\rho+2)}}^{2(\rho+2)} + \varepsilon (c_4 - \varepsilon d M_4) \|v\|_{l^{2(\rho+2)}}^{2(\rho+2)} + \varepsilon \left[ 2 - \varepsilon d M_5 \right] H(t).
\]
We can find positive constants \( \lambda_1, \lambda_2, \lambda_3 \) and \( M_6 \) such that (3.40) becomes
\[
L'(t) \geq (1 - \varepsilon M_6) H'(t) + 2 \varepsilon \left( \|u_t\|_2^2 + \|v_t\|_2^2 \right) + \varepsilon \lambda_1 \|u_t\|_{l^{2(\rho+2)}}^{2(\rho+2)} + \varepsilon \lambda_2 \|v_t\|_{l^{2(\rho+2)}}^{2(\rho+2)} + \varepsilon \lambda_3 H(t). \] (3.40)
Once \( M_1, M_2, M_3, M_4 \) and \( M_5 \) are fixed, we pick \( \varepsilon \) small enough so that \((1 - \varepsilon M_6) \geq 0 \) and
\[
L(0) = H(0) + \int_{\Omega} [u_0 u_t + v_0 v_t] dx > 0.
\]
Consequently, there exists $\Gamma > 0$ such that (3.40) becomes
\[
L'(t) \geq e^\Gamma \left( H(t) + ||u||^2 + ||v||^2 + ||u||^{2(p+2)} + ||v||^{2(p+2)} \right).
\] (3.41)

Thus, the functional $L(t)$ is strictly positive and increasing for all $t \geq 0$.

Now, by Holder’s and Young’s inequalities, we estimate
\[
\int_\Omega uu_t(x,t)dx + \int_\Omega vv_t(x,t)dx
\leq k ||u||^2 + \frac{1}{4k} ||u||^2 + l ||v||^2 + \frac{1}{4l} ||v||^2, k, l > 0,
\]
and since $-1 < \rho$ for all $n \geq 1$, we have
\[
\int_\Omega uu_t(x,t)dx + \int_\Omega vv_t(x,t)dx
\leq c_{11} \left( ||u||^{2(p+2)} + ||u||^2 + ||v||^{2(p+2)} + ||v||^2 \right).
\] (3.42)

Also, by noting that
\[
L(t) = H(t) + e \int_\Omega (uu_t + vv_t)(x,t)dx
\leq c_{12} \left( H(t) + \int_\Omega (uu_t(x,t) + vv_t(x,t))dx \right)
\leq c_{13} \left( H(t) + ||u||^{2(p+2)} + ||v||^{2(p+2)} + ||u||^2 + ||v||^2 \right), \forall t \geq 0,
\]
and combining with (3.43) and (3.41), we arrive at
\[
\frac{dL(t)}{dt} \geq \xi L(t), \xi > 0, \forall t \geq 0.
\] (3.44)

Integration of (3.44) between 0 and $t$ gives us $L(t) \geq L(0) \exp(\zeta t)$ and from (3.28), (3.27) and for $\varepsilon$ small enough, we have
\[
L(t) \leq H(t) \leq \frac{c_1}{2(\rho + 2)} \left( ||u||^{2(p+2)} + ||v||^{2(\rho + 2)} \right), \forall t > 0.
\]

then,
\[
L(0) \exp(\xi t) \leq \frac{c_1}{2(\rho + 2)} \left( ||u||^{2(p+2)} + ||v||^{2(\rho + 2)} \right), \forall t > 0.
\]

This completes the proof $\square$

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