Weakly Compatible Mappings along with $CLR_S$ property in Fuzzy Metric Spaces

Saurabh Manro$^1$, S. S. Bhatia$^1$, Sanjay Kumar$^2$, Poom Kumam$^3$, Sumitra Dalal$^4$

$^1$Thapar University School of Mathematics and Computer Applications Patiala, Punjab, India
$^2$Deenbandhu Chhotu Ram University of Science and Technology Murthal (Sonepat), India
$^3$Department of Mathematics, Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), Bang Mod, Thung khrwai, Bangkok 10140, Thailand
$^4$Department of Mathematics, Faculty of Science, Jazan University, Saudi Arabia

Abstract

The aim of this work is to use newly introduced property, which is so called "common limit in the range ($CLR_S$)" for four self-mappings, and prove some theorems which satisfy this property. Moreover, we establish some new existence of a common fixed point theorem for generalized contractive mappings in fuzzy metric spaces by using this new property and give some examples to support our results. Ours results does not require condition of closeness of range and so our theorems generalize, unify, and extend many results in literature. Our results improve and extend the results of Cho et al. [4], Pathak et al. [20] and Imdad et al.[10] besides several known results.

Keywords: Weakly Compatible Maps, Fuzzy metric space, property ($E_A$), Common property ($E_A$), $CLR_S$ property, $CLR_{ST}$ property, Common limit in range property.

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1 Introduction

The concept of fuzzy sets was introduced by Zadeh [28] in 1965. In 1975, Kramosil and Michalek [14] gave the notion of probabilistic metric spaces due to Menger [17]. On the other hand, fixed point theory is one of the most famous mathematical theories with application in several branches of science. Fixed point theory in fuzzy metric spaces has been developing since the work of Heilpern [9]. He introduced the concept of fuzzy contraction mappings and proved some fixed point theorems for fuzzy contraction mappings in metric linear spaces, which is a fuzzy extension of the Banach contraction principle. In [6, 7], George and Veeramani introduced and studied the notion of fuzzy metric spaces that constitutes a modification of the one due to Kramosil and Michalek. Many authors have contributed to the development of this theory and apply to fixed point theory, for instance [1, 3-5, 8, 10, 11, 15, 16, 18-20, 22, 24-27].

In 1976, Jungck [12] introduced the notion of commuting mappings. Afterward, Sessa [23] gave the notion of weakly

*Corresponding author. Email address: sauravmanro@hotmail.com
2 Preliminaries

The concept of triangular norms (t-norms) is originally introduced by Menger [17] in study of statistical metric spaces.

Definition 2.1 (Schweizer and Sklar [21]). A binary operation \( * : [0, 1] \times [0, 1] \to [0, 1] \) is called a continuous triangular norm (t-norm) if it satisfies the following conditions:

(i) \( * \) is associative and commutative;

(ii) \( * \) is continuous;

(iii) \( a * 1 = a \) for all \( a \in [0, 1] \);

(iv) \( a * b \leq c * d \), whenever \( a \leq c \) and \( b \leq d \), for all \( a, b, c, d \in [0, 1] \).

Three basic examples of continuous t-norms are \( a *_1 b = \min\{a, b\}, a *_2 b = ab \) and \( a *_3 b = \max\{a + b - 1, 0\} \).

Definition 2.2 (Kramosil and Michalek [14]). A fuzzy metric space is a triple \((X, M, *)\), where \(X\) is a non-empty set, \( * \) is a continuous t-norm and \( M \) is a fuzzy set on \( X \times X \times [0, +\infty) \), satisfying the following properties:

\[
\begin{align*}
\text{(KM-1)} & \quad M(x, y, 0) = 0 \text{ for all } x, y \in X; \\
\text{(KM-2)} & \quad M(x, y, t) = 1 \text{ for all } t > 0 \text{ iff } x = y; \\
\text{(KM-3)} & \quad M(x, y, t) = M(y, x, t) \text{ for all } x, y \in X \text{ and for all } t > 0; \\
\text{(KM-4)} & \quad M(x, y, \cdot) : [0, +\infty) \to [0, 1] \text{ is left continuous for all } x, y \in X; \\
\text{(KM-5)} & \quad M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \text{ for all } x, y, z \in X \text{ and for all } t, s > 0.
\end{align*}
\]

We denote such space as KM-fuzzy metric space.

Lemma 2.1 ([14]). In a KM-fuzzy metric space \((X, M, *)\), \( M(x, y, \cdot) \) is non-decreasing for all \( x, y \in X \).

Definition 2.3 ([6]). Let \((X, M, *)\) be a fuzzy metric space. Then a sequence \( \{x_n\}_{n \in \mathbb{N}} \) is said to be

(i) convergent to \( x \in X \), that is, \( \lim_{n \to +\infty} x_n = x \), if \( \lim_{n \to +\infty} M(x_n, x, t) = 1 \) for all \( t > 0 \);

(ii) Cauchy sequence if \( \lim_{n \to +\infty} M(x_n + p, x_n, t) = 1 \) for all \( t > 0 \) and \( p > 0 \).

Definition 2.4 ([6]). A fuzzy metric space is said to be complete if and only if every Cauchy sequence in \( X \) is convergent.
Definition 2.5 ([13]). Two self-mappings \( f \) and \( g \) of a fuzzy metric space \( (X, M, *) \) are said to be compatible if 
\[
\lim_{n \to +\infty} M(fgx_n, gfx_n) = 1
\]
for all \( t > 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that 
\[
\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = z
\]
for some \( z \in X \).

Definition 2.6 ([13]). Two self-mappings \( f \) and \( g \) of a fuzzy metric space \( (X, M, *) \) are said to be non-compatible if there exists at least one sequence \( \{x_n\} \) in \( X \) such that 
\[
\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = z
\]
for some \( z \in X \), but for some \( t > 0 \), either 
\[
\lim_{n \to +\infty} M(fgx_n, gfx_n) \neq 1
\]
or the limit does not exist.

Definition 2.7 ([8]). A pair \( (f, g) \) of self-mappings of a non-empty set \( X \) is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if \( fz = gz \) for some \( z \in X \), then \( fgz = gfz \).

If two self-mappings \( f \) and \( g \) of a fuzzy metric space \( (X, M, *) \) are compatible then they are weakly compatible but the converse need not be true.

Definition 2.8 ([2]). A pair \( (f, g) \) of self-mappings of a fuzzy metric space \( (X, M, *) \) is said to satisfy the property \((E.A)\) if there exists a sequence \( \{x_n\} \) in \( X \) such that 
\[
\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = z,
\]
for some \( z \in X \).

From Definition 2.8, it is easy to see that any two non-compatible self-mappings of a fuzzy metric space \( (X, M, *) \) satisfy the property \((E.A)\) but the reverse need not be true.

Definition 2.9 ([11]). Two pairs \((A, S)\) and \((B, T)\) of self-mappings of a fuzzy metric space \( (X, M, *) \) are said to satisfy the common property \((E.A)\) if there exist two sequences \( \{x_n\}, \{y_n\} \) in \( X \) such that 
\[
\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} By_n = \lim_{n \to +\infty} Ty_n = z,
\]
for some \( z \in X \).

Definition 2.10 ([25]). A pair of self-mappings \( (f, g) \) of a fuzzy metric space \( (X, M, *) \) is said to satisfy the common limit in the range of \( g \) property (CLR\(_g\)) if there exists a sequence \( \{x_n\} \) in \( X \) such that 
\[
\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = gz,
\]
for some \( z \in X \).

With a view to extend the \((CLR_g)\) property to two pair of self mappings, very recently Imdad et al. [10] define the \((CLR_{ST})\) property (with respect to mappings \( S \) and \( T \)) as follows:

Definition 2.11. [10] Two pairs \((A, S)\) and \((B, T)\) of self-mappings of a fuzzy metric space \( (X, M, *) \) are said to satisfy the \((CLR_{ST})\) property (with respect to mappings \( S \) and \( T \)) if there exist two sequences \( \{x_n\}, \{y_n\} \) in \( X \) such that 
\[
\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} By_n = \lim_{n \to +\infty} Ty_n = Sz,
\]
for some \( z \in S(X) \) and \( z \in T(X) \).

Inspired by Sintunavarat et al. [25] and Imdad et al. [10], Manro et al. [15] introduced the following notion:

Definition 2.12 ([15]). Two pairs \((A, S)\) and \((B, T)\) of self-mappings of a fuzzy metric space \( (X, M, *) \) are said to share the common limit in the range of \( S \) property if there exist two sequences \( \{x_n\}, \{y_n\} \) in \( X \) such that 
\[
\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} By_n = \lim_{n \to +\infty} Ty_n = Sz,
\]
for some \( z \in X \).
Example 2.1 ([8]). Let \((X, M, \ast)\) be a fuzzy metric space with \(X = [-1, 1]\) and for all \(x, y \in X\), define \(M(x, y, t) = e^{-|x-y|^t}\) if \(t > 0\) and \(M(x, y, 0) = 0\). Define self mappings \(A, B, S\) and \(T\) on \(X\) by \(Ax = \frac{x}{2}, Bx = \frac{1}{x}, Sx = x, Tx = -x\) for all \(x \in X\). Then with sequences \(\{x_n = \frac{1}{n}\}\) and \(\{y_n = \frac{1}{n}\}\) in \(X\), one can easily verify that
\[
\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} By_n = \lim_{n \to +\infty} Ty_n = S0 = 0.
\]
This shows that the pairs \((A, S)\) and \((B, T)\) share the common limit in the range of \(S\) property.

Definition 2.13 ([8]). Two families of self-mappings \(\{A_i\}\) and \(\{S_j\}\) are said to be pairwise commuting if:
1. \(A_iA_j = A_jA_i\), \(i, j \in \{1, 2, \ldots, m\}\),
2. \(S_iS_j = S_jS_i\), \(i, j \in \{1, 2, \ldots, m\}\),
3. \(A_iS_j = S_jA_i\), \(i \in \{1, 2, \ldots, m\}\), \(j \in \{1, 2, \ldots, n\}\).

Lemma 2.2 ([16]). Let \((X, M, \ast)\) be a fuzzy metric space, where \(\ast\) is a continuous \(t\)-norm. If there exists a constant \(k \in (0, 1)\) such that \(M(x, y, kt) \geq M(x, y, t)\), for all \(x, y \in X\) and \(t > 0\), then \(x = y\).

3 Main Results

Lemma 3.1. Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, \ast)\) satisfying the followings:
(i) the pair \((A, S)\) (or \((B, T)\)) satisfies the common limit in the range of \(S\) property (or \(T\) property);
(ii) there exists a constant \(k \in (0, 1)\) such that
\[
(M(Ax, By, kt))^2 \geq \min((M(Sx, Ty, t))^2, M(Sx, Ax, t).M(Ty, By, t), M(Sx, By, 2t)).
\]
for any \(x, y \in X\) and \(t > 0\);
(iii) \(A(X) \subseteq T(X)\) (or \(B(X) \subseteq S(X)\)).
Then the pairs \((A, S)\) and \((B, T)\) share the common limit in the range of \(S\) property.

Proof. Suppose that the pair \((A, S)\) satisfies the common limit in the range of \(S\) property, then there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = z\) for some point \(z \in X\). Since \(A(X) \subseteq T(X)\), therefore for each \(\{x_n\}\), there exist \(\{y_n\}\) in \(X\) such that \(Ax_n = Ty_n\). Thus \(\lim_{n \to +\infty} Ty_n = z\). Hence we have \(\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} Ty_n = z\). Now, we assert that \(\lim_{n \to +\infty} By_n = z\). From (3.1), we get
\[
(M(Ax_n, By_n, kt))^2 \geq \min((M(Sx_n, Ty_n, t))^2, M(Sx_n, Ax_n, t).M(Ty_n, By_n, t), M(Sx_n, By_n, 2t)).
\]
Taking limit as \(n \to +\infty\), we get
\[
(M(Sz, \lim_{n \to +\infty} By_n, kt))^2 \geq \min((M(Sz, Sz, t))^2, M(Sz, Sz, t).M(Sz, \lim_{n \to +\infty} By_n, t), M(Sz, lim_{n \to +\infty} By_n, 2t)).
\]
by Lemma 2.2, we have therefore, \(\lim_{n \to +\infty} By_n = z\). Then the pairs \((A, S)\) and \((B, T)\) share the common limit in the range of \(S\) property.
Theorem 3.1. Let $A, B, S$ and $T$ be self mappings of a fuzzy metric space $(X, M, *)$ satisfying inequality (3.1). Suppose that

(i) the pair $(A, S)$ (or $(B, T)$) satisfies the common limit in the range of $S$ property (or $T$ property);

(ii) $A(X) \subseteq T(X)$ ( or $B(X) \subseteq S(X)$).

Then the pairs $(A, S)$ and $(B, T)$ have a point of coincidence each. Moreover, $A, B, S$ and $T$ have a unique common fixed point provided that both the pairs $(A, S)$ and $(B, T)$ are weakly compatible.

Proof. In view of Lemma 3.1, the pairs $(A, S)$ and $(B, T)$ share the common limit in the range of $S$ property, that is there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$
limit_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} By_n = Sz \text{ for some } z \in X.
$$

Firstly, we assert that $Az = Sz$. By (3.1), we have

$$
(M(Az, By_n, kt))^2 \geq \min((M(Sz, Ty_n, t))^2, M(Sz, Az, t), M(Ty_n, By_n, t), M(Sz, By_n, 2r), M(Ty_n, Az, t), M(Ty_n, Az, t)).
$$

Proceeding limit as $n \to \infty$, we get

$$
(M(Az, Sz, kt))^2 \geq \min((M(Sz, Sz, t))^2, M(Sz, Az, t), M(Sz, Sz, t), M(Sz, Sz, 2r), M(Sz, Az, t), M(Sz, Az, t), M(Sz, Sz, 2r), M(Sz, Sz, t)) = 1
$$

$$
(M(Az, Sz, kt))^2 \geq (M(Az, Sz, t))^2
$$

By Lemma 2.2, we have $Az = Sz$.

Since, $A(X) \subseteq T(X)$, there exist $v \in X$ such that $Az = Tv$.

Secondly, we assert that $Bv = Tv$. By (3.1), we get

$$
(M(Az, Bv, kt))^2 \geq \min((M(Sz, Bv, t))^2, M(Sz, Az, t), M(Tv, Bv, t), M(Sz, Bv, 2r), M(Tv, Az, t), M(Tv, Az, t), M(Sz, Bv, 2r), M(Tv, Bv, t))
$$

$$
(M(Tv, Bv, kt))^2 \geq \min((M(Sz, Sz, t))^2, M(Sz, Sz, t), M(Tv, Bv, t), M(Tv, Bv, 2r), M(Tv, Tv, t), M(Tv, Tv, t), M(Tv, Bv, 2r), M(Tv, Bv, t)) = 1
$$

From Lemma 2.2, we have $Tv = Bv = Az = Sz$.

Since the pairs $(A, S)$ and $(B, T)$ are weakly compatible and $Az = Sz$ and $Tv = Bv$, therefore, $ASz = SAz = AAz = SSz, BTv = T Bv = TT v = BBv$.

Finally, we assert that $AAz = Az$. Again by (3.1), we have

$$
(M( AAz, Bv, kt))^2 \geq \min((M(SAz, Ty_n, t))^2, M(SAz, AAz, t), M(Tv, Bv, t), M(SAz, Bv, 2r), M(Tv, AAz, t), M(SAz, Bv, 2r), M(Tv, Bv, t))
$$

$$
(M( AAz, Az, kt))^2 \geq \min((M(AAz, Az, t))^2, M(AAz, AAz, t), M(Az, Az, t), M(AAz, Az, 2r), M(Az, AAz, t), M(AAz, Az, 2r), M(Az, Az, t))
$$
\[(M(AA_z, Az, kt))^2 \geq \min((M(AA_z, Az, t))^2, 1, M(AA_z, Az, 2t), M(Az, AA_z, t), M(Az, AA_z, t), M(Az, AA_z, 2t)) \]

\[(M(AA_z, Az, kt))^2 \geq (M(AA_z, Az, t))^2 \]

which again by Lemma 2.2 gives, \(AA_z = Az = SAz\) which gives, \(Az\) is common fixed point of \(A\) and \(S\). Similarly, one can easily prove that \(BBv = Bv = TBv\), that is \(Bv\) is common fixed point of \(B\) and \(T\). As \(Az = Bv\), therefore \(Az\) is common fixed point of \(A, B, S\) and \(T\). The uniqueness of common fixed point is an easy consequence of inequality (3.1).

Our result (Theorem 3.1) improve and extend the results of Cho et al. [4], Pathak et al.[20] and Imdad et. al. [10] besides several known results.

By choosing \(A, B, S\) and \(T\) suitably, one can derive corollaries involving two or three mappings.

**Corollary 3.1.** Let \(A\) and \(S\) be self mappings of a fuzzy metric space \((X, M, *)\) satisfying:

(i) the pair \((A,S)\) satisfies the common limit in the range of \(S\) property;

(ii) \(A(X) \subseteq S(X)\);

(iii) there exists a constant \(k \in (0, 1)\) such that

\[(M(Ax, Ay, kt))^2 \geq \min((M(Sx, Sy, t))^2, M(Sx, Ax, t), M(Sy, Ay, t), M(Sx, Ay, 2t), M(Sy, Ax, t), M(Sx, Ay, 2t), M(Sy, Ay, t)) \]

for any \(x, y \in X\) and \(t > 0\).

Then \(A\) and \(S\) have a point of coincidence. Moreover, \(A\) and \(S\) have a unique common fixed point provided that \(A\) and \(S\) are weakly compatible.

**Proof.** Taking \(B = A\) and \(T = S\) in Theorem 3.1.

**Corollary 3.2.** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, *)\) satisfying inequality (3.1). Suppose that

(i) the pairs \((A,S)\) and \((B,T)\) satisfies the common limit in the range of \(S\) property.

(ii) \(A(X) \subseteq T(X)\) (or \(B(X) \subseteq S(X)\)).

Then the pairs \((A,S)\) and \((B,T)\) have a point of coincidence each. Moreover, \(A, B, S\) and \(T\) have a unique common fixed point provided that both the pairs \((A,S)\) and \((B,T)\) are weakly compatible.

**Proof.** Proof easily follows on same lines of Theorem 3.1 using Lemma 3.1.

**Theorem 3.2.** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, *)\) satisfying inequality (3.1). Suppose that

(i) the pair \((A, S)\) (or \((B, T)\)) satisfies property \((E.A.)\) and \(S(X)\) is a closed subspace of \(X\);

(ii) \(A(X) \subseteq T(X)\) (or \(B(X) \subseteq S(X)\)).

Then the pairs \((A, S)\) and \((B, T)\) have a point of coincidence each. Moreover, \(A, B, S\) and \(T\) have a unique common fixed point provided that both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

**Proof.** Suppose pair \((A, S)\) satisfy property \((E.A.)\), there exist a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = p\) for some \(p \in X\). It follows from \(S(X)\) is a closed subspace of \(X\) that \(p = Sz\) for some \(z \in X\) and then the pair \((A, S)\) satisfies the common limit in the range of \(S\) property. By Theorem 3.1, we get \(A, B, S\) and \(T\) have a unique common fixed point.

**Corollary 3.3.** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, *)\) satisfying inequality (3.1). Suppose that

(i) the pairs \((A, S)\) and \((B, T)\) satisfies common property \((E.A.)\) and \(S(X)\) is a closed subspace of \(X\);

(ii) \(A(X) \subseteq T(X)\) (or \(B(X) \subseteq S(X)\)).

Then the pairs \((A, S)\) and \((B, T)\) have a point of coincidence each. Moreover, \(A, B, S\) and \(T\) have a unique common fixed point provided that both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.
Proof. Since the pairs \((A, S)\) and \((B, T)\) satisfies common property \((E.A.)\), there exists two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} By_n = p\) for some \(p \in X\). It follows from \(S(X)\) is a closed subspace of \(X\) that \(p = Sz\) for some \(z \in X\) and then the pairs \((A, S)\) and \((B, T)\) share the common limit in the range of \(S\) property. By Theorem 3.1, we get \(A, B, S\) and \(T\) have a unique common fixed point. 

Since the pair of non compatible mappings implies to the pair satisfying property \((E.A.)\), we get the following corollary.

**Corollary 3.4.** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, *)\) satisfying inequality (3.1). Suppose that

(i) the pair \((A, S)\) (or \((B, T)\)) is non compatible mappings and \(S(X)\) is a closed subspace of \(X\);

(ii) \(A(X) \subseteq T(X)\) (or \(B(X) \subseteq S(X)\)).

Then the pairs \((A, S)\) and \((B, T)\) have a point of coincidence each. Moreover, \(A, B, S\) and \(T\) have a unique common fixed point provided that both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

**Proof.** Since the pair \((A, S)\) is non compatible mappings, we get \(A\) and \(S\) satisfy property \((E.A.)\). Therefore, by Theorem 3.2, we get \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

**Remark 3.1.** The property \((E.A.)\) (common property \((E.A.)\)) is an essential tool to claim the existence of common fixed points of some mappings. However these properties require the condition of closedness of \(S(X)\). Note that Theorem 3.1 weakens the condition of closed subspace of \(S(X)\). Therefore it is most interesting to used common limit in the range of \(S\) property as another auxiliary tool to claim the existence of a common fixed point. However, all the main results in this paper are some of the choices for claim that the existence of common fixed point in fuzzy metric spaces. Our result may be the motivation to other authors for extending and improving these results to suitable tools for these problems.

As an application of Theorems 3.1, we prove a common fixed point theorem for four finite families of mappings on fuzzy metric spaces. While proving our result, we utilize Definition 2.13 which is a natural extension of commutativity condition to two finite families.

**Theorem 3.3.** Let \(\{A_1, A_2, \ldots, A_m\}, \{B_1, B_2, \ldots, B_n\}, \{S_1, S_2, \ldots, S_p\}\) and \(\{T_1, T_2, \ldots, T_q\}\) be four finite families of self-mappings of a fuzzy metric space \((X, M, *)\) such that \(A = A_1A_2 \cdots A_m, B = B_1B_2 \cdots B_n, S = S_1S_2 \cdots S_p\) and \(T = T_1T_2 \cdots T_q\) satisfy the conditions of Theorem 3.1. Then

(a) the pairs \((A, S)\) and \((B, T)\) have a point of coincidence each;

(b) \(\{A_i\}, \{B_j\}, \{S_k\}\) and \(\{T_r\}\) have a unique common fixed point provided that the pairs of families \(\{(A_i), \{S_k\}\}\) and \(\{(B_j), \{T_r\}\}\) commute pairwise, for all \(i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p\) and \(r = 1, \ldots, q\).

**Proof.** Since the pairs of families \(\{(A_i), \{S_k\}\}\) and \(\{(B_j), \{T_r\}\}\) commute pairwise, we first show that \(AS = SA\). In fact, we have

\[
AS = (A_1A_2 \cdots A_m)(S_1S_2 \cdots S_p) = (A_1A_2 \cdots A_{m-1})(A_mS_1S_2 \cdots S_p) = \cdots = (A_1A_2 \cdots A_{m-2})(A_mS_1S_2 \cdots S_pA_m) = \cdots = (S_1S_2 \cdots S_pA_1A_2 \cdots A_m) = SA.
\]

Similarly one can prove that \(BT = TB\), and hence, obviously the pair \((A, S)\) is compatible and \((B, T)\) is weakly compatible. Now, using Theorem 3.1, we conclude that \(A, S, B\) and \(T\) have a unique common fixed point in \(X\), say \(z\).
Now, we need to prove that $z$ remains the fixed point of all the component mappings. To this aim, consider

$$A(A_i z) = ((A_1 A_2 \cdots A_m) A_i) z = (A_1 A_2 \cdots A_{m-1}) (A_m A_i) z$$

$$= (A_1 A_2 \cdots A_{m-1}) (A_m A_i z) = (A_1 A_2 \cdots A_{m-2}) (A_m A_{m-1} A_i) z$$

$$= (A_1 A_2 \cdots A_{m-2}) (A_{m-1} A_i) z = \cdots = A_1 (A_2 \cdots A_m) z$$

$$= A_1 (A_2 \cdots A_m) z = A_1 z = z.$$

Similarly, one can prove that $A(S_i z) = S_i (A z_i) = S_i z_i, S_j (S_i z_i) = S_j z_i, S_i (S_j z_i) = A_i (S_j z_i) = A_i z_i, B(B_j z_i) = B_j (B_j z_i) = B_j z_i, B(T z_i) = T_i (B_j z_i) = T_i z_i, T_1 (T_i z_i) = T_1 z_i$ and $T (B_j z_i) = B_j (T_1 z_i) = B_j z_i$, which show that for all $i, j, k$ and $r$ $A_i z$ and $S_k z_i$ are other fixed points of the pair $(A, S)$ whereas $B_j z_i$ and $T_i z_i$ are other fixed points of the pair $(B, T)$. Since $A, B, S$ and $T$ have a unique common fixed point, then we get $z = A_i z = S_k z_i = B_j z_i = T_k z_i$ for all $i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p$ and $r = 1, \ldots, q$. Thus $z$ is the unique common fixed point of $(A_i), \{B_j\}, \{S_k\} \text{ and } \{T_i\}$. □

Now we prove common fixed point theorem using lower semi continuous function, $\psi : [0, 1] \to [0, 1]$ such that $\psi(t) > t$ for all $t \in (0, 1)$ along with $\psi(0) = 0$ and $\psi(1) = 1$, for integral mappings satisfying contractive conditions.

**Theorem 3.4.** Let $A, B, S$ and $T$ be self mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions (i) and (ii) of Theorem 3.1 and for all $x, y \in X, t \geq 0$

$$\int_0^t \phi(s) ds \geq \psi(\int_0^{\min \{M(S(x, A y) z), M(S(x, A y) z), M(T(y, B y) z), M(T(y, B y) z), M(T(y, A y) z), M(T(y, A y) z)\}} \phi(s) ds) \tag{3.2}$$

where $\phi : R^+ \to R^+$ is a Lebesgue integrable function which is summable and satisfies $0 < \int_0^s \phi(t) ds < 1$ for all $0 < \epsilon < 1$ and $\int_0^1 \phi(t) ds = 1$.

Then the pairs $(A, S)$ and $(B, T)$ have a point of coincidence each. Moreover, $A, B, S$ and $T$ have a unique common fixed point provided that both the pairs $(A, S)$ and $(B, T)$ are weakly compatible.

**Proof.** Suppose that the pair $(A, S)$ satisfies the common limit in the range of $S$ property, then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = z \in X$. Since $A(X) \subseteq T(X)$, therefore for each $\{x_n\}$, there exist $\{y_n\}$ in $X$ such that $A x_n = T y_n$. Thus $\lim_{n \to \infty} T y_n = z$. Hence we have $\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T y_n = z$. Now, we assert that $\lim_{n \to \infty} B y_n = z$. Suppose not, then applying inequality (3.2), we get

$$\int_0^t \phi(s) ds \geq \psi(\int_0^{\min \{M(S(x, A y) z), M(S(x, A y) z), M(T(y, B y) z), M(T(y, B y) z), M(T(y, A y) z), M(T(y, A y) z)\}} \phi(s) ds)$$

taking $n \to \infty$, we have

$$\int_0^t \phi(s) ds \geq \psi(\int_0^{\min \{M(S(z, z) z), M(S(z, z) z), M(T(z, B y) z), M(T(z, B y) z), M(T(z, A y) z), M(T(z, A y) z)\}} \phi(s) ds)$$

$$= \psi(\int_0^{\min \{M(S(z, lim_{n \to \infty} B y_n), z), M(S(z, lim_{n \to \infty} B y_n), z)\}} \phi(s) ds)$$

$$= \psi(\int_0^{\min \{M(S(z, lim_{n \to \infty} B y_n), z)\}} \phi(s) ds)$$

$$> \int_0^{\min \{M(S(z, lim_{n \to \infty} B y_n), z)\}} \phi(s) ds$$

which is a contradiction, therefore $\lim_{n \to \infty} B y_n = z$. Then the pairs $(A, S)$ and $(B, T)$ share the common limit in the range of $S$ property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n =$
\[ \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} By_n = Sz \text{ for some } z \in X. \]

Firstly, we assert that \( Az = Sz \). Suppose not, then by (3.2), we have
\[
\int_0^{M(Az, Bv, t) + \min [M(Sz, Bv, t), M(Az, T v, t)]} \phi(s)ds \geq \psi(\int_0^{\min [M(Sz, Ty_n, t), M(Sz, Az, t), M(Ty_n, By_n, t), M(Sz, By_n, t), M(Ty_n, Az, t)]} \phi(s)ds)
\]

taking \( n \to \infty \), we have
\[
\int_0^{M(Az, Sz, t)} \phi(s)ds \geq \psi(\int_0^{\min [M(Sz, Sz, t), M(Sz, Az, t), M(Sz, Sz, t), M(Sz, Sz, t), M(Sz, Az, t)]} \phi(s)ds)
\]
\[= \psi(\int_0^{\min [1, M(Sz, Az, t), 1, 1, M(Sz, Az, t)]} \phi(s)ds) \]
\[= \psi(\int_0^{M(Sz, Az, t)} \phi(s)ds) \]
\[> \int_0^{\min [M(Sz, Az, t), t]} \phi(s)ds \]

which is a contradiction and therefore, \( Az = Sz \). Since, \( A(X) \subseteq T(X) \), there exist \( v \in X \) such that \( Az = Tv \). Secondly, we assert that \( Bv = Tv \). Suppose not, then by (3.2), we get
\[
\int_0^{M(Az, Bv, t) + \min [M(Sz, Bv, t), M(Az, T v, t)]} \phi(s)ds \geq \psi(\int_0^{\min [M(Sz, Ty_v, t), M(Sz, Az, t), M(Ty_v, By_v, t), M(Sz, By_v, t), M(Ty_v, Az, t)]} \phi(s)ds)
\]
\[
\int_0^{M(Tv, Bv, t)} \phi(s)ds \geq \psi(\int_0^{\min [M(Tv, Ty_v, t), M(Tv, Bv, t), M(Tv, Bv, t), M(Tv, Bv, t), M(Tv, T v, t)]} \phi(s)ds)
\]
\[
= \psi(\int_0^{\min [1, 1, M(Tv, Bv, t), 1, 1] M(Tv, Bv, t), 1]} \phi(s)ds) \]
\[
= \psi(\int_0^{M(Tv, Bv, t)} \phi(s)ds) \]
\[> \int_0^{M(Tv, Bv, t)} \phi(s)ds \]

a contradiction and therefore \( Tv = Bv = Az = Sz \). Since the pairs \((A, S)\) and \((B, T)\) are weakly compatible and \( Az = Sz \) and \( Tv = Bv \), therefore, \( ASz = SAz = AAz = SSz \), \( BTv = T Bv = TTv = BBv \). Finally, we assert that \( AAz = Az \). Suppose not, then again by (3.2), we have
\[
\int_0^{M(Az, Bv, t) + \min [M(Sz, Bv, t), M(Az, T v, t)]} \phi(s)ds \geq \psi(\int_0^{\min [M(Sz, Az, t), M(Sz, Az, t), M(Tv, Bv, t), M(Sz, Az, t), M(Tv, AAz, t)]} \phi(s)ds)
\]
\[
\int_0^{M(AAz, Az, t)} \phi(s)ds \geq \psi(\int_0^{\min [MAAz, Az, t], MAAz, Az, t), M(Bv, Bv, t), M(AAz, Az, t), M(Bv, AAz, t)]} \phi(s)ds)
\]
\[
\psi\left(\int_0^{\min\{M(AAz;Az), 1.1, M(AAz;Az), M(AAz;AAz)\}} \phi(s)ds\right)
\]
\[
= \psi\left(\int_0^{M(AAz;Az)} \phi(s)ds\right)
\]
\[
\geq \int_0^{M(AAz;Az)} \phi(s)ds
\]
which is contradiction and therefore \(AAz = Az = SAz\) which gives, \(Az\) is common fixed point of \(A\) and \(S\). Similarly, one can easily prove that \(BBv = Bv = TBv\); that is \(Bv\) is common fixed point of \(B\) and \(T\). As \(Az = Bv\), therefore, \(Az\) is common fixed point of \(A, B, S\) and \(T\). The uniqueness of common fixed point is an easy consequence of inequality (3.2).

**Remark 3.2.** Notice that results similar to Theorems 3.2, 3.3 and Corollaries 3.1, 3.2, 3.3, 3.4 can also be outlined in respect to Theorem 3.4, but we may omit the details with a view to avoid any repetition.

**Example 3.1.** Let \((X, M, *)\) be a fuzzy metric space where \(X = [0, 2)\) with a \(t\)-norm defined by \(a * b = \min\{a, b\}\). Define \(M(x, y, t) = \frac{t}{t + |x - y|}\) if \(t > 0\) and \(M(x, y, 0) = 0\). Define \(A, B, S\) and \(T\) by \(A(X) = B(X) = 1\) and \(Sx = 1\) if \(x \in Q\); \(Sx = \frac{2}{3}\) otherwise and \(Tx = 1\) if \(x \in Q\) and \(Tx = \frac{1}{3}\) otherwise. Clearly, the pairs \((A, S)\) and \((B, T)\) satisfies all conditions of Theorem 3.1 and shares the common limit in the range of \(S\) property and \(A(X) \subseteq T(X)\). \(A, B, S\) and \(T\) have a unique common fixed point \(x = 1\).

**Remark 3.3.** One can obtain the similar results in setting of probabilistic metric spaces.

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