Common fixed point theorems in modular G-metric spaces

B. Azadifar¹, M. Maramaei¹, Gh. Sadeghi¹*

(1) Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

Abstract
The purpose of this paper is to prove the existence of the unique common fixed point theorems of a pair of weakly compatible mappings satisfying $\Phi$–maps in modular G–metric spaces.

Keywords: modular $G$-metric space, weakly compatible self-maps, contractive mappings satisfying $\Phi$-maps; common fixed point.

1 Introduction and Preliminaries

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a major role in mathematics and applied sciences, such as optimization, mathematical models and economic theories.

There were many authors introduced the generalization of metric spaces such as $2$–metric spaces [4] and $D$–metric spaces [3]. In [5] Mustafa and Sims found that most of the claim concerning the fundamental topological properties of $D$–metric spaces are incorrect. So, they introduced a generalization of metric spaces ($G$–metric spaces). The notion of a modular metric on an arbitrary set an the corresponding modular space, more general than a metric space were introduced and studied recently by Chistyakof [2]. Recently, the authors introduce the notion of modular $G$–metric spaces and obtain some fixed point theorems of contractive mappings defined on modular G–metric spaces [1]. In the sequel, we collect some basic facts and introduce some notations related to modular G–metric spaces. For further details and proofs, we refer the reader to [1].

Definition 1.1. Let $X$ be a nonempty set, and let $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, y, z) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality),

then the function $G$ is called a $G$-metric on $X$, and the pair $(X, G)$ is a $G$-metric space.

Definition 1.2. Let $X$ be a nonempty set, and let $\nu : (0, \infty) \times X \times X \times X \rightarrow [0, \infty]$ be a function satisfying:

(V1) $\nu(\lambda, y, z) = 0$ for all $x, y \in X$ and $\lambda > 0$ if $x = y = z$,

(V2) $\nu(\lambda, x, y) > 0$ for all $x, y \in X$ and $\lambda > 0$ with $x \neq y$,

*Corresponding author. Email address: ghadir54@gmail.com
then the function $\nu_\lambda$ is called a modular G-metric on $X$.

Example 1.1. The following indexed objects $\nu$ are simple examples of modulars on a set $X$. Let $\lambda > 0$ and $x, y, z \in X$, we have:
(a) $\nu_\lambda(x, y, z) = \infty$ if $x \neq y \neq z$, $\nu_\lambda(x, y, z) = 0$ if $x = y = z$; and if $(X, G)$ is a (pseudo)metric space with (pseudo)metric $G$, then we also have:
(b) $\nu_\lambda(x, y, z) = \frac{G(x, y, z)}{\psi(A)}$, where $\psi : (0, \infty) \rightarrow (0, \infty)$ is a nondecreasing function;
(c) $\nu_\lambda(x, y, z) = \infty$ if $\lambda \leq G(x, y, z)$, and $\nu_\lambda(x, y, z) = 0$ if $\lambda > G(x, y, z)$;
(d) $\nu_\lambda(x, y, z) = \infty$ if $\lambda < G(x, y, z)$, and $\nu_\lambda(x, y, z) = 0$ if $\lambda \geq G(x, y, z)$.

Remark 1.1. Note that for $x, y, z \in X$ the function $0 < \lambda \mapsto \nu_\lambda(x, y, z) \in [0, \infty]$ is nonincreasing on $(0, \infty)$. Suppose $0 < \mu < \lambda$, then (V1) and (V5) imply
$$\nu_\lambda(x, y, z) \leq \nu_{\lambda-\mu}(x, x, x) + \nu_\mu(x, y, z) = \nu_\mu(x, y, z).$$

It follows that each point $\lambda > 0$ the right limit $\nu_{\lambda+0}(x, y, z) = \lim_{\mu \rightarrow \lambda+0}\nu_\mu(x, y, z)$ and left limit $\nu_{\lambda-0}(x, y, z) = \lim_{\mu \rightarrow \lambda-0}\nu_\mu(x, y, z)$ exist in $[0, \infty)$ and following two inequalities hold:
$$\nu_{\lambda+0}(x, y, z) \leq \nu_\lambda(x, y, z) \leq \nu_{\lambda-0}(x, y, z).$$

Proposition 1.1. Let $(X, \nu)$ be a modular G-metric space induced by metric modular $\nu$, for any $x, y, z, a \in X$ it follows that:
(1) If $\nu_\lambda(x, y, z) = 0$ for all $\lambda > 0$, then $x = y = z$.
(2) $\nu_\lambda(x, y, z) \leq \nu_2(x, x, y) + \nu_2(x, x, z)$ for all $\lambda > 0$.
(3) $\nu_\lambda(x, y, z) \leq 2\nu_2(x, x, y)$ for all $\lambda > 0$.
(4) $\nu_\lambda(x, y, z) \leq \nu_2(x, a, z) + \nu_2(a, y, z)$ for all $\lambda > 0$.
(5) $\nu_\lambda(x, y, z) \leq \frac{2}{\lambda} \nu_2(x, y, a) + \nu_2(x, a, z) + \nu_2(a, y, z)$ for all $\lambda > 0$.
(6) $\nu_\lambda(x, y, z) \leq \left( \nu_2(x, a, a) + \nu_2(y, a, a) + \nu_2(z, a, a) \right)$ for all $\lambda > 0$.

Definition 1.3. Let $(X, \nu)$ be a modular G-metric space then for $x_0 \in X$ and $r > 0$, the $\nu$-ball with center $x_0$ and radius $r > 0$ is
$$B_\nu(x_0, r) = \{ y \in X : \nu_\lambda(x_0, y, y) < r \text{ for all } \lambda > 0 \}.$$

Definition 1.4. Let $(X, \nu)$ be a modular G-metric space.
(i) The sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ is said to be $\nu$-convergent if for all $\epsilon > 0$, there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that $\nu_\lambda(x_n, x_n, x) < \epsilon$, for any $n, m \geq n_0$ and $\lambda > 0$.
(ii) The sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ is said to be $\nu$-Cauchy if for all $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $\nu_\lambda(x_n, x_m, x) < \epsilon$, for any $n, m, l \geq n_0$ and $\lambda > 0$.
(iii) $X$ is said to be $\nu$-complete if every $\nu$-Cauchy in $X$ is a $\nu$-convergent sequence in $X$.

Proposition 1.2. Let $(X, \nu)$ be a modular G-metric space and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $X$. Then the following are equivalent:
(1) $\{x_n\}_{n \in \mathbb{N}}$ is $\nu$-convergent to $x$.
(2) $\nu_\lambda(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda > 0$.
(3) $\nu_\lambda(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda > 0$.
(4) $\nu_\lambda(x_m, x_n, x) \rightarrow 0$ as $m, n \rightarrow \infty$ for all $\lambda > 0$. 
Proposition 1.3. [1] Let \((X, \nu)\) be a modular G-metric space and \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence in \(X\). Then the following are equivalent:

1) \(\{x_n\}_{n \in \mathbb{N}}\) is \(\nu\)-Cauchy.

2) For every \(\epsilon > 0\), there exist \(n_\epsilon \in \mathbb{N}\) such that \(\nu_k(x_n, x_{n+m}, x_m) < \epsilon\) for any \(n, m \geq n_\epsilon\) and \(k > 0\).

Definition 1.5. Let \(g\) and \(h\) be single-valued self mappings on a set \(X\). If \(w = gx = hx\) for some \(x \in X\), then \(x\) is called a coincidence point of \(g\) and \(h\), and \(w\) is called a point of coincidence of \(g\) and \(h\).

Definition 1.6. A pair of maps \(g\) and \(h\) is called weakly compatible pair if they commute at coincidence point.

Proposition 1.4. Let \(g\) and \(h\) be weakly compatible self mappings on a set \(X\). If \(g\) and \(h\) have a unique point of coincidence \(w = gx = hx\), then \(w\) is the unique common fixed point of \(g\) and \(h\).

Proof. Since \(w = gx = hx\), \(g\) and \(h\) are weakly compatible, we have \(gw = ghx = hgx = hw\), i.e. \(gw = hw\) is a point of coincidence of \(g\) and \(h\). But \(w\) is the only point of coincidence of \(g\) and \(h\), so \(w = gw = hw\). Moreover if \(z = gz = hz\), then \(z\) is a point of coincidence of \(g\) and \(h\), and therefore \(z = w\) by uniqueness. Thus \(w\) is the unique common fixed point of \(g\) and \(h\).

2. Common fixed point theorems of a pair of weakly compatible mappings

Let \(\Phi\) be the set of all function \(\phi\) such that \(\phi : [0, +\infty) \rightarrow [0, +\infty)\) is a nondecreasing function satisfying \(\lim_{t \to +\infty} \phi(t) = 0\) for all \(t \in (0, +\infty)\). If \(\phi \in \Phi\), then \(\phi\) is called a \(\Phi\)-map, [6]. Moreover, if \(\phi\) is a \(\Phi\)-map then

i) \(\phi(t) < t\) for all \(t \in (0, +\infty)\);

ii) \(\phi(0) = 0\).

Throughout this paper, unless otherwise stated, we assume that \(\phi\) is a \(\Phi\)-map.

Theorem 2.1. Let \((X, \nu)\) be a modular G-metric space. Suppose that the mappings \(g, h : X_\nu \rightarrow X_\nu\) satisfy either

\[
v_k(gx, gy, gz) \leq \phi(\max\{v_k(hx, gx, gx), v_k(hy, gy, gy), v_k(hz, gz, gz)\}),
\]

or

\[
v_k(gx, gy, gz) \leq \phi(\max\{v_k(hx, hx, hx), v_k(hy, hy, hy), v_k(hz, hz, hz)\})
\]

for all \(x, y, z \in X_\nu\) and \(k > 0\). If the range of \(h\) contains the range of \(g\) and \(h(X_\nu)\) is complete subspace of \(X_\nu\), then \(g\) and \(h\) have a unique point of coincidence in \(X_\nu\). Moreover if \(g\) and \(h\) are weakly compatible, then \(g\) and \(h\) have a unique common fixed point.

Proof. Assume that \(g\) and \(h\) satisfy the condition (2.1). Let \(x_0\) be an arbitrary point in \(X_\nu\). Since the range of \(h\) contains the range of \(g\), there is \(x_1 \in X_\nu\) such that \(hx_1 = gx_0\). By continuing the process as before, we can construct a sequence \(\{hx_n\}\) such that \(hx_{n+1} = gx_n\) for all \(n \in \mathbb{N}\). If there is \(n \in \mathbb{N}\) such that \(hx_n = hx_{n+1}\), then \(g\) and \(h\) have a point of coincidence. Thus we can suppose that \(hx_n \neq hx_{n+1}\) for all \(n \in \mathbb{N}\). Therefore, for each \(n \in \mathbb{N}\), we obtain that

\[
v_k(hx_n, hx_{n+1}, hx_{n+1}) = v_k(gx_{n+1}, gx_{n+1}, gx_{n+1}) \leq \phi(\max\{v_k(hx_{n+1}, gx_{n+1}, gx_{n+1}), v_k(hx_n, gx_n, gx_n)\}) \leq \phi(\max\{v_k(hx_{n+1}, gx_{n+1}, gx_{n+1}), v_k(hx_n, gx_n, gx_n)\}) \leq \phi(\max\{v_k(hx_{n+1}, hx_n, hx_n), v_k(hx_n, hx_{n+1}, hx_{n+1})\}).
\]

If \(\max\{v_k(hx_{n+1}, hx_n, hx_n), v_k(hx_n, hx_{n+1}, hx_{n+1})\} = v_k(hx_n, hx_{n+1}, hx_{n+1}),\) then

\[
v_k(hx_n, hx_{n+1}, hx_{n+1}) \leq \phi(v_k(hx_n, hx_{n+1}, hx_{n+1})) < v_k(hx_n, hx_{n+1}, hx_{n+1}),
\]

which leads to a contradiction. This implies that

\[
v_k(hx_n, hx_{n+1}, hx_{n+1}) \leq \phi(v_k(hx_{n+1}, hx_n, hx_n)).
\]
That is, for each $n \in \mathbb{N}$, we have

$$\nu_\lambda(h_{x_n}, h_{x_{n+1}}, h_{x_{n+1}}) = \nu_\lambda(g_{x_{n-1}}, g_{x_n}, g_{x_n})$$

$$\leq \phi(\nu_\lambda(h_{x_{n-1}}, h_{x_n}, h_{x_n}))$$

$$\leq \phi^2(\nu_\lambda(h_{x_{n-2}}, h_{x_{n-1}}, h_{x_{n-1}}))$$

$$\vdots$$

$$\leq \phi^n(\nu_\lambda(h_{x_0}, h_{x_1}, h_{x_1}))$$

We will show that $\{h_{x_n}\}$ is $G$-Cauchy. Let $\varepsilon > 0$.

Since $\lim_{n \to \infty} \phi^n(\nu_\lambda(h_{x_0}, h_{x_1}, h_{x_1})) = 0$ and $\phi(\varepsilon) < \varepsilon$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\phi^n(\nu_\lambda(h_{x_0}, h_{x_1}, h_{x_1})) < \varepsilon - \phi(\varepsilon) \quad \text{for all} \quad n \geq n_\varepsilon.$$

This implies that

$$\nu_\lambda(h_{x_n}, h_{x_{n+1}}, h_{x_{n+1}}) \quad \text{for all} \quad n \geq n_\varepsilon. \quad (2.3)$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$\nu_\lambda(h_{x_n}, h_{x_m}, h_{x_m}) < \varepsilon \quad \text{for all} \quad m \geq n \geq n_\varepsilon. \quad (2.4)$$

by induction on $m$. Since $\varepsilon - \phi(\varepsilon) < \varepsilon$ and by (2.3), we obtain that (2.4) holds for $m = n + 1$. Suppose that (2.4) holds for $m = k$. Therefore, for $m = k + 1$, we have

$$\nu_\lambda(h_{x_n}, h_{x_{k+1}}, h_{x_{k+1}}) \leq \nu_\lambda(h_{x_n}, h_{x_{n+1}}, h_{x_{n+1}}) + \nu_\lambda(h_{x_{n+1}}, h_{x_{k+1}}, h_{x_{k+1}})$$

$$\leq \varepsilon - \phi(\varepsilon) + \nu_\lambda(g_{x_{n+1}}, g_{x_{n+1}}, g_{x_{k+1}})$$

$$\leq \varepsilon - \phi(\varepsilon) + \phi(\max\{\nu_\lambda(h_{x_n}, h_{x_{n+1}}, h_{x_{n+1}}), \nu_\lambda(h_{x_{k+1}}, h_{x_{k+1}}, h_{x_{k+1}})\})$$

$$\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon.$$

Thus (2.4) holds for all $m \geq n \geq n_\varepsilon$. It follows that $\{h_{x_n}\}$ is $\nu$-Cauchy. By the completeness of $h(X_\nu)$, we obtain that $\{h_{x_n}\}$ is $\nu$-convergent to some $q \in h(X_\nu)$. So there exists $p \in X_\nu$ such that $hp = q$. We will show that $hp = gp$. Suppose that $hp \neq gp$. By (2.1) we have

$$\nu_\lambda(h_{x_n}, gp, gp) = \nu_\lambda(g_{x_{n-1}}, gp, gp)$$

$$\leq \phi(\max\{\nu_\lambda(h_{x_{n-1}}, h_{x_n}, h_{x_n}), \nu_\lambda(hp, gp, gp)\}).$$

Case 1.

$$\max\{\nu_\lambda(h_{x_{n-1}}, h_{x_n}, h_{x_n}), \nu_\lambda(hp, gp, gp)\} = \nu_\lambda(h_{x_{n-1}}, h_{x_n}, h_{x_n}),$$

we obtain that

$$\nu_\lambda(h_{x_n}, gp, gp) \leq \phi(\nu_\lambda(h_{x_{n-1}}, h_{x_n}, h_{x_n})) < \nu_\lambda(h_{x_{n-1}}, h_{x_n}, h_{x_n}).$$

By taking $n \to \infty$, we have $\nu_\lambda(hp, gp, gp) = 0$ and so $hp = gp$.

Case 2.

$$\max\{\nu_\lambda(h_{x_{n-1}}, h_{x_n}, h_{x_n}), \nu_\lambda(hp, gp, gp)\} = \nu_\lambda(hp, gp, gp),$$

we obtain that

$$\nu_\lambda(hp, gp, gp) \leq \phi(\nu_\lambda(hp, gp, gp)).$$

By taking $n \to \infty$, we have

$$\nu_\lambda(hp, gp, gp) \leq \phi(\nu_\lambda(hp, gp, gp)) < \nu_\lambda(hp, gp, gp).$$
which leads to a contradiction. Therefore \( hp = gp \). We now show that \( g \) and \( h \) have a unique point of coincidence. Suppose that \( hq = gq \) for some \( q \in X_v \). By applying (2.1), it follows that

\[
\nu_\lambda(hp, hp, hq) = \nu_\lambda(gp, gp, gq) \\
\leq \phi(\max\{ \nu_\lambda(hp, gp, gp), \nu_\lambda(hp, gp, gp), \nu_\lambda(hq, gq, gq) \}) \\
= 0.
\]

Therefore \( hp = hq \). This implies that \( g \) and \( h \) have a unique point of coincidence. By proposition 1.4, we can conclude that \( g \) and \( h \) have a unique common fixed point. The proof using (2.2) is similar.

**Corollary 2.1.** Let \((X, \nu)\) be a modular G-metric space. Suppose that the mapping \( g, h : X_v \rightarrow X_v \) satisfy either

\[
\nu_\lambda(gx, gy, gz) \leq k(\max\{ \nu_\lambda(hx, gx, gx), \nu_\lambda(hy, gy, gy), \nu_\lambda(hz, gz, gz) \})
\]

or

\[
\nu_\lambda(gx, gy, gz) \leq k(\max\{ \nu_\lambda(hx, hx, gx), \nu_\lambda(hy, hy, gy), \nu_\lambda(hz, hz, gz) \})
\]

for all \( x, y, z \in X_v \) where \( 0 \leq k \leq 1 \). If the range of \( h \) contains the range of \( g \) and \( h(X_v) \) is a complete subspace of \( X_v \), then \( g \) and \( h \) have a unique point of coincidence in \( X_v \). Moreover if \( g \) and \( h \) are weakly compatible, then \( g \) and \( h \) have a unique common fixed point.

**Proof.** Define \( \phi : [0, \infty) \rightarrow [0, \infty) \) by \( \phi(t) = kt \). Therefore \( \phi \) is a nondecreasing function and \( \lim_{t \rightarrow \infty} \phi^p(t) = 0 \) for all \( t \in [0, \infty) \). It follows that the contractive conditions in Theorem 2.1 are now satisfied. This completes the proof.

**Example 2.1.** Let \( X = [0, 2] \), \( \nu_\lambda(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\} \) and \( \phi(t) = \frac{t}{2} \). Therefore \( \phi \) is a \( \phi \)-map. Define \( g, h : X \rightarrow X \) by

\[
gx = 1 \quad \text{and} \quad hx = 2 - x
\]

We obtain that \( g \) and \( h \) satisfy (2.1) and (2.2) in Theorem 2.1. Indeed, we have

\[
\nu_\lambda(gx, gy, gz) = 0,
\]

\[
\phi(\max\{ \nu_\lambda(hx, gx, gx), \nu_\lambda(hy, gy, gy), \nu_\lambda(hz, gz, gz) \}) = \frac{1}{2}(\max\{|1 - x|, |1 - y|, |1 - z|},
\]

and

\[
\phi(\max\{ \nu_\lambda(hx, hx, gx), \nu_\lambda(hy, hy, gy), \nu_\lambda(hz, hz, gz) \}) = \frac{1}{2}(\max\{|1 - x|, |1 - y|, |1 - z|}.
\]

It is obvious that the range of \( h \) and \( h(X) \) is a complete subspace of \((X, \nu)\). Furthermore, \( g \) and \( h \) are weakly compatible. Thus all assumptions in Theorem 2.1 are satisfied. This implies that \( g \) and \( h \) have a unique common fixed point which is \( x = 1 \).

**Theorem 2.2.** Let \((X, \nu)\) be a modular G-metric space. Suppose that the mapping \( g, h : X_v \rightarrow X_v \) satisfy

\[
\nu_\lambda(gx, gy, gz) \leq \phi(\nu_\lambda(hx, hy, hz)),
\]

for all \( x, y, z \in X_v \) and \( \lambda > 0 \). If \( g(X_v) \subseteq h(X_v) \) and \( h(X_v) \) is a complete subspace of \( X_v \), then \( g \) and \( h \) have a unique point of coincidence in \( X_v \). Moreover if \( g \) and \( h \) are weakly compatible, then \( g \) and \( h \) have a unique common fixed point.

**Proof.** Let \( x_0 \) be an arbitrary point in \( X_v \). Since \( g(X_v) \subseteq h(X_v) \) there is \( x_1 \in X_v \) such that \( hx_1 = gx_0 \). By continuing the process as before, we can construct a sequence \( \{hx_n\}_{n \in \mathbb{N}} \) such that \( hx_{n+1} = gx_n \) for all \( n \in \mathbb{N} \). If there is \( n \in \mathbb{N} \) such
that $h_{x_{n+1}} = h_{x_n}$, then $g$ and $h$ have a a point of coincidence. Thus we can suppose that $h_{x_{n+1}} \neq h_{x_n}$ for all $n \in \mathbb{N}$. Therefore for each $n \in \mathbb{N}$, we obtain that

$$v_\lambda(h_{x_n}, h_{x_{n+1}}, h_{x_{n+1}}) = v_\lambda(g_{x_{n-1}}, g_{x_n}, g_{x_n}) \leq \phi(v_\lambda(h_{x_{n-1}}, h_{x_n}, h_{x_n})) \leq \phi^2(v_\lambda(h_{x_{n-2}}, h_{x_{n-1}}, h_{x_{n-1}})) \leq \cdots \leq \phi^n(v_\lambda(h_{x_0}, h_{x_1}, h_{x_1})),$$

for all $\lambda > 0$. We will show that $\{h_{x_n}\}_{n \in \mathbb{N}}$ is $\nu$-Cauchy. Let $\epsilon > 0$. Since $\lim_{n \to \infty} \phi^n(v_\lambda(h_{x_0}, h_{x_1}, h_{x_1})) = 0$ and $\phi(\epsilon) < \epsilon$, there exists $n_\epsilon \in \mathbb{N}$, such that

$$\phi^n(v_\lambda(h_{x_0}, h_{x_1}, h_{x_1})) < \epsilon - \phi(\epsilon) \quad \text{for all} \quad n \geq n_\epsilon.$$

for all $\lambda > 0$. This implies that

$$v_\lambda(h_{x_n}, h_{x_{n+1}}, h_{x_{n+1}}) < \epsilon - \phi(\epsilon) \quad \text{for all} \quad n \geq n_\epsilon, \quad (2.6)$$

for all $\lambda > 0$. Let $m, n \in \mathbb{N}$ with $m > n$. Then

$$v_\lambda(h_{x_n}, h_{x_m}, h_{x_m}) < \epsilon \quad \text{for all} \quad m \geq n \geq n_\epsilon, \quad (2.7)$$

by induction on $m$ and for all $\lambda > 0$. Since $\epsilon - \phi(\epsilon) < \epsilon$ and by inequality (2.6), we obtain that (2.7) holds for $m = n + 1$. Suppose that (2.7) holds for $m = k$. Therefore for $m = k + 1$ we have

$$v_\lambda(h_{x_n}, h_{x_{k+1}}, h_{x_{k+1}}) \leq v_\lambda(h_{x_n}, h_{x_{n+1}}, h_{x_{n+1}}) + v_\lambda(h_{x_{n+1}}, h_{x_{k+1}}, h_{x_{k+1}}) \leq \epsilon - \phi(\epsilon) + v_\lambda(g_{x_k}, g_{x_k}, g_{x_k}) \leq \epsilon - \phi(\epsilon) + \phi(v_\lambda(h_{x_n}, h_{x_k}, h_{x_k})) \leq \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon,$$

for all $\lambda > 0$. Thus (2.7) holds for all $m \geq n \geq n_\epsilon$ and $\lambda > 0$. It follows that $\{h_{x_n}\}_{n \in \mathbb{N}}$ is $\nu$-Cauchy. By the completeness of $h(X_\nu)$, we obtain that $\{h_{x_n}\}_{n \in \mathbb{N}}$ is $\nu$-convergent to some $q \in h(X_\nu)$. So there exists $p \in X_\nu$ such that $hp = q$. We will show that $hp = gq$. By (2.5), we obtain

$$v_\lambda(hp, hp, gp) \leq v_\lambda(hp, hp, h_{x_{n+1}}) + v_\lambda(h_{x_{n+1}}, h_{x_{n+1}}, gp) \leq v_\lambda(hp, hp, h_{x_{n+1}}) + \phi(v_\lambda(h_{x_n}, h_{x_n}, hp)) \leq v_\lambda(hp, hp, h_{x_{n+1}}) + v_\lambda(h_{x_n}, h_{x_n}, hp),$$

for all $\lambda > 0$. By taking $n \to \infty$, we have $v_\lambda(hp, hp, gp) = 0$ and so $gq = hq$. We now show that $g$ and $h$ have a a point of coincidence. Suppose that $gq = hq$ for some $q \in X_\nu$. Assume that $hp \neq hq$. By applying (2.5), it follows that

$$v_\lambda(hp, hp, hq) = v_\lambda(gp, gp, gq) \leq \phi(v_\lambda(hp, hp, gq)) \leq v_\lambda(hp, hp, gq),$$

for all $\lambda > 0$. Which leads to a contraction. Therefore $hp = hq$. This implies that $g$ and $h$ have a unique point of coincidence. By Proposition 1.4, we can conclude that $g$ and $h$ have a unique common fixed point.

By setting $h$ to be the identity function on $X_\nu$, we immediately have the following corollary.
Corollary 2.2. Let $X$ be a $\nu$-complete modular $G$-metric space. Suppose that the mapping $g : X \to X$ satisfy

$$\nu_\lambda(gx, gy, gz) \leq \phi(\nu_\lambda(x, y, z)),$$

for all $x, y, z \in X$ and $\lambda > 0$. Then $g$ has a unique fixed point.

Theorem 2.3. Let $(X, \nu)$ be a modular $G$-metric space. Suppose that the mapping $g, h : X \to X$ satisfy

$$\nu_\lambda(gx, gy, gz) \leq \phi(\nu_\lambda(hx, hy, hz)), \tag{2.8}$$

for all $x, y, z \in X$ and $\lambda > 0$. If $g(X) \subset h(X)$ and $h(X)$ is a complete subspace of $X$, then $g$ and $h$ have a unique point of coincidence in $X$. Moreover if $g$ and $h$ are weakly compatible, then $g$ and $h$ have a unique common fixed point.

Proof. Let $x_0$ be an arbitrary point in $X$. Since $g(X) \subset h(X)$ there is $x_1 \in X$ such that $hx_1 = gx_0$. By continuing the process as before, we can construct a sequence $\{hx_n\}_{n \in \mathbb{N}}$ such that $hx_{n+1} = gx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $hx_n = x$, then $g$ and $h$ have a a point of coincidence. Thus we can suppose that $hx_{n+1} \neq hx_n$ for all $n \in \mathbb{N}$. Therefore for each $n \in \mathbb{N}$, we obtain that

$$\nu_\lambda(hx_n, hx_{n+1}, hx_{n+1}) = \nu_\lambda(gx_{n-1}, gx_n, gx_n) \leq \phi(\max\{\nu_\lambda(hx_{n-1}, gx_n, gx_n), \nu(hx_{n-1}, hx_n, hx_n)\}, \nu_\lambda(hx_n, hx_n, gx_n), \nu_\lambda(gx_{n-1}, hx_n, hx_n)) \leq \phi(\max\{\nu_\lambda(hx_n, hx_n, hx_n), \nu_\lambda(hx_n, hx_n, hx_n)\}),$$

for all $\lambda > 0$. If $\max\{\nu_\lambda(hx_{n-1}, hx_n, hx_n), \nu_\lambda(hx_{n-1}, hx_n, hx_n)\} = \nu_\lambda(hx_n, hx_{n+1}, hx_{n+1})$, for all $\lambda > 0$ then $\nu_\lambda(hx_n, hx_{n+1}, hx_{n+1}) \leq \phi(\nu_\lambda(hx_n, hx_{n+1}, hx_{n+1})) < \nu_\lambda(hx_n, hx_{n+1}, hx_{n+1})$, for all $\lambda > 0$ which is a contradiction. This implies that

$$\nu_\lambda(hx_n, hx_{n+1}, hx_{n+1}) \leq \phi(\nu_\lambda(hx_{n}, hx_{n}, hx_{n})), \quad (\lambda > 0).$$

That is for each $n \in \mathbb{N}$, we have

$$\nu_\lambda(hx_n, hx_{n+1}, hx_{n+1}) = \nu_\lambda(gx_{n-1}, gx_n, gx_n) \leq \phi(\nu_\lambda(hx_{n-1}, hx_n, hx_n)) \leq \phi^2(\nu_\lambda(hx_{n-2}, hx_{n-1}, hx_{n-1})) \leq \cdots \leq \phi^n(\nu_\lambda(hx_0, hx_1, hx_1)),$$

for all $\lambda > 0$. We will show that $\{hx_n\}_{n \in \mathbb{N}}$ is $\nu$-Cauchy. Let $\varepsilon > 0$. Since $\lim_{n \to \infty} \phi^n(\nu_\lambda(hx_0, hx_1, hx_1)) = 0$ and $\phi(\varepsilon) < \varepsilon$, there exists $n_\varepsilon \in \mathbb{N}$, such that

$$\phi^n(\nu_\lambda(hx_0, hx_1, hx_1)) < \varepsilon - \phi(\varepsilon) \quad for \ all \ n \geq n_\varepsilon.$$

for all $\lambda > 0$. This implies that

$$\nu_\lambda(hx_0, hx_1, hx_1) < \varepsilon - \phi(\varepsilon) \quad for \ all \ n \geq n_\varepsilon, \tag{2.9}$$

for all $\lambda > 0$. Let $m, n \in \mathbb{N}$ with $m > n$.

$$\nu_\lambda(hx_n, hx_m, hx_m) < \varepsilon \quad for \ all \ m \geq n \geq n_\varepsilon, \tag{2.10}$$
by induction on $m$ and for all $\lambda > 0$. Since $\varepsilon - \phi(\varepsilon) < \varepsilon$ and by inequality (2.9), we obtain that (2.10) holds for $m = n + 1$. Suppose that (2.10) holds for $m = k$. Therefore for $m = k + 1$ we have

$$v_{\lambda}(hx_n, hx_{k+1}, hx_{k+1}) \leq v_{\lambda}(hx_n, hx_{n+1}, hx_{n+1}) + v_{\lambda}(hx_{n+1}, hx_{k+1}, hx_{k+1})$$

$$\leq \varepsilon - \phi(\varepsilon) + v_{\lambda}(g_{x_n}, g_{x_k}, g_{x_k})$$

$$\leq \varepsilon - \phi(\varepsilon) + \phi(\max\{v_{\lambda}(hx_n, hx_n, hx_n), v_{\lambda}(hx_{n+1}, hx_{n+1}, hx_{n+1})\})$$

$$\leq \varepsilon - \phi(\varepsilon) + \phi(\max\{v_{\lambda}(hx_n, hx_n, hx_n), v_{\lambda}(hx_{n+1}, hx_{n+1}, hx_{n+1})\})$$

for all $\lambda > 0$. Thus (2.10) holds for all $m \geq n \geq n_e$ and $\lambda > 0$. It follows that $\{hx_n\}_{n \in \mathbb{N}}$ is $v$-Cauchy. By the completeness of $h(X_n)$, we obtain that $\{hx_n\}_{n \in \mathbb{N}}$ is $v$-convergent to some $q \in h(X_n)$. So there exists $p \in X_n$ such that $hp = q$. We will show that $hp = gp$. By (2.11), we obtain

$$v_{\lambda}(hp, hp, gp) \leq v_{\lambda}(hp, hp, hx_n) + v_{\lambda}(hx_n, hx_n, gp)$$

$$\leq v_{\lambda}(hp, hp, hx_n) + v_{\lambda}(g_{x_n-1}, g_{x_n-1}, gp)$$

$$\leq v_{\lambda}(hp, hp, hx_n) + \phi(\max\{v_{\lambda}(hx_{n-1}, hx_{n-1}, hp), v_{\lambda}(hx_{n-1}, g_{x_n-1}, g_{x_n-1})\})$$

$$\leq v_{\lambda}(hp, hp, hx_n) + \phi(\max\{v_{\lambda}(hx_{n-1}, hx_{n-1}, hp), v_{\lambda}(hx_{n-1}, hx_{n-1}, hp)\})$$

for all $\lambda > 0$. 

Case 1. If

$$\max\{v_{\lambda}(hx_{n-1}, hx_{n-1}, hp), v_{\lambda}(hx_{n-1}, hx_{n-1}, hx_n), v_{\lambda}(hx_{n-1}, hx_{n-1}, hp)\}$$

$$= v_{\lambda}(hx_{n-1}, hx_{n-1}, hp),$$

for all $\lambda > 0$. We obtain that

$$v_{\lambda}(hp, hp, gp) < v_{\lambda}(hp, hp, hx_n) + v_{\lambda}(hx_{n-1}, hx_{n-1}, hp),$$

for all $\lambda > 0$. By taking $n \rightarrow \infty$, we have $v_{\lambda}(hp, gp, gp) = 0$, for all $\lambda > 0$. Whence $hp = gp$.

Case 2. If

$$\max\{v_{\lambda}(hx_{n-1}, hx_{n-1}, hp), v_{\lambda}(hx_{n-1}, hx_{n-1}, hx_n), v_{\lambda}(hx_{n-1}, hx_{n-1}, hp)\}$$

$$= v_{\lambda}(hx_{n-1}, hx_{n-1}, hp),$$

for all $\lambda > 0$. We obtain that

$$v_{\lambda}(hp, hp, gp) < v_{\lambda}(hp, hp, hx_n) + v_{\lambda}(hx_{n-1}, hx_{n-1}, hx_n),$$

for all $\lambda > 0$. By taking $n \rightarrow \infty$, we have $v_{\lambda}(hp, gp, gp) = 0$, for all $\lambda > 0$. Whence $hp = gp$.

Case 3. If

$$\max\{v_{\lambda}(hx_{n-1}, hx_{n-1}, hp), v_{\lambda}(hx_{n-1}, hx_{n-1}, hx_n), v_{\lambda}(hx_{n-1}, hx_{n-1}, hp)\}$$

$$= v_{\lambda}(hx_{n-1}, hx_{n-1}, hp),$$

for all $\lambda > 0$. We obtain that

$$v_{\lambda}(hp, hp, gp) < v_{\lambda}(hp, hp, hx_n) + v_{\lambda}(hx_{n-1}, hx_{n-1}, hx_n),$$

for all $\lambda > 0$. By taking $n \rightarrow \infty$, we have $v_{\lambda}(hp, gp, gp) = 0$, for all $\lambda > 0$. Whence $hp = gp$. 
for all $\lambda > 0$. We obtain that
\[ \nu_{\lambda}(hp, gp, gp) < \nu_{\lambda}(hp, hp, hx_{n}) + \nu_{\lambda}(hx_{n}, hx_{n-1}, hp), \]
for all $\lambda > 0$. By taking $n \to \infty$, we have $\nu_{\lambda}(hp, gp, gp) = 0$, for all $\lambda > 0$. Whence $hp = gp$. We show that $g$ and $h$ have a unique point of coincidence. Suppose that $gq = hq$ for some $q \in X_{\nu}$. Assume that $hp \neqhq$. By applying (2.11), it follows that
\[ \nu_{\lambda}(hp, hp,hq) = \phi(\max\{\nu_{\lambda}(hp, hp, gp), \nu_{\lambda}(hp, gp, gp), \nu_{\lambda}(gp, gp, gp)\}) \]
for all $\lambda > 0$, which leads to a contradiction. Therefore $hp =hq$. This implies that $g$ and $h$ have a unique point of coincidence. By Proposition 1.4, we can conclude that $g$ and $h$ have a unique common fixed point. \hfill \Box

Consequently, if we suppose that $h$ is the identity function on $X_{\nu}$, then we obtain the following corollary.

**Corollary 2.3.** Let $X_{\nu}$ be a $\nu$-complete modular G-metric space. Suppose that the mapping $g : X_{\nu} \to X_{\nu}$ satisfies
\[ \nu_{\lambda}(gx, gy, gz) \leq \phi(\max\{\nu_{\lambda}(x, y, z), \nu_{\lambda}(x, gx, gx), \nu_{\lambda}(y, gy, gy), \nu_{\lambda}(gx, y, z)\}), \] (2.11)
for all $x, y, z \in X_{\nu}$ and $\lambda > 0$. Then $g$ has a unique fixed point.

**References**


