Continuity of quantization of an anti-Wick operators and compactness of anti-Wick operators

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Abstract
We study continuity of anti-Wick quantization with symbol in $L^p(\mathbb{R}^{2n})$ acting on $L^q(\mathbb{R}^n)$. To this end, we apply the Wigner transform and Short-time Fourier transform. In continue we study compactness and Hilbert-Schmidt properties of the anti-Wick operators when $q \in \left[ \frac{2p}{p+1}, \frac{2p}{p-1} \right]$ for every $p \in [1, \infty)$.

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1 Introduction
The corresponding principle of quantum mechanics asks that for a given physical system there should be a systematic way to convert observable into quantum observable. We use for this the rather vague term of quantization. For many given systems a true, manageable quantization is problematic, but there are important situations in which commonly accepted solutions exist. A common feature of all types of quantization proposed in the literature is that they establish a correspondence between self-adjoint operators, i.e. quantum observable and classical observable, i.e. real functions on phase space. The theory of Weyl operators is a vast subject of remarkable interest both in mathematical analysis and physics. It originates in quantum physics as a quantization rule associating a self-adjoint operator $\mathcal{O}_W[a]$ with a classical observable $a(x, \xi)$ on phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. In the theory of partial differential equations Weyl operators have been studied as a particular type of pseudo-differential operators and they have proved to be a useful technique in a quantity of problems like elliptic theory, spectral asymptotic, regularity problem[1, 2, 3, 4, 9, 18, 19, 20]. In general in the frame of pseudo-differential calculus the correspondence between symbols and operators does not fulfill this requirement, that is, the operator fails to be self-adjoint when the symbol is real valued. However, this is true for the Weyl and anti-Wick symbols, as is shown for instance in [21]. M.W.Wong analyzed in [27] the case of Weyl quantization with symbol in $L^p(\mathbb{R}^{2n})$ and gave conditions for the operators to be continuous, compact and Hilbert-schmidt. In this paper we also consider symbols in $L^p(\mathbb{R}^{2n})$ that corresponding operator acting on $L^p(\mathbb{R}^n)$ and we study the behaviour of the corresponding anti-Wick operators. As operators acting on $L^2(\mathbb{R}^n)$, Weyl and anti-Wick operators have been investigated mainly in the case where its symbols is a smooth function belong to specific symbol classes. The literature is so vast that we do not even attempt to give exhaustive references but, limiting ourselves to some classical and some recent contributions we indicate as bibliography [3, 7, 12, 16].

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In connection to Weyl operator is the Wigner transform introduced in 1932 by Wigner with the aim of defining a joint probability density for position and momentum in quantum mechanics [17]. If \( f, g \) are functions in \( \mathcal{S}(\mathbb{R}^n) \), then Wigner transform is defined by

\[
\text{Wig}(f, g)(x, \xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x + \frac{t}{2})g(x - \frac{t}{2}) \, dt.
\]  

Weyl operators and Wigner transform are related as [27]

\[
(\text{OP}_W[a]u,v)_{L^2(\mathbb{R}^n)} = (\text{Wig}(v, u))_{L^2(\mathbb{R}^n)},
\]

where for simplicity one can suppose again \( a \in \mathcal{S}(\mathbb{R}^n), u, v \in \mathcal{S}(\mathbb{R}^n) \), with natural extension to various distributional cases. If on the phase space \( \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n} \), \( z = (x, \xi) \), we consider [21]

\[
\Gamma_m = \{ a(z) \in C^m(\mathbb{R}^{2n}) : |\partial_\xi^m a(z)| \leq C_m(z)^{m-\rho(|z|)}, \}
\]

with \( m \in \mathbb{R}, \rho \in (0,1), (z) = \sqrt{1+|z|^2}, \) then a satisfactory pseudo-differential calculus, both in the case of Weyl and anti-Wick operators, has been developed, see for example [21, 24, 27].

Given a function \( a(z), z = (x, \xi) \in \mathbb{R}^{2n} \), consider the Weyl operator:

\[
\text{OP}_W[a]u(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi} a\left(\frac{x+y}{2}, \xi\right)u(y)dyd\xi,
\]

as well the anti-Wick operator:

\[
\text{OP}_{\text{AW}}[a]u(x) = \int_{\mathbb{R}^{2n}} a(z)P_zudz,
\]

where \( P_zu(x) = (u, \Phi_z)\Phi_z(x) \) are orthogonal projection in \( L^2(\mathbb{R}^n) \) for

\[
\Phi_z(x) = e^{ix \cdot \eta} e^{-|x|^2/\gamma^2} \phi(x - y),
\]

\( \lambda \in \mathbb{R}, x \in \mathbb{R}^n, z = (y, \eta) \in \mathbb{R}^{2n} \) and \( \phi(x) = \pi^{-\frac{n}{2}} e^{-|x|^2/2}, x \in \mathbb{R}^n \). In [12], the relation between Weyl and anti-Wick operators was given by the equality

\[
\text{OP}_{\text{AW}}[a] = \text{OP}_W[(2\pi)^{-n}(a * \sigma)],
\]

where \( f * g(x) = \int f(y)g(x-y)dy \). It means that every anti-Wick operator is also a Weyl operator with symbol given by the convolution between anti-Wick symbol and Gaussian function \( \sigma(z) = 2^ne^{-|z|^2} \). The converse is true only modulo regularizing operators see.

2 Main results

In this section we give a direct proof of the continuity of the quantization map

\[
Q : a \in L^p(\mathbb{R}^{2n}) \mapsto \text{OP}_{\text{AW}}[a] \in B(L^q(\mathbb{R}^n)).
\]

In paper [12], for the case \( \text{OP}_{\text{AW}}[a] \in B(L^2(\mathbb{R}^n)) \), it was proved the same result as a consequence of the Calderon-Vaillancourt theorem. Finally, we show the compactness of \( \text{OP}_{\text{AW}}[a] \in B(L^q(\mathbb{R}^n)) \). We can apply the short-time Fourier transform, (for short STFT). Another useful transform widely used in time-frequency analysis, is as follows:

\[
V_\xi f(x,\xi) = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi it \cdot \xi} \, dt,
\]

where \( f \) and \( g \) belong to suitable spaces such that the integral make sense. For details and properties of the STFT for example we refer to [22] and [25]. A straightforward computation show that

\[
\text{Wig}(f, g)(x, \xi) = 2^{n} e^{i2\pi x \cdot \xi} V_\xi f(2x, 2\xi),
\]

where \( \bar{g}(x) = g(-x) \).
Proposition 2.1. [23] Let $E_1, E_2$ and $E_3$ be Banach spaces such that $E_2$ is a reflexive space. Then

(i) $V : (f, g) \in E_2^2 \times E_1 \mapsto V_f g \in E^*$, is continuous.

(ii) $W_{\sigma} : (f, g) \in E_3^2 \times E_1 \mapsto W_{\sigma}(f, g) \in E^*$ is continuous.

Proposition 2.2. The following statements are equivalent:

(i) $Q : a \in L^p(\mathbb{R}^n) \mapsto OP_{AW}[a] \in \mathcal{B}(L^q(\mathbb{R}^n))$ for which $q \neq 1, \infty$, is continuous.

(ii) $W_{\sigma} : L^q(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$, where $\frac{1}{q} + \frac{1}{r} = 1$ and $\frac{1}{r} + \frac{1}{p} = 1$, is continuous.

Proof. For first $a \in L^p(\mathbb{R}^n)$, $u \in L^q(\mathbb{R}^n)$ and Gaussian function $\sigma(\cdot)$ we define $F : L^q(\mathbb{R}^n) \rightarrow \mathbb{C}$ such that

$$F(v) = W_{\sigma,v}(a \ast \sigma) \in \mathbb{C}.$$ 

The functional $F$ is conjugate-linear and by the continuity of $W_{\sigma,v}(\cdot)$ and $W_{\sigma}(\cdot)$ on $L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ respectively,

$$|F(v)| = |W_{\sigma,v}(a \ast \sigma)| = |W_{\sigma,v}(a \ast \sigma)| \leq C \|a \ast \sigma\|_{L^p(\mathbb{R}^n)} \|W_{\sigma,v}\|_{L^q(\mathbb{R}^n)}$$

$$\leq C \|a\|_{L^p(\mathbb{R}^n)} \|\sigma\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)} \|u\|_{L^q(\mathbb{R}^n)}.$$

Therefore, $F$ belongs to $L^q(\mathbb{R}^n)$ since $L^q(\mathbb{R}^n)$ is reflexive. Then there exists a unique $w \in L^q(\mathbb{R}^n)$ for which

$$W_{\sigma,v}(a \ast \sigma) = v(w).$$

Since the element $w \in L^q(\mathbb{R}^n)$ depends on symbol $a$, Gaussian function $\sigma$ and $u \in L^q(\mathbb{R}^n)$,

$$OP_{AW}[a]u = OP_{W}[a \ast \sigma] := w.$$ 

So,

$$(OP_{AW}[a]u, v) = (OP_{W}[a \ast \sigma]u, v) = (a \ast \sigma, W_{\sigma}(v, u))$$

$$= W_{\sigma,v}(a \ast \sigma).$$

Using the inequality (2.3) and relation (2.5),

$$\|OP_{AW}[a]\|_{B(L^1(\mathbb{R}^n))} = \|OP_{W}[a \ast \sigma]\|_{B(L^1(\mathbb{R}^n))} = \|W_{\sigma}(a \ast \sigma)\|_{L^p(\mathbb{R}^n)}$$

$$\leq C \|a \ast \sigma\|_{L^p(\mathbb{R}^n)} \leq C \|a\|_{L^p(\mathbb{R}^n)}.$$ 

For $a \in L^p(\mathbb{R}^n)$, $u \in L^q(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$,

$$(OP_{AW}[a]u, v) = (OP_{W}[a \ast \sigma]u, v) = (a \ast \sigma, W_{\sigma}(v, u)).$$

By relation (2.4) we have that $W_{\sigma}$ is linear in $v$ and conjugate-linear in $u, a \ast \sigma$. Moreover, by the continuity of $v$ on $L^q(\mathbb{R}^n)$, $OP_{W}[a \ast \sigma]$ on $L^q(\mathbb{R}^n)$ and $Q$ on $L^p(\mathbb{R}^n)$ we obtain

$$|W_{\sigma,v}(a \ast \sigma)| = |(OP_{W}[a \ast \sigma]u, v)| = |(OP_{AW}[a]u, v)|$$

$$\leq C \|v\|_{L^q(\mathbb{R}^n)} \|OP_{AW}[a]\|_{L^1(\mathbb{R}^n)} \leq C \|v\|_{L^q(\mathbb{R}^n)} \|a \ast \sigma\|_{L^p(\mathbb{R}^n)} \|u\|_{L^q(\mathbb{R}^n)},$$

so, for every fixed $u \in L^q(\mathbb{R}^n), v \in L^q(\mathbb{R}^n)$, $W_{\sigma,v}(\cdot)$ belong to $L^q(\mathbb{R}^n)$ and

$$\|W_{\sigma,v}(\cdot)\|_{L^q(\mathbb{R}^n)} \leq C \|v\|_{L^q(\mathbb{R}^n)} \|u\|_{L^q(\mathbb{R}^n)},$$

which gives the desired continuity of $W_{\sigma}$ on $L^q(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$. 


Proposition 2.3. Let \( a \in L^p(\mathbb{R}^n) \), \( p \in [1, 2] \). Then \( OP_{AW}[a] : L^q(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n) \) for \( p \leq q \leq p' \) such that \( \frac{1}{p} + \frac{1}{q'} = 1 \) and

\[
\| OP_{AW}[a] \|_{B(L^q(\mathbb{R}^n))} \leq 2^n \left( \frac{p}{p'} \right) ^{\frac{1}{q'}} \| a \|_{L^p}.
\]  

(2.16)

Proof. Using the theorem (2.5) in [26] and by relation (2.9) we obtain that

\[
\| (OP_{AW}[a]u, v) \| \leq \int |a(z)| \| (u, \Phi_z) \| (\Phi_z, v) dz \leq \| a \|_{L^p} \| \text{Wig}(\Phi_z, u) \|_{L^{q'}} \| \text{Wig}(\Phi_z, v) \|_{L^{q'}}
\]

\[
\leq \| a \|_{L^p} 2^n \left( \frac{p}{p'} \right) ^{\frac{1}{q'}} \| u \|_{L^{q'}} \| v \|_{L^{q'}}.
\]

Therefore, the anti-Wick operator \( OP_{AW}[a] : L^q(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n) \) is continuous when \( p \in [1, 2] \).

Proposition 2.4. Let \( a \in L^p(\mathbb{R}^n) \), \( p \in [1, \infty] \). Then the quantization map \( Q : L^p(\mathbb{R}^{2n}) \to B(L^q(\mathbb{R}^n)) \) with \( q = \frac{2p}{p+1} \) is continuous and

\[
\| Q(a) \|_{B(L^q(\mathbb{R}^n))} \leq C \| a \|_{L^p}.
\]  

(2.17)

Proof. From proposition 2.3 with \( q = 1 \), \( Q : L^1(\mathbb{R}^{2n}) \to B(L^1(\mathbb{R}^n)) \) with \( Q(a) = OP_{AW}[a] \) for \( a \in L^p(\mathbb{R}^{2n}) \),

\[
\| Q(a)u \|_{B(L^1(\mathbb{R}^{2n}), B(L^1(\mathbb{R}^n)))} \leq \| a \|_{L^1} \| u \|_{L^1}.
\]

For \( p = \infty \) then \( q = 2 \), so by theorem (2.4) in [26] the quantization map \( Q : L^\infty(\mathbb{R}^{2n}) \to B(L^2(\mathbb{R}^n)) \) given by \( Q(a) = OP_{AW}[a] \) is continuous and

\[
\| Q(a)u \|_{B(L^\infty(\mathbb{R}^{2n}), B(L^2(\mathbb{R}^n)))} \leq \| a \|_{L^1} \| u \|_{L^2},
\]

where \( \| . \|_{B(L^p(\mathbb{R}^{2n}), B(L^q(\mathbb{R}^n)))} \) denotes the norm in the \( L^p(\mathbb{R}^{2n}) \) into \( B(L^q(\mathbb{R}^n)) \), for \( 1 \leq p, q \leq \infty \). The result in case \( 1 < p < \infty \) follows from interpolation theory.

Proposition 2.5. Let \( a \in L^p(\mathbb{R}^n) \), \( p \in [1, \infty] \). Then the quantization map \( Q : L^p(\mathbb{R}^{2n}) \to B(L^q(\mathbb{R}^n)) \) with \( q = \frac{2p}{p+1} \) is continuous and

\[
\| Q(a) \|_{B(L^q(\mathbb{R}^n))} \leq C \| a \|_{L^p}.
\]  

(2.18)

Proof. We know that \( (OP_{AW}[a])^* = OP_{AW}[a] \) (see [21]), so the result follows from duality and proposition 2.4.

Theorem 2.1. Let \( a \in L^p(\mathbb{R}^{2n}) \) and \( p \in [1, \infty] \). Then the quantization map

\[
Q : L^p(\mathbb{R}^{2n}) \to B(L^q(\mathbb{R}^n))
\]

such that \( Q(a) = OP_{AW}[a] \) is continuous when \( q \in \left[ \frac{2p}{p+1}, \frac{2p}{p-1} \right] \) and we obtain

\[
\| OP_{AW}[a] \|_{B(L^q(\mathbb{R}^n))} \leq C \| a \|_{L^p(\mathbb{R}^{2n})}.
\]  

(2.19)

Proof. This is direct by using the proposition 2.4 and 2.5.

Theorem 2.2. Suppose that \( p \in [1, \infty] \), \( a \in L^p(\mathbb{R}^{2n}) \). Then

\[
OP_{AW}[a] \in B(L^q(\mathbb{R}^n)),
\]

for \( q \in \left[ \frac{2p}{p+1}, \frac{2p}{p-1} \right] \) is a compact operator.
Proof. In the first case we consider $1 < p < \infty$. Suppose that $a \in \mathcal{F}(\mathbb{R}^2)$. Then $\text{OP}_a : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R}^2)$, so

$$\text{OP}_a : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R}^2).$$

Using proposition (4.1) in [26], $\text{OP}_a : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is compact operator. Since for $1 < p < \infty$, $\mathcal{F}(\mathbb{R}^2)$ is dense in $L^p(\mathbb{R}^2)$ and $\mathcal{F}(\mathbb{R}^2)$ is dense in $L_2^p(\mathbb{R}^n)$ and $L_2^2(\mathbb{R}^n)$, by applying standard density arguments to theorem 2.1 we obtain the compactness of $\text{OP}_a$ for every symbol $a \in L^p(\mathbb{R}^2)$. For case $p = 1$, let $a \in L^1_{\text{comp}}(\mathbb{R}^2)$. We will show that $\text{OP}_a(B)$ is compact operator in $L^1(\mathbb{R}^2)$ where

$$B = \{ u \in L^1(\mathbb{R}^n) \mid \| u \|_{L^1} \leq 1 \}.$$

Let us consider a sequence $\{ u_n \}$ in $B$. \{OP$_a[u_n]$\} is uniformly bounded Since,

$$\| \text{OP}_a[u_n] \|_{L^1} \leq \sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} |a(z)| \|u_n, \Phi_z\| \| \Phi_z(x) \| dz$$

$$\leq \| u_n \|_{L^1} \| a \|_{L^1} \leq \| a \|_{L^1}.$$

Then by the Ascoli-Arzela theorem, for compact set $K \subset \mathbb{R}^n$, \{OP$_a[u_n]$\} is compact in $\mathcal{C}(K)$. Hence, there exists a subsequence $\{ u_{n_k} \}$ such that the sequence \{OP$_a[u_{n_k}]$\} converges pointwise for any $x \in \mathbb{R}^n$ to a function $u(x)$. Therefore, for any sequence $\{ u_{n_k} \}$,

$$| \text{OP}_a[u_{n_k}(x)] | = | \int_{\mathbb{R}^n} a(z)P_zu_{n_k}(x)dz | \leq \int_{\mathbb{R}^n} |a(z)| |P_zu_{n_k}(x)| dz$$

$$= \int_{\mathbb{R}^n} |a(z)| |u_{n_k}(x), \Phi_z(x)\| | \Phi_z(x) \| dz \leq \| u_{n_k} \|_{L^1} \| a \|_{L^1} \leq \| a \|_{L^1}.$$

Applying Lebesgue's dominated converges theorem, $\text{OP}_a[u_{n_k}] \to u$ in $L^1(\mathbb{R}^n)$. Hence, $\text{OP}_a(B)$ is compact in $L^1(\mathbb{R}^n)$ in the case that the symbol $a$ is compactly supported. If support of $a$ is not compact then we use a sequence $\{ a_{n} \} \in L^1_{\text{comp}}(\mathbb{R}^n)$ such that $a_n \to a$ in $L^1(\mathbb{R}^n)$. Then $\text{OP}_a(a_n) \to \text{OP}_a[a]$ in $B(L^1(\mathbb{R}^n))$, so $\text{OP}_a[a]$ is compact in $B(L^1(\mathbb{R}^n))$. In the case $p = \infty$ we obtain that $\text{OP}_a[a]$ is compact by duality argument.

Remark 2.1. Let us note that in theorem 2.1, the case of non-continuity of the quantization map $Q$ means that this map cannot be everywhere defined on $L^p(\mathbb{R}^2)$, i.e. there exists a symbol $\sigma \in L^p(\mathbb{R}^2)$ for which $\text{OP}_a[a]$ is not bounded on $L^p(\mathbb{R}^n)$. This is actually an immediate consequence of the following theorem.

**Theorem 2.3.** The anti-Wick quantization

$$Q : a \in L^p(\mathbb{R}^2) \mapsto \text{OP}_a[a] \in B(L^p(\mathbb{R}^n))$$

is continuous if and only if it is everywhere defined.

Proof. Let $\{ a_n \}_{n \in \mathbb{N}}$ be sequence such that for Gaussian function $\sigma$, $a_n * \sigma \to a * \sigma$ in $L^p(\mathbb{R}^2)$ and there exists $A \in B(L^q(\mathbb{R}^n))$ for which $\text{OP}_a[a_n] \to A$ in $B(L^q(\mathbb{R}^n))$. Then $a_n * \sigma \to a * \sigma$ in $\mathcal{F}(\mathbb{R}^2)$ and so

$$(\text{OP}_a[a_n]u, v) = (\text{OP}_a[a, \sigma]u, v) = (a_n * \sigma, \text{Wig}(v, u)) = \text{Wig}_{v,u}(a_n * \sigma).$$

Using proposition 2.2, $\text{Wig}$ is continuous so $\text{Wig}_{v,u}(a_n * \sigma) \to \text{Wig}_{v,u}(a * \sigma)$. Therefore, $\text{OP}_a[a] = A$. The graph of the anti-Wick quantization map $Q$ is then closed and the assertion is an immediate consequence of the closed graph theorem.

**Theorem 2.4.** Let $a \in L^p(\mathbb{R}^2)$, $p \in [1, \infty)$. Then the operator $\text{OP}_a[a] : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is compact.
Proof. Let $a \in L^p(\mathbb{R}^{2n})$ and consider the set $M = \{ z \in \mathbb{R}^{2n} : |z| \leq D \}$. Then we can approximate $a$ with $a \chi_M$ the characteristic function of the set $M$ that has compact support. Let $\varphi \in C_0^\infty(\mathbb{R}^{2n})$ and consider the associated approximate identity $\varphi_n$, $n \in \mathbb{N}$. Then $a \chi_M * \varphi_n \in C_0^\infty(\mathbb{R}^{2n})$ so that the kernel of the operator $OP_W[a \chi_M * \varphi_n]$ is in $\mathcal{F}(\mathbb{R}^n)$ [12]. By theorem (24.1) in [21], $OP_W[b] = OP_W[b * \sigma]$. If we set $b = a \chi_M * \varphi_n$ it follows that both $OP_W[b * \sigma]$ is regularizing operator and $OP_W[a \chi_M * \varphi]$ on $\mathcal{F}(\mathbb{R}^n)$ and therefore, a compact operator on $L^p(\mathbb{R}^n)$. Since $b$ approximates $a$ in $L^p(\mathbb{R}^{2n})$, $OP_W[a]$ is compact as $D \to \infty$. □

2.1 Hilbert-Schmidt

Let $f$ and $g$ be in $L^p(\mathbb{R}^n)$. Then we define the function $f \otimes g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$(f \otimes g)(x,y) = f(x)g(y),$$

$x,y \in \mathbb{R}^n$. We call $f \otimes g$ the tensor product of $f$ and $g$ [27].

**Proposition 2.6.** [27] Let $f$ and $g$ be in $L^p(\mathbb{R}^n)$. Then $f \otimes g$ belongs to $L^p(\mathbb{R}^{2n})$ and

$$\|f \otimes g\|_{L^p(\mathbb{R}^{2n})} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}.$$  \hspace{1cm} (2.20)

We denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ the Fourier transform with respect to first and second variables respectively $f \in L^p(\mathbb{R}^n)$.

**Proposition 2.7.** [27] Let $f$ and $g$ be in $L^p(\mathbb{R}^n)$. Then

(i) $\mathcal{F}_1(f \otimes g) = (\mathcal{F}f) \otimes g$.

(ii) $\mathcal{F}_2(f \otimes g) = f \otimes (\mathcal{F}g)$.

Now we define the linear operators $K, T : L^p(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ by

$$(Tf)(x,y) = f(x + \frac{y}{2}, x - \frac{y}{2})$$ \hspace{1cm} (2.21)

and

$$(Kf)(x,y) := (T^{-1}\mathcal{F}_2 f)(y,x),$$ \hspace{1cm} (2.22)

for any $f \in L^p(\mathbb{R}^{2n})$.

**Proposition 2.8.** The linear operator $K$ on $L^p(\mathbb{R}^{2n})$ defined by (2.16) has the following properties:

(i) $K = T^{-1}\mathcal{F}_2^{-1}.$

(ii) $Wig(f,g) = K^{-1}(f \otimes g)$, for all $f,g \in L^p(\mathbb{R}^n)$.

**Proof.** For part (i) we show that $K(f \otimes g) = T^{-1}\mathcal{F}_2^{-1}(f \otimes g)$, for all $f,g \in L^p(\mathbb{R}^n)$. But for any $f$ and $g$ in $L^p(\mathbb{R}^n)$ by definition of operator $K$ and proposition (6.2) in [27] and according to this fact that $T : L^p(\mathbb{R}^{2n}) \to L^p(\mathbb{R}^{2n})$ is unitary and

$$(T^{-1}f)(x,y) = f(\frac{x+y}{2}, x - y),$$

for any $x,y \in \mathbb{R}^n$, we obtain that

$$(K(f \otimes g))(x,y) = T^{-1}\mathcal{F}_2(f \otimes g)(y,x) = T^{-1}(f \otimes \mathcal{F}g)(y,x)$$
for all \( x, y \in \mathbb{R}^n \), and
\[
T^{-1} \mathcal{F}_2^{-1}(f \otimes g)(x, y) = T^{-1}(f \otimes \mathcal{F}_2^{-1} g)(x, y) = f\left(\frac{x + y}{2}\right)\left(\mathcal{F}_2 g\right)(x - y),
\]
for any \( x, y \in \mathbb{R}^n \). Therefore, by the two estimates and \((i)\) is true.

\[(ii)\] By definitions of the Wigner transform, tensor product and twisting operator \( T \),
\[
Wig(f, g)(x, \xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ip \cdot \xi} f(x + \frac{p}{2}) g(x - \frac{p}{2}) dp = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ip \cdot \xi} T(f \otimes \bar{g})(x, p) dp = \mathcal{F}_2 T(f \otimes \bar{g})(x, \xi),
\]
for any \( x, \xi \in \mathbb{R}^n \), and for all \( f \) and \( g \) in \( \mathcal{S}(\mathbb{R}^n) \). Therefore, by corollary (3.4) in [27] and above estimate;
\[
Wig(f, g) = \mathcal{F}_2 T(f \otimes \bar{g}),
\]
for each \( f, g \in L^p(\mathbb{R}^n) \). Using part \((i)\) and above relation we obtain
\[
Wig(f, g) = K^{-1}(f \otimes \bar{g}),
\]
where \( K^{-1} = \mathcal{F}_2 T \).

**Theorem 2.5.** \( a \in L^p(\mathbb{R}^{2n}), p \in [1, 2] \). Then \( \text{OP}_{\text{IW}}[a] : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \) with \( p \leq q \leq p' \), is Hilbert-Schmidt operator with kernel
\[
\mathcal{K}(x, y) = (2\pi)^{-\frac{n}{2}} (K(a \ast \sigma))(y, x).
\]

**Proof.** Let \( f \) and \( g \) in \( L^p(\mathbb{R}^n) \). By theorem (24.1) in [21] \( \text{OP}_{\text{IW}}[a] = \text{OP}_{\text{W}}[a \ast \sigma] \). Then by relations (1.2), (2.5) we obtain
\[
(\text{OP}_{\text{W}}[a \ast \sigma] f, g) = (a \ast \sigma, \text{Wig}(f, g)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (a \ast \sigma)(x, \xi) \text{Wig}(f, g)(x, \xi) dx d\xi =
\]
\[
(2\pi)^{-\frac{n}{2}} (\text{Wig}(f, g), (a \ast \sigma)) = (2\pi)^{-\frac{n}{2}} (K^{-1}(f \otimes \bar{g}), (a \ast \sigma)) = (2\pi)^{-\frac{n}{2}} (f \otimes \bar{g}, (K(a \ast \sigma))').
\]
Therefore, by definition of Hilbert-Schmidt operator we obtain
\[
(\text{OP}_{\text{W}}[a \ast \sigma] f, g) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) \bar{g}(y) \mathcal{K}(a \ast \sigma)(y, x) dx dy = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{K}(a \ast \sigma)(y, x) f(x) dx \bar{g}(y) dy = (S_{\mathcal{K}} f, g),
\]
for all \( f, g \in L^p(\mathbb{R}^n) \), where \( \mathcal{K} \) is the function on \( \mathbb{R}^{2n} \) defined by
\[
\mathcal{K}(x, y) = (2\pi)^{-\frac{n}{2}} (K(a \ast \sigma))(y, x),
\]
for any \( x, y \in \mathbb{R}^n \). Hence, \( \text{OP}_{\text{IW}}[a] = \text{OP}_{\text{W}}[a \ast \sigma] \) is Hilbert-Schmidt operator for all \( a \in L^p(\mathbb{R}^{2n}) \).

**Proposition 2.9.** For all \( p > 2 \), there exists \( a \in L^p(\mathbb{R}^{2n}) \) such that \( \text{OP}_{\text{IW}}[a] \) is not Hilbert-Schmidt operator.
**Proof.** Suppose that there exists \( p > 2 \) for which \( \text{OP}_{W}[a] \) is Hilbert-Schmidt operator for each \( a \in L^p(\mathbb{R}^{2n}) \). Then the pseudo-differential operators \( \text{OP}_{W}[b = a \ast \sigma] \) is Hilbert-Schmidt operator for each \( a \in L^p(\mathbb{R}^{2n}) \) and Gaussian function \( \sigma \). Then for all \( a \in L^p(\mathbb{R}^{2n}) \) the convolution \( c = b \ast \sigma \) should belong to \( L^2(\mathbb{R}^{2n}) \), since every symbol \( c \in \Gamma_0^0 \) satisfies
\[
c \in L^2(\mathbb{R}^{2n}) \iff \text{OP}_{W}[c] \in \text{HS}(L^2(\mathbb{R}^{2n})).
\]
Therefore, for \( p > 2 \) the map \( a \in L^p(\mathbb{R}^{2n}) \mapsto c = (a \ast \sigma) \ast \sigma \in L^2(\mathbb{R}^{2n}) \) is linear and bounded; moreover it commutes with translations, so by a property of the convolution transform it would be the null operator [12]. This is a contradiction. \( \square \)

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