JH-Operators and Occasionally Weakly g-Biased Pairs in Fuzzy Symmetric Spaces

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Abstract

We introduce the notions of JH-operators and occasionally weakly g-biased mappings in fuzzy symmetric spaces to prove common fixed point theorems for self-mappings satisfying a generalized mixed contractive condition. We also prove analogous results for two pairs of JH-operators by assuming symmetry only on the set of points of coincidence. These results unify, extend and complement many results existing in the recent literature. We give also an application of our results to product spaces.

Keywords: Fuzzy metric space; Fuzzy symmetric space; JH-operators; Occasionally weakly g-biased pairs.

1 Introduction

The concept of fuzzy metric space was introduced in different ways (see, i.e., [9, 13]) and further using these notions, many authors [3, 4, 10, 11, 15, 17, 18, 22, 24, 25, 26, 27, 28, 30] proved theorems to assure the existence of fixed points. Here, we use the notion of fuzzy metric space established by George and Veeramani [9]. The main reason of our interest in fuzzy metric spaces is its important application in engineering problems, specifically, fuzzy metric spaces are applied in quantum particle physics, in concern with both string and E-infinity theories which were given and studied by El Naschie [5, 6, 7, 8]. More recently, fuzzy metrics have been applied also to color image filtering, improving some filters by replacing some classical metrics [12, 19, 20, 21], and this is a promising field for applied research. Now, to improve results and applications in this direction, it can be of a certain interest the attempt to weaken the requirements on the fuzzy metric space and on the involved mappings.

Motivated by this intent, we present a paper with the following structure: after the preliminary section on fuzzy metric spaces, in section 3, we prove a fixed point theorem for a pair of JH-operators without using the triangle inequality or the symmetry for the metric function. Moreover, we prove an analogous result for occasionally weakly g-biased mappings in fuzzy symmetric spaces. In section 4, we prove several fixed point theorems for two pairs of JH-operators with the assumption of symmetry only on the set of points of coincidence of the mappings. Finally, in section 5, we give an application of our results to product spaces.
2 Preliminaries

In this section, we collect some relevant definitions, results and examples.

Definition 2.1 ([31]). A fuzzy set \( A \) on \( X \) is a function with domain \( X \) and values in \([0, 1]\).

Definition 2.2 ([25]). A continuous t-norm is a binary operation \( T \) on \([0, 1]\) satisfying the following conditions:

(i) \( T \) is commutative and associative;
(ii) \( T(a, 1) = a \) for all \( a \in [0, 1] \);
(iii) \( T(a, b) \leq T(c, d) \) whenever \( a \leq c \) and \( b \leq d \) \((a, b, c, d \in [0, 1])\);
(iv) \( T : [0, 1] \times [0, 1] \to [0, 1] \) is continuous.

A simple and intuitive example of continuous t-norm is the minimum norm given by \( T_M(a, b) = \min\{a, b\} \), with \( a, b \in [0, 1] \).

In a fuzzy setting, the establishment of a Hausdorff topology was given by George and Veeramani in [9] using the following notion of fuzzy metric space:

Definition 2.3. A fuzzy metric space is a triple \((X, M, *)\), where \( X \) is a nonempty set, \( * \) is a continuous t-norm and \( M \) is a fuzzy set on \( X^2 \times (0, +\infty) \) such that, for all \( x, y \in X \) and \( t > 0 \), the following properties hold:

\[
\text{(GV-1) } M(x, y, t) > 0; \\
\text{(GV-2) } M(x, y, t) = 1 \iff x = y; \\
\text{(GV-3) } M(x, y, t) = M(y, x, t); \\
\text{(GV-4) } M(x, y, \cdot) : (0, +\infty) \to (0, 1] \text{ is continuous;} \\
\text{(GV-5) } M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \text{ for all } z \in X \text{ and } s > 0.
\]

We will refer to these spaces as GV-fuzzy metric spaces. For more recent considerations on fuzzy topology, the reader can refer to [29] and the references cited therein.

Remark 2.1. If only \((GV-1)-(GV-3)\) hold, then the triple \((X, M, *)\) is called a fuzzy symmetric space.

Definition 2.4 ([13]). A fuzzy metric \( M \) on \( X \) is said to be stationary if \( M \) does not depend on \( t \), i.e., the function \( M(y) = M(x,y,t) \) is constant.

Example 2.1 ([13]). Let \((X, d)\) be a metric space. Denote \( a * b = ab \) for all \( a, b \in [0, 1] \) and let \( M_d \) be a fuzzy set on \( X^2 \times (0, +\infty) \) defined as

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)}. 
\]

Then \((X, M_d, *)\) is a fuzzy metric space. This fuzzy metric (induced by a metric \( d \)) is called the standard fuzzy metric.

In 1994, Mishra et al. [18] introduced the concept of compatible mappings in fuzzy metric spaces akin to concept of compatible mappings in metric spaces as follows:

Definition 2.5. Let \( f \) and \( g \) be self-mappings of a fuzzy metric space \((X, M, *)\). The pair \( \{f, g\} \) is said to be compatible if \( \lim_{n \to +\infty} M(fx_n, gx_n, t) = 1 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = u \) for some \( u \in X \) and for each \( t > 0 \).

Definition 2.6. Let \( f \) and \( g \) be self-mappings of a fuzzy metric space \((X, M, *)\). A point \( x \in X \) is called a coincidence point of \( f \) and \( g \) iff \( fx = gx \). We call \( w = fx = gx \) a point of coincidence of \( f \) and \( g \).

Definition 2.7. Let \( f \) and \( g \) be self-mappings of a fuzzy metric space \((X, M, *)\). The pair \( \{f, g\} \) is said to be weakly compatible if \( f \) and \( g \) commute at their coincidence points, i.e., \( fgx = gfx \) whenever \( fx = gx \) for \( x \in X \).
It is known that a pair \( \{f, g\} \) of compatible mappings is weakly compatible but, in general, the converse is not true.

In 2008, Al-Thagafi and Shahzad [1] introduced, in the setting of metric spaces, an even weaker condition which they called occasionally weak compatibility.

**Definition 2.8.** Two self-mappings \( f \) and \( g \) of a fuzzy metric space \((X, M, *)\) are occasionally weakly compatible (owc) iff there exists a point \( x \) which is coincidence point of \( f \) and \( g \) at which \( f \) and \( g \) commute.

Let \( X \) be a non-empty set and \( M : X^2 \times (0, +\infty) \to (0, 1] \) be a function satisfying the condition \( M(x, y, t) = 1 \) iff \( x = y \), for all \( x, y \in X \). We recall that for a set \( A \subseteq X \), the diameter of \( A \) can be defined by

\[
\delta_M(A) = \inf \{\min\{M(x, y, t) : x, y \in A\} : t \in \mathbb{R}_{>0}\}.
\]

Let \( C(f, g, t) \) and \( PC(f, g, t) \) denote the set of coincidence points and points of coincidence, respectively, of the pair \( \{f, g\} \) with respect to \( t \).

Now, we introduce the concepts of \( \mathcal{P} \)-operators and \( \mathcal{J} \)-operators in fuzzy metric spaces (see [14, 23] for the same concepts in metric spaces).

**Definition 2.9.** Two self-mappings \( f \) and \( g \) of a fuzzy metric space \((X, M, *)\) are called \( \mathcal{P} \)-operators if there is a point \( u \in X \) such that \( u \in C(f, g, t) \) and \( M(u, f u, t) \geq \delta_M(C(f, g, t)) \).

It is easy to verify that occasionally weakly compatible and weakly compatible mappings which have coincidence points are \( \mathcal{P} \)-operators.

**Definition 2.10.** Two self-mappings \( f \) and \( g \) of a fuzzy metric space \((X, M, *)\) are called \( \mathcal{J} \)-operators if there is a point \( w = fx = gx \) in \( PC(f, g, t) \) such that \( M(w, x, t) \geq \delta_M(PC(f, g, t)) \).

**Example 2.2.** Let \( X = [0, 1] \) and define for all \( x, y \in X \) and \( t > 0 \), the symmetric function \( M : X^2 \times (0, +\infty) \to (0, 1] \) given by

\[
M(x, y, t) = \frac{t}{t + |x - y|}.
\]

Define also \( f, g : X \to X \) by

\[
fx = \begin{cases} 
  x^2 & \text{if } x \in (0, 1] \\
  \frac{2}{3} & \text{if } x = 0 
\end{cases},
\]

\[
gx = \begin{cases} 
  x^2 & \text{if } x \in (0, 1] \\
  \frac{2}{3} & \text{if } x = 0 
\end{cases}.
\]

Now, \( C(f, g, t) = \{0, 1/2\} \) and \( PC(f, g, t) = \{1/4, 3/4\} \), and so \( f \) and \( g \) are \( \mathcal{J} \)-operators. In fact, we have

\[
M(f(1/2), 1/2, t) = M(1/4, 1/2, t) = \frac{t}{t + 1/4} > \frac{t}{t + 1/2} = \delta_M(PC(f, g, t)).
\]

Clearly, \( f \) and \( g \) are not occasionally weakly compatible mappings.

**Definition 2.11.** Two self-mappings \( f \) and \( g \) of a fuzzy metric space \((X, M, *)\) are called weakly \( g \)-biased iff \( M(g fx, gx, t) \geq M(f gx, fx, t) \) for all \( x \in X \) and \( t > 0 \).

**Definition 2.12.** Two self-mappings \( f \) and \( g \) of a fuzzy metric space \((X, M, *)\) are called occasionally weakly \( g \)-biased iff there exists some \( x \in X \) such that \( fx = gx \) and \( M(g fx, gx, t) \geq M(f gx, fx, t) \) for all \( t > 0 \).

It is obvious that occasionally weakly compatible and \( g \)-biased mappings are occasionally weakly \( g \)-biased mappings, but, in general, the converse is not true.
3 Fixed point theorems for a pair of mappings

Our first result is a fixed point theorem for a pair of \(\mathcal{F} \mathcal{H}\)-operators. To prove this theorem, it is not needed to use the triangle inequality and the symmetry of \(M\). Moreover, if \(M\) is not a symmetric function, we consider the following definition.

**Definition 3.1.** Let \(X\) be a non-empty set and \(M : X^2 \times (0, +\infty) \to (0, 1]\) be a function satisfying the condition \(M(x, y, t) = 1\) iff \(x = y\), for all \(x, y \in X\). Two self-mappings \(f\) and \(g\) on \(X\) are called \(\mathcal{F} \mathcal{H}\)-operators iff there is a point \(w = fx = gx\) in \(PC(f, g, t)\) such that

\[
\min\{M(w, x, t), M(x, w, t)\} \geq \delta_{eq}(PC(f, g, t)).
\]

**Theorem 3.1.** Let \(X\) be a non-empty set and \(M : X^2 \times (0, +\infty) \to (0, 1]\) be a function satisfying the condition \(M(x, y, t) = 1\) iff \(x = y\), for all \(x, y \in X\). Suppose that \(f\) and \(g\) are \(\mathcal{F} \mathcal{H}\)-operators on \(X\), and for all \(x, y \in X\) with \(gx \neq fy\) and for each \(t > 0\) we have:

\[
M(fx, fy, t) \geq \phi(M(gx, gy, t)),
\]

where \(p \geq 0\) and \(\phi : [0, 1] \to [0, 1]\) is a nondecreasing function satisfying the condition \(\phi(s) > s\) for each \(s \in [0, 1]\) and \(\phi(1) = 1\). Then \(f\) and \(g\) have a unique common fixed point.

**Proof.** By hypothesis there exists a point \(x \in X\) such that \(w = fx = gx\). Suppose that there exists another point \(y \in X\) such that \(z = fy = gy\). If \(w \neq z\), then by (3.1), we have

\[
M(w, z, t) = M(fx, fy, t) \geq pM(fy, gy, t) \frac{1 - M(fx, gx, t)}{1 - M(fy, gy, t)} + \phi(M(gx, gy, t)),
\]

where \(p \geq 0\) and \(\phi : [0, 1] \to [0, 1]\) is a nondecreasing function satisfying the condition \(\phi(s) > s\) for each \(s \in [0, 1]\) and \(\phi(1) = 1\). Then \(f\) and \(g\) have a unique common fixed point.

Let \(\alpha = \min\{M(w, z, t), M(z, w, t)\} > 0\) and \(\alpha \neq 1\), then we have

\[
M(w, z, t) \geq \phi(\alpha) > \alpha.
\]

Similarly, we get

\[
M(z, w, t) = M(fy, fx, t) \geq pM(fx, gx, t) \frac{1 - M(fy, gy, t)}{1 - M(fx, gx, t)} + \phi(M(gx, fy, t)),
\]

where \(p \geq 0\) and \(\phi : [0, 1] \to [0, 1]\) is a nondecreasing function satisfying the condition \(\phi(s) > s\) for each \(s \in [0, 1]\) and \(\phi(1) = 1\). Then \(f\) and \(g\) have a unique common fixed point. It follows that \(M(z, w, t) \geq \phi(\alpha) > \alpha\). So, we conclude that

\[
\alpha = \min\{M(w, z, t), M(z, w, t)\} \geq \phi(\alpha) > \alpha,
\]

a contradiction. Hence \(w = fx = fy = z\). Therefore, there exists a unique element \(w \in X\) such that \(w = fx = gx\). Thus, \(\delta(\mathcal{F} \mathcal{H}(f, g, t)) = 1\), implies that \(M(x, w, t) = 1\) and hence \(x\) is a unique common fixed point of \(f\) and \(g\). \(\square\)

Following arguments similar to those used before, it is easy to prove the following result.
Theorem 3.2. Theorem 3.1 remains true if the contractive condition (3.1) is replaced by the following condition:

\[ M(fx, fy, t) \geq aM(gx, gy, t) + b \min\{M(fx, gx, t), M(fy, gy, t)\} + c \min\{M(gx, gy, t), M(fx, fx, t), M(gy, fy, t)\}, \]

for all \( x, y \in X \) where \( a, b, \) and \( c \) are real numbers such that \( 0 < a + c < 1 \) and \( b > 1 \).

Example 3.1. Let \( X = [0, 3] \) and define \( M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t} \) for all \( x, y \in X \) and \( t > 0 \). Also define the functions \( f \) and \( g \) by

\[
fx = \begin{cases} 
3x^2 & \text{if } x \in [0, 1], \\
0 & \text{otherwise}
\end{cases}, \quad gx = \begin{cases} 
1 + 2x^2 & \text{if } x \in [0, 1], \\
1 & \text{otherwise}
\end{cases}.
\]

Now, \( C(f, g, t) = \{1\} \) and \( PC(f, g, t) = \{3\} \), and so \( f \) and \( g \) are occasionally weakly \( g \)-biased. In fact, we have

\[
M(gf(1), g(1), t) = \frac{1 + t}{3 + t} > \frac{t}{3 + t} = M(fg(1), f(1), t).
\]

Moreover, we have:

- \( fg(1) = 0 \neq 1 = gf(1) \), and hence, \( f \) and \( g \) are not occasionally weakly compatible;
- \( M(f(1), 1, t) = M(3, 1, t) = \frac{1 + t}{3 + t} < \frac{3 + t}{3 + t} = 1 = \delta(PC(f, g, t)) \), and hence, \( f \) and \( g \) are not \( JH \)-operators;
- \( M(1, f(1), t) = M(1, 3, t) = \frac{1 + t}{3 + t} < \frac{3 + t}{3 + t} = 1 = \delta(C(f, g, t)) \), and hence, \( f \) and \( g \) are not \( P \)-operators.

Motivated by Example 3.1, we give an analogous of Theorem 3.1 considering a pair \( \{f, g\} \) of occasionally weakly \( g \)-biased mappings instead of a pair of \( JH \)-operators.

Theorem 3.3. Let \( X \) be a non-empty set and \( M : X^2 \times (0, +\infty) \rightarrow [0, 1] \) be a symmetric function satisfying the condition \( M(x, y, t) = 1 \) iff \( x = y \), for all \( x, y \in X \). Suppose that \( f \) and \( g \) are occasionally weakly \( g \)-biased, and for all \( x, y \in X \) with \( gx \neq fy \) and for each \( t > 0 \) we have:

\[
M(fx, fy, t) \geq pM(fy, gy, t) \frac{1 - M(fx, fx, t)}{1 - M(fy, fy, t)} + \phi(\min\{M(gx, gy, t), M(gx, gy, t), M(gx, gy, t)\}),
\]

where \( p \geq 0 \) and \( \phi : [0, 1] \rightarrow [0, 1] \) is a nondecreasing function satisfying the condition \( \phi(s) > s \) for each \( s \in [0, 1] \) and \( \phi(1) = 1 \). Then \( f \) and \( g \) have a unique common fixed point.

Proof. By hypothesis there exists a point \( u \in X \) such that \( fu = gu \) and \( M(gfu, gu, t) \geq M(fgu, fu, t) \). We claim that \( fu \) is a unique common fixed point of \( f \) and \( g \). We first assert that \( fu \) is a fixed point of \( f \). If \( ffu \neq fu \), then by using (3.2), we have

\[
M(ffu, fu, t) \geq pM(fu, gu, t) \frac{1 - M(ffu, gu, t)}{1 - M(fu, gu, t)} + \phi(\min\{M(gfu, gu, t), M(fu, gu, t)\}) \geq \phi(\min\{M(ffu, fu, t), M(ffu, fu, t)\}) \geq M(ffu, fu, t),
\]

which is a contradiction. Therefore, \( ffu = fu = fgu \). Moreover, \( M(gfu, gu, t) \geq M(fgu, fu, t) = 1 \) implies \( gfu = gu = fu = ffu \), and so \( fu \) is a common fixed point of \( f \) and \( g \). Uniqueness follows easily from (3.2), then we omit the details.
4 Fixed point theorems for two pairs of JH-operators

In this section, we prove several fixed point theorems for four self-mappings. We prove our first result with the help of an altering distance function, that is a control function used to alter the distance between two points. We recall also that a control function \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) is a continuous, monotonically increasing function that satisfies the condition \( \phi(2s) \leq 2\phi(s) \) and \( \phi(s) = 0 \) if \( s = 0 \) (see [16]).

**Theorem 4.1.** Let \( X \) be a non-empty set and \( M : X^2 \times (0, +\infty) \rightarrow [0, 1] \) be a function such that \( M(x, y) = 1 \) if and only if \( x = y \), for all \( x, y \in X \). Suppose that \( f, g, S \) and \( T \) are self-mappings on \( X \), \( \{ f, S \} \) and \( \{ g, T \} \) are two pairs of JH-operators and

\[
M(z, w, t) = M(w, z, t), \tag{4.3}
\]

whenever \( w \) and \( z \) are points of coincidence of \( \{ f, S \} \) and \( \{ g, T \} \), respectively. Suppose also that, for all \( x, y \in X \) such that \( fx \neq gy \) and for each \( t > 0 \), we have

\[
\phi(M(fx, gy, t)) \geq \psi(M_\phi(x, y, t)), \tag{4.4}
\]

where

\[
M_\phi(x, y, t) = \begin{cases} 
M(fy, Ty, t) & \phi(M(Sx, ty, t)) + \min\{\phi(M(Sx, Ty, t)) \} \\
\phi(M(Sx, fx, t)) & \phi(M(fy, Ty, t)) \\
\phi((M(fx, Ty, t) + M(Sx, gy, t))/2) & \phi(M(fy, Ty, t)) = \phi(M(w, z, t)),
\end{cases}
\tag{4.5}
\]

with \( p \geq 0 \) and \( \psi : [0, +\infty) \rightarrow [0, +\infty) \) is a nondecreasing function such that \( \psi(s) > s \), for each \( s > 0 \). Then \( f, g, S \) and \( T \) have a unique common fixed point.

**Proof.** By hypothesis there exist points \( x, y \in X \) such that \( w = fx = Sx \) and \( z = gy = Ty \). We claim that \( fx = gy \). If not, then by (4.3), (4.4) and (4.5), we have

\[
\phi(M(w, z, t)) = \phi(M(fx, gy, t)) \geq \psi(M_\phi(x, y, t)) = \psi(M_\phi(x, y, t)) = \phi(M(w, z, t)),
\]

that is a contradiction and so we conclude that \( fx = gy \). Moreover, if there is another point \( u \in X \) such that \( fu = Su \), then using (4.3), (4.4) and (4.5), it follows that \( fx = fu \). Therefore, there exists a unique element \( w \in X \) such that \( w = fx = Sx \). Thus, \( \delta(PC(f, g, t)) = 1 \), implies that \( M(x, w, t) = 1 \) and hence \( x = w \) is the unique common fixed point of \( f \) and \( S \). Using similar arguments, we obtain that \( y = z \) is the unique common fixed point of \( g \) and \( T \). It follows easily that \( z = w \) is the unique common fixed point of \( f, g, S \) and \( T \).

**Example 4.1.** Let \( X = \{0, 1, 2, 3\} \) and define \( \phi, \psi : [0, +\infty) \rightarrow [0, +\infty) \) by

\[
\phi(s) = 4s, \; \psi(s) = \begin{cases} \sqrt{s} & \text{if } s \leq 1 \\ s^2 & \text{if } s > 1 \end{cases}
\]

and \( M : X^2 \times (0, +\infty) \rightarrow [0, 1] \) by

\[
M(x, y, t) = \begin{cases} 1 & \text{if } x \neq y \text{ and } 0 < t \leq 2 \\ \frac{|x-y|}{2} & \text{if } x \neq y \text{ and } t > 2 \\ 1 & \text{if } x = y \text{ and } t > 0
\end{cases}
\]

Define \( f, g : X \rightarrow X \) as \( f(0) = f(1) = f(2) = 0, f(3) = 1, g(0) = g(1) = 0, g(2) = 1 \) and \( g(3) = 2 \). Now, \( C(f, g, t) = \{0, 1\} \) and \( PC(f, g, t) = \{0\} \). Clearly, \( f \) and \( g \) are JH-operators. Assuming also \( T = f \) and \( S = g \), condition (4.4) is satisfied with \( p = 0 \). Thus, all the hypotheses of Theorem 4.1 hold and \( 0 \) is the unique common fixed point of \( f \) and \( g \).
Following arguments similar to those used in the proofs of Theorems 3.1 and 4.1, it is easy to prove the following result.

**Theorem 4.2.** Let $X$ be a non-empty set and $M : X^2 \times (0, +\infty) \to (0, 1]$ be a function such that $M(x, y, t) = 1$ iff $x = y$, for all $x, y \in X$. Suppose that $f, g, S$ and $T$ are self-mappings on $X$. $\{f, S\}$ and $\{g, T\}$ are two pairs of $\mathcal{H}$-operators and

$$M(z, w, t) = M(w, z, t), \quad (4.6)$$

whenever $w$ and $z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively. Suppose also that, for all $x, y \in X$ such that $fx \neq gy$ and for each $t > 0$, we have

$$(M(fx, gy, t))^k > p(M(gy, Ty, t))^k 1 - (M(fx, Sx, t))^k$$

whenever $w$ and $z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively. Suppose also that, for all $x, y \in X$ such that $fx \neq gy$ and for each $t > 0$, we have

$$(M(fx, gy, t))^k > p(M(gy, Ty, t))^k 1 - (M(fx, Sx, t))^k$$

whenever $w$ and $z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively. Suppose also that, for all $x, y \in X$ such that $fx \neq gy$ and for each $t > 0$, we have

$$(M(fx, gy, t))^k > p(M(gy, Ty, t))^k 1 - (M(fx, Sx, t))^k$$

whenever $w$ and $z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively. Suppose also that, for all $x, y \in X$ such that $fx \neq gy$ and for each $t > 0$, we have

$$(M(fx, gy, t))^k > p(M(gy, Ty, t))^k 1 - (M(fx, Sx, t))^k$$

whenever $w$ and $z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively. Suppose also that, for all $x, y \in X$ such that $fx \neq gy$ and for each $t > 0$, we have

$$(M(fx, gy, t))^k > p(M(gy, Ty, t))^k 1 - (M(fx, Sx, t))^k$$

where $0 < a < 1, k \geq 1$ and $p \geq 0$. Then $f, g, S$ and $T$ have a unique common fixed point.

To state and prove our next result, we introduce implicit relations, that are an useful tool to cover several contractive conditions rather than a single contractive condition [2].

**Theorem 4.3.** Let $X$ be a non-empty set and $M : X^2 \times (0, +\infty) \to (0, 1]$ be a function such that $M(x, y, t) = 1$ iff $x = y$, for all $x, y \in X$. Suppose that $f, g, S$ and $T$ are self-mappings on $X$. $\{f, S\}$ and $\{g, T\}$ are two pairs of $\mathcal{H}$-operators and

$$M(z, w, t) = M(w, z, t), \quad (4.8)$$

whenever $w$ and $z$ are points of coincidence of $\{f, S\}$ and $\{g, T\}$, respectively. Suppose also that, for all $x, y \in X$, for which $fx \neq gy$ and for each $t > 0$, we have

$$M(fx, gy, t) \geq pM(gy, Ty, t) \frac{1 - M(fx, Sx, t)}{1 - M(fx, gy, t)} + \Psi(M(Sx, Ty, t), M(fx, Sx, t), M(gy, Ty, t), M(fx, Ty, t))$$

where $p \geq 0$ and $\Psi : [0, +\infty)^5 \to [0, +\infty)$ satisfies the following condition:

$$\text{if } u \in [0, +\infty) \text{ is such that } u \geq \Psi(u, 1, 1, a, u), \text{ then } u = 1.$$  

Then $f, g, S$ and $T$ have a unique common fixed point.

**Proof.** By hypothesis there exist points $x, y \in X$ such that $w = fx = 5x$ and $z = gy = Ty$. We claim that $fx = gy$. If not, then by (4.8) and (4.9), we have

$$(M(fx, gy, t)) \geq \Psi(M(fx, gy, t), 1, 1, M(fx, gy, t), M(fx, gy, t)).$$

By (4.10) we have $M(fx, gy, t) = 1$ and so $fx = gy$. Suppose that there is another point $u \in X$ such that $fu = Su$. Then, using (4.8) and (4.9), we get $fu = fx$. Hence $w = fx = 5x$ is the unique point of coincidence of $f$ and $S$. Thus, $\delta(PC(f, g, t)) = 1$, implies that $M(x, w, t) = 1$ and hence $x = w$ is the unique common fixed point of $f$ and $S$. By repeated use of (4.8) and (4.9), it follows that $w$ is the unique common fixed point of $f, g, S$ and $T$. \qed

**Example 4.2.** Define $\Psi : [0, +\infty)^5 \to [0, +\infty)$ as

$$\Psi(t_1, t_2, t_3, t_4, t_5) = \max\{t_1t_4, t_2t_3, t_5\}.$$  

Clearly, $\Psi$ satisfies condition (4.10). In fact, $u \geq \max\{u^2, 1, u\} \Leftrightarrow u = 1$.

**Remark 4.1.** The theorems in this section unify, extend and complement many results in the literature. See, for example, Theorems 1 and 2 of [2], Theorem 3.1 of [26] and Theorem 4.5 of [24].
5 Application to product spaces

In this section we give an application of our results to the product space \( X \times X \). To this aim, we have to use the following corollary, that is an immediate consequence of Theorem 4.3, assuming \( f = T \) and \( g = S \).

**Corollary 5.1.** Let \( X \) be a non-empty set and \( M : X^2 \times (0, +\infty) \to (0, 1) \) be a function such that \( M(x, y, t) = 1 \) iff \( x = y \), for all \( x, y \in X \). Suppose that \( f \) and \( g \) are self-mappings on \( X \), \( \{f, g\} \) is a pair of \( J \mathcal{H} \)-operators and \( M(z, w, t) = M(w, z, t) \), whenever \( w \) and \( z \) are points of coincidence of \( \{f, g\} \). Suppose also that, for all \( x, y \in X \) such that \( f x \neq gy \) and for each \( t > 0 \), we have

\[
M(f x, gy, t) \geq p M(gy, f x, t) + \frac{1 - M(f x, gy, t)}{1 - M(f x, gy, t)} + \Psi(M(gy, f x, t), M(f x, gy, t), M(gy, f x, t), M(f x, gy, t)),
\]

where \( p \geq 0 \) and \( \Psi : [0, +\infty)^5 \to [0, +\infty) \) satisfies condition (4.10). Then \( f \) and \( g \) have a unique common fixed point.

Now, we are ready to state and prove our last theorem.

**Theorem 5.1.** Let \( X \) be a non-empty set and \( M : X^2 \times (0, +\infty) \to (0, 1) \) be a function such that \( M(x, y, t) = 1 \) iff \( x = y \), for all \( x, y \in X \). Suppose that \( F \) and \( G \) are two mappings on the product \( X \times X \) with values in \( X \), \( \{F(\cdot, y), G(\cdot, y)\} \) is a pair of \( J \mathcal{H} \)-operators for each \( y \in X \) and \( \{F(z(\cdot), y), G(z(\cdot), y)\} \) is a pair of \( J \mathcal{H} \)-operators for each \( z : X \to X \). Suppose also that, for all \( x, y, u, v \in X \) such that \( F(x, y) \neq G(u, v) \) and for each \( t > 0 \), we have

\[
M(F(x, y), G(u, v), t) \geq p M(G(u, v), F(u, v), t) + \frac{1 - M(F(x, y), G(u, v), t)}{1 - M(F(x, y), G(u, v), t)} + \Psi(M(G(u, v), F(u, v), t), M(F(x, y), G(u, v), t), M(G(u, v), F(u, v), t), M(F(x, y), G(u, v), t)),
\]

where \( p \geq 0 \) and \( \Psi : [0, +\infty)^5 \to [0, +\infty) \) satisfies condition (4.10). Then there exists a unique point \( w \in X \), such that \( F(w, w) = G(w, w) = w \).

**Proof.** Fix \( y = v \in X \) and let \( f, g : X \to X \) be such that \( F(x, y) = f x \) and \( G(u, y) = gu \), for all \( x, u \in X \). Then, condition (5.12) reduces to condition (5.11) and so, by Corollary 5.1, \( \{f, g\} \) has a unique common fixed point \( z(y) \), that is \( f(z(y)) = z(y) = g(z(y)) \). Now, we can apply Corollary 5.1 to the self-mappings \( F(z(\cdot), y) \) and \( G(z(\cdot), y) \) on \( X \) and so we deduce that there exists a unique point \( w \) such that \( F(z(w), w) = G(z(w), w) = z(w) = w \). This completes the proof.

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