

The Existence and the stability of solutions for equilibrium problems with lower and upper bounds

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Abstract

In this paper, we study a class of equilibrium problems with lower and upper bounds. We obtain some existence results of solutions for equilibrium problems with lower and upper bounds by employing some classical fixed-point theorems. We investigate the stability of the solution sets for the problems, and establish sufficient conditions for the upper semicontinuity, lower semicontinuity and continuity of the solution set mapping $S : \Lambda_1 \times \Lambda_2 \rightarrow 2^X$ in a Hausdorff topological vector space, in the case where a set K and a mapping f are perturbed respectively by parameters λ and μ .

Keywords: Equilibrium problem, Upper semicontinuity, Lower semicontinuity, Solution set mapping.

1 Introduction

Let X be a Hausdorff topological vector space, K a nonempty subset of X , and let $f : K \times K \rightarrow \mathbb{R}$ be a function. The equilibrium problem is to find $\bar{x} \in K$, such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in K \quad (1.1)$$

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Equilibrium problems include variational inequality problems as well as fixed point problems, complementarity problems, optimization, saddle point problems and Nash equilibrium problems as special cases. Equilibrium problems provide us with a systematic framework to study a wide class of problems arising in finance economics, optimization and operation research et al., which motivate the extensive concern. In recent years, equilibrium problems have been deeply and thoroughly researched. See, for example, [1]-[9].

In 1999, Isac, Sehgal and Singh [10] raised the following open problem, which is the equilibrium problem with lower and upper bounds.

Let X be a locally convex topological vector space, K a nonempty closed subset of X , $f : K \times K \rightarrow \mathbb{R}$ a functional, and let α, β be real numbers with $\alpha \leq \beta$. The problem is to find $\bar{x} \in K$, such that

$$\alpha \leq f(\bar{x}, y) \leq \beta, \quad \forall y \in K \quad (1.2)$$

As to the equilibrium problem with lower and upper bounds, the existence, in particular, the stability of solution sets, are of considerable interest at present. For the existence of solutions of such problems, Li [11] gave some answers to the open problem (1.2) by introducing and using the concept of extremal subsets. Chadli et al. [12] derived some results by employing a fixed point theorem and the FKKM theorem. Zhang [13] obtained some existence theorems by using the concept of (α, β) -convexity. Ding [14] also drew some conclusions in this aspect. As to the stability of solution sets for equilibrium problems, Bianchi and Pini [15] considered equilibrium problems in vector metric spaces, where the functional f and the set K are perturbed by the parameters ε and η respectively. Li et al. [16] studied the stability of solutions of generalized vector quasi-variational inequality problems. Recently, Anh and Khanh [17] as well as Huang et al. [18] established sufficient conditions for the solution set of parametric multi-valued vector quasi-equilibrium problems with fixed cone to be semicontinuous.

However, the stability of solution sets for equilibrium problems with lower and upper bounds have been rarely studied up to now.

In this paper, we obtain some existence results of equilibrium problems with lower and upper bounds by employing some classical fixed point theorems. We investigate the stability of the solution sets for the problems in a Hausdorff topological vector space, in the case where a set K and a mapping f are perturbed respectively by parameters λ and μ . Finally, we study the stability of the solution sets in a vector metric space, in the particular case where K is fixed, and f is perturbed by a parameter ε .

2 Preliminaries

Let X, Y denote topological vector spaces. For a nonempty subset K of X , let $co(K)$ denote the convex hull of K , and $\langle K \rangle$ denote the family of all nonempty finite subsets of K . 2^X denotes the family of all nonempty subsets of X , and \mathbb{R} denotes the set of real numbers.

Definition 2.1. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, the subset $\{(x, y) \in X \times Y : y \in F(x)\}$ of $X \times Y$ is called the graph of F , denoted by $graph(F)$.

Definition 2.2. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, $F^{-1} : Y \rightarrow X$ is defined as follows: $x \in F^{-1}(y)$ if and only if $y \in F(x)$, F^{-1} is called the inverse mapping of F .

Definition 2.3. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, F is called closed, if the graph of F is a closed subset of $X \times Y$.

Definition 2.4. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, F is called upper semicontinuous at x_0 , if for every open set W containing $F(x_0)$, there exists a neighborhood U of x_0 , such that for every $x \in U$, we have $F(x) \subset W$. F is called upper semicontinuous in X , if F is upper semicontinuous at every point of X .

Definition 2.5. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, F is called lower semicontinuous at x_0 , if for every open set V with $V \cap F(x_0) \neq \emptyset$, there exists a neighborhood U of x_0 , such that for every $x \in U$, we have $F(x) \cap V \neq \emptyset$, F is called lower semicontinuous in X , if F is lower semicontinuous at every point of X .

Definition 2.6. [13] Let K be a nonempty convex subset of X , $f : K \times K \rightarrow \mathbb{R}$ a functional, and let $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$, $f(x, y)$ is called (α, β) -convex related to y , if for every finite set $\{y_1, \dots, y_n\} \subset K$, and for every $y_0 \in \text{co}\{y_1, \dots, y_n\}$, there exists an element $i \in \{1, 2, \dots, n\}$, such that $\alpha \leq f(y_0, y_i) \leq \beta$.

Definition 2.7. [16] Let $f : K \times K \rightarrow \mathbb{R}$ be a functional, f is called pseudomonotone, if $f(x, y) \geq 0$ implies $f(y, x) \leq 0$.

Definition 2.8. Let $f : X \rightarrow \mathbb{R}$ be a functional, f is called quasiconcave (qcv), if for every $x, y \in X$, $0 \leq t \leq 1$, we have $f((1-t)x + ty) \geq \min\{f(x), f(y)\}$. f is called quasiconvex (qcv), if $-f$ is quasiconcave.

The following lemma and the concept of upper semicontinuity, see [19].

Lemma 2.1. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, if Y is a compact space, and F is closed, then F is upper semicontinuous in X .

Lemma 2.2. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, and F is compact-valued, then F is upper semicontinuous at $x_0 \in X$ if and only if for every $\{x_n\} \subset X$, $x_n \rightarrow x_0$, and for every $y_n \in F(x_n)$, there exists $y_0 \in F(x_0)$, $\{y_{n_i}\} \subset \{y_n\}$, such that $y_{n_i} \rightarrow y_0$.

Lemma 2.3. Let $F : X \rightarrow 2^Y$ be a set-valued mapping, then F is lower semicontinuous at $x_0 \in X$ if and only if for every $\{x_n\} \subset X$, $x_n \rightarrow x_0$, and for every $y_0 \in F(x_0)$, there exists $y_n \in F(x_n)$, such that $y_n \rightarrow y_0$.

Lemma 2.4. Let $f : X \rightarrow \mathbb{R}$ be a functional, f is lower semicontinuous if and only if for every $c \in \mathbb{R}$, $\{x \in X : f(x) \leq c\}$ is a closed subset of X . f is upper semicontinuous if and only if $-f$ is lower semicontinuous.

Lemma 2.5. Let $f : K \times K \rightarrow \mathbb{R}$ be a functional, if for every $y \in Y$, $f(\cdot, y)$ is upper semicontinuous, then $u(x) = \inf_{y \in Y} f(x, y)$ is upper semicontinuous in X . If for every $y \in Y$, $f(\cdot, y)$ is lower semicontinuous, then $u(x) = \sup_{y \in Y} f(x, y)$ is lower semicontinuous in X .

Lemma 2.6. Let $F, G : X \rightarrow 2^Y$ be set-valued mappings. If the following conditions hold:

- (i) For every $x \in X$, $F(x) \cap G(x) \neq \emptyset$;
- (ii) F is upper semicontinuous at x_0 ;
- (iii) $F(x_0)$ is compact;
- (iv) The graph of G is closed,

Then the set-valued mapping $F \cap G : x \rightarrow F(x) \cap G(x)$ is upper semicontinuous at x_0 .

Lemma 2.7. *Let X be a Hausdorff topological vector space, K a nonempty subset of X , D a nonempty compact convex subset of K , and let $F : K \rightarrow 2^D$ be a nonempty set-valued mapping, if for every $y \in K$, $F^{-1}(y)$ is open, then there exists $\hat{x} \in K$, such that $\hat{x} \in \text{co}(F(\hat{x}))$.*

Lemma 2.8. *Let X be a Hausdorff topological vector space, K a nonempty compact convex subset of X , and let $F : K \rightarrow 2^K$ be a set-valued mapping. If the following conditions hold:*

- (i) *For every $x \in K$, $F(x)$ is convex;*
- (ii) *For every $y \in K$, $F^{-1}(y)$ contains an open subset O_y of K (O_y may be \emptyset);*
- (iii) *$K = \cup_{y \in K} O_y$,*

Then there exists $\hat{x} \in K$, such that $\hat{x} \in F(\hat{x})$.

Lemma 2.9. [22] *Let X be a Hausdorff topological vector space, C, K nonempty convex subsets of X , and let $F : C \rightarrow 2^X$ be a set-valued mapping. If the following conditions hold:*

- (i) *$C \subset K \subset F(C)$;*
- (ii) *$F(C)$ is a compact subset of X ;*
- (iii) *For every $x \in C$, $F(x)$ is open,*

Then there exists $\hat{x} \in C$, such that $\hat{x} \in \text{co}(F^{-1}(\hat{x}))$.

Lemma 2.10. [14] *Let X be a Hausdorff topological vector space, K a nonempty compact convex subset of X , $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f : K \times K \rightarrow \mathbb{R}$ be a functional. If the following conditions hold:*

- (i) *For every $y \in K$, $f(x, y)$ is continuous related to x ;*
- (ii) *For every $x \in K$, $f(x, y)$ is (α, β) -convex related to y ,*

Then there exists $\bar{x} \in K$, such that $\alpha \leq f(\bar{x}, y) \leq \beta$ for every $y \in K$.

3 The existence of solutions

In this section, the solution existence of Problem (1.2) will be studied by employing some fixed point theorems.

Throughout this section, all topological spaces are assumed to be Hausdorff.

Theorem 3.1. *Let K be a nonempty subset of X , $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f, g_1, g_2 : K \times K \rightarrow \mathbb{R}$ be functionals. Assume that the following conditions hold:*

- (i) *For every $x \in K$, $g_1(x, x) \geq \alpha$ and $g_2(x, x) \leq \beta$;*
- (ii) *For each $x \in K$, $\text{co}(\{y \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\})$
 $\subset \{y \in K : g_1(x, y) < \alpha \text{ or } g_2(x, y) > \beta\}$;*
- (iii) *There exists a compact convex subset D of K , such that for every $x \in K$,
 $\{y \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\} \subset D$;*
- (iv) *For every $y \in K$, $\{x \in K : \alpha \leq f(x, y) \leq \beta\}$ is a closed subset of K ,*

Then there exists $\hat{x} \in K$, such that $\alpha \leq f(\hat{x}, y) \leq \beta$ for every $y \in K$.

Proposition 3.1. *Let the set-valued mappings $F, G : K \rightarrow 2^K$ be defined by*

$$F(x) = K \setminus \{y \in K : \alpha \leq f(x, y) \leq \beta\}, \quad \forall x \in K$$

$$G(x) = \{y \in K : g_1(x, y) < \alpha\} \cup \{y \in K : g_2(x, y) > \beta\}, \quad \forall x \in K$$

Suppose to the contrary that the result of the theorem is not true. Equivalently, $F(x)$ is nonempty for every $x \in K$. Condition (ii) implies that $\text{co}(F(x)) \subset G(x)$ for any $x \in K$.

By Condition (iii), there exists a compact convex set $D \subset K$, such that $F(x) \subset D$ for each $x \in K$. Condition (iv) indicates that $F^{-1}(y) = \{x \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ is open in K for every $y \in K$. Then from Lemma 2.10, it follows that there exists $\hat{x} \in K$, such that $\hat{x} \in \text{co}(F(\hat{x}))$. Since $\text{co}(F(x)) \subset G(x)$ for every $x \in K$, so $\hat{x} \in G(\hat{x})$ which implies that $g_1(\hat{x}, \hat{x}) < \alpha$ or $g_2(\hat{x}, \hat{x}) > \beta$. This contradicts Condition (i). Thus, there exists $\hat{x} \in K$, such that $\alpha \leq f(\hat{x}, y) \leq \beta$, for every $y \in K$.

Corollary 3.1. Let K be a nonempty compact convex subset of X , $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f : K \times K \rightarrow \mathbb{R}$ be a functional. Assume that the following conditions hold:

- (i) For every $x \in K$, $\alpha \leq f(x, x) \leq \beta$;
- (ii) For each $x \in K$, $\{y \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ is convex;
- (iii) For every $y \in K$, $f(x, y)$ is continuous related to x ,

Then there exists $\hat{x} \in K$, such that $\alpha \leq f(\hat{x}, y) \leq \beta$ for every $y \in K$.

Proposition 3.2. Let $g_1 = g_2 = f$, then Condition (i) and (ii) are equivalent to Condition (i) and (ii) of Theorem 3.1. Clearly, Condition (iii) of Theorem 3.1 is satisfied, since K is a nonempty compact convex subset of X . Since for every $y \in K$, $f(x, y)$ is continuous related to x , Condition (iv) of Theorem 3.1 is also satisfied. Hence, by Theorem 3.1, the proof is completed.

Theorem 3.2. Let K be a nonempty compact convex subset of X , $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f, g_1, g_2 : K \times K \rightarrow \mathbb{R}$ be functionals. Assume that the following conditions hold:

- (i) For each $x \in K$, $g_1(x, x) \geq \alpha$ and $g_2(x, x) \leq \beta$;
- (ii) For every $x \in K$, $\{y \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\} \subset \{y \in K : g_1(x, y) < \alpha \text{ or } g_2(x, y) > \beta\}$;
- (iii) For every $x \in K$, $\{y \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ is convex;
- (iv) For each $y \in K$, there exists an open subset O_y of K , such that $O_y \subset \{x \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$;
- (v) $K = \cup_{y \in K} O_y$,

Then there exists $\hat{x} \in K$, such that $\alpha \leq f(\hat{x}, y) \leq \beta$ for every $y \in K$.

Proposition 3.3. Let the set-valued mappings $F, G : K \rightarrow 2^K$ be defined by

$$F(x) = K \setminus \{y \in K : \alpha \leq f(x, y) \leq \beta\}, \quad \forall x \in K$$

$$G(x) = \{y \in K : g_1(x, y) < \alpha\} \cup \{y \in K : g_2(x, y) > \beta\}, \quad \forall x \in K$$

Suppose to the contrary that the result of the theorem is not true. Equivalently, $F(x)$ is nonempty for every $x \in K$. Condition (ii) implies that $F(x) \subset G(x)$ for every $x \in K$. By Condition (iii), $F(x)$ is convex for each $x \in K$. From Condition (iv), for every $y \in K$, there exists an open set $O_y \subset K$, such that $O_y \subset F^{-1}(y)$. Since $K = \cup_{y \in K} O_y$, Lemma 2.8 indicates that there exists $\hat{x} \in K$, such that $\hat{x} \in F(\hat{x})$, and since $F(x) \subset G(x)$ for all $x \in K$, so $\hat{x} \in G(\hat{x})$ which shows that $g_1(\hat{x}, \hat{x}) < \alpha$ or $g_2(\hat{x}, \hat{x}) > \beta$. This contradicts Condition (i). Thus, there exists $\hat{x} \in K$, such that $\alpha \leq f(\hat{x}, y) \leq \beta$ for every $y \in K$.

Remark 3.1. In Theorem 3.2, if $g_1 = g_2 = f$, then Condition (i) implies that $\alpha \leq f(x, x) \leq \beta$ for all $x \in K$ and Condition (ii) can be omitted.

Theorem 3.3. Let K be a nonempty convex subset of X , $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f, g_1, g_2 : K \times K \rightarrow \mathbb{R}$ be functionals. Assume that the following conditions hold:

- (i) For every $x \in K$, $g_1(x, x) \geq \alpha$ and $g_2(x, x) \leq \beta$;
- (ii) For each $y \in K$, $\text{co}(\{x \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\})$

- $\subset \{x \in K : g_1(x, y) < \alpha \text{ or } g_2(x, y) > \beta\};$
- (iii) For every $y \in K$, $\{x \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ is nonempty;
 - (iv) $\cup_{x \in K} \{y \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ is compact in X ;
 - (v) For each $x \in K$, $\{y \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ is open in X ,
- Then there exists $\hat{x} \in K$, such that $\alpha \leq f(\hat{x}, y) \leq \beta$ for every $y \in K$.

Proposition 3.4. Let the set-valued mappings $F, G : K \rightarrow 2^K$ be defined by

$$F(x) = K \setminus \{y \in K : \alpha \leq f(x, y) \leq \beta\}, \quad \forall x \in K$$

$$G(x) = \{y \in K : g_1(x, y) < \alpha\} \cup \{y \in K : g_2(x, y) > \beta\}, \quad \forall x \in K$$

Suppose to the contrary that the result of the theorem is not true. Equivalently, $F(x)$ is nonempty for every $x \in K$. By Condition (ii), $co(F^{-1}(y)) \subset G^{-1}(y)$ for every $y \in K$. Condition (iii) indicates that $F^{-1}(y)$ is nonempty for each $y \in K$. From Condition (iv), $F(K)$ is compact in X . Condition (v) implies that $F(x)$ is an open subset of X for all $x \in K$. Then by Lemma 2.9, there exists $\hat{x} \in K$, such that $\hat{x} \in co(F^{-1}(\hat{x}))$, and since $co(F^{-1}(y)) \subset G^{-1}(y)$ for all $y \in K$, so $\hat{x} \in G^{-1}(\hat{x})$ which shows that $g_1(\hat{x}, \hat{x}) < \alpha$ or $g_2(\hat{x}, \hat{x}) > \beta$. This contradicts Condition (i). Therefore, there exists $\hat{x} \in K$, such that $\alpha \leq f(\hat{x}, y) \leq \beta$ for every $y \in K$.

Remark 3.2. In Theorem 3.3, if $g_1 = g_2 = f$, then Condition (i) implies that $\alpha \leq f(x, x) \leq \beta$ for all $x \in K$, and Condition (ii) indicates that $\{x \in K : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ is convex.

4 The Stability of solution sets

Let X, Λ_1, Λ_2 be topological vector spaces, $K : \Lambda_1 \rightarrow 2^X$ a nonempty set-valued mapping, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f : X \times X \times \Lambda_2 \rightarrow \mathbb{R}$ be a functional. Given $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, we consider:

Find $\bar{x} \in K(\lambda)$, such that $\alpha \leq f(\bar{x}, y, \mu) \leq \beta, \quad \forall y \in K(\lambda)$.

For every given $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, we denote by $S(\lambda, \mu)$ the solution set of this problem.

In this section, We will discuss the stability of the solution sets for Problem (1.2) in a Hausdorff topological vector space, in the case where a set K and a mapping f are perturbed respectively by parameters λ and μ , that is the semicontinuity and the continuity of the solution mapping $S : \Lambda_1 \times \Lambda_2 \rightarrow 2^X$. Then, we study the upper semicontinuity of the solution set mapping in a vector metric space, in the particular case where K is fixed, and f is perturbed by a parameter ε .

In the following, all topological spaces are assumed to be Hausdorff.

First, we discuss the upper semicontinuity of the solution set mapping Problem (1.2).

Theorem 4.1. Let $K : \Lambda_1 \rightarrow 2^X$ be a nonempty set-valued mapping, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$ and let $f : X \times X \times \Lambda_2 \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) $K(\cdot)$ is continuous in Λ_1 , and $K(x)$ is nonempty compactly convex for every $x \in \Lambda_1$;
- (ii) $f(\cdot, \cdot, \cdot)$ is continuous in $X \times X \times \Lambda_2$;
- (iii) For each $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, and for each $x \in K(\lambda)$, $f(x, y, \mu)$ is (α, β) -convex related to y ,

Then (a) For every $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, $S(\lambda, \mu) \neq \emptyset$;

(b) The solution set mapping $S : \Lambda_1 \times \Lambda_2 \rightarrow 2^X$ is upper semicontinuous in $\Lambda_1 \times \Lambda_2$.

Proposition 4.1. (a) For every $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, $K(\lambda)$ and $f(\cdot, \cdot, \mu)$ satisfy the conditions of Lemma 2.10, then by Lemma 2.10, it follows that $S(\lambda, \mu) \neq \emptyset$.

(b) For every $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, $S(\lambda, \mu) = \{\bar{x} \in K(\lambda) : \alpha \leq f(\bar{x}, y, \mu) \leq \beta, \forall y \in K(\lambda)\}$. By Condition (ii), $S(\lambda, \mu)$ is a closed subset of $K(\lambda)$, and since $K(\lambda)$ is compact, so $S(\lambda, \mu)$ is compact.

Next, we prove S is upper semicontinuous. Let $\{(\lambda_n, \mu_n)\} \subset \Lambda_1 \times \Lambda_2$, $(\lambda_n, \mu_n) \rightarrow (\lambda, \mu)$, $x_n \in S(\lambda_n, \mu_n)$. By Lemma 2.2, we only need to prove that there exists $x \in S(\lambda, \mu)$, and $\{x_{n_i}\} \subset \{x_n\}$, such that $x_{n_i} \rightarrow x$. Since $x_n \in K(\lambda_n)$ and K is upper semicontinuous, then Lemma 2.2 implies that there exists $x \in K(\lambda)$, and $\{x_{n_i}\} \subset \{x_n\}$, such that $x_{n_i} \rightarrow x$. Next, we prove $x \in S(\lambda, \mu)$. Suppose $x \notin S(\lambda, \mu)$, then there exists $y \in K(\lambda)$, such that

$$f(x, y, \mu) < \alpha \quad \text{or} \quad f(x, y, \mu) > \beta \tag{4.3}$$

By the lower semicontinuity of K and Lemma 2.3, for the above y , we know that there exists $\{y_{n_i}\}$, such that $y_{n_i} \in K(\lambda_{n_i})$, and $y_{n_i} \rightarrow y$. Since $x_{n_i} \in S(\lambda_{n_i}, \mu_{n_i})$, then $\alpha \leq f(x_{n_i}, y_{n_i}, \mu_{n_i}) \leq \beta$, and from the continuity of f , we have $\alpha \leq f(x, y, \mu) \leq \beta$. This contradicts (4.3). The proof is completed.

When f is fixed, K is perturbed by a parameter ε , we have:

Corollary 4.1. Let Λ be a Hausdorff topological vector space, $K : \Lambda \rightarrow 2^X$ a nonempty set-valued mapping, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f : X \times X \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) $K(\cdot)$ is continuous in Λ , and $K(x)$ is nonempty compactly convex for every $x \in \Lambda$;
- (ii) $f(\cdot, \cdot)$ is continuous in $X \times X$;
- (iii) For every $\varepsilon \in \Lambda$, and for every $x \in K(\varepsilon)$, $f(x, y)$ is (α, β) -convex related to y ,

Then (a) For every $\varepsilon \in \Lambda$, $S(\varepsilon) \neq \emptyset$;

- (b) The solution set mapping $S : \Lambda \rightarrow 2^X$ is upper semicontinuous in Λ .

When K is fixed, and f is perturbed by a parameter ε , we have:

Corollary 4.2. Let Λ be a Hausdorff topological vector space, K a nonempty compact convex subset of X , $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f : X \times X \times \Lambda \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) $f(\cdot, \cdot, \cdot)$ is continuous in $X \times X \times \Lambda$;
- (ii) For every $\varepsilon \in \Lambda$, and for every $x \in K$, $f(x, y, \varepsilon)$ is (α, β) -convex related to y ;

Then (a) For every $\varepsilon \in \Lambda$, $S(\varepsilon) \neq \emptyset$;

- (b) The solution set mapping $S : \Lambda \rightarrow 2^X$ is continuous in Λ .

Theorem 4.2. Let $K : \Lambda_1 \rightarrow 2^X$ be a nonempty set-valued mapping, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$ and let $f : X \times X \times \Lambda_2 \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) $K(\cdot)$ is continuous in Λ_1 , and $K(x)$ is nonempty compactly convex for every $x \in \Lambda_1$;
- (ii) For each $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, and for each $x \in K(\lambda)$, $\alpha \leq f(x, x, \mu) \leq \beta$;
- (iii) For every $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, and for every $x \in K(\lambda)$,
 $\{y \in K(\lambda) : f(x, y, \mu) < \alpha \text{ or } f(x, y, \mu) > \beta\}$ is convex;
- (iv) $f(\cdot, \cdot, \cdot)$ is continuous in $X \times X \times \Lambda_2$,

Then (a) For every $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, $S(\lambda, \mu) \neq \emptyset$;

- (b) The solution set mapping $S : \Lambda_1 \times \Lambda_2 \rightarrow 2^X$ is upper semicontinuous in $\Lambda_1 \times \Lambda_2$.

Proposition 4.2. For every $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, $K(\lambda)$ and $f(\cdot, \cdot, \mu)$ satisfy the conditions of Corollary 3.1, then by Corollary 3.1, we have $S(\lambda, \mu) \neq \emptyset$.

(b) The proof is similar to the proof of Theorem 4.1.

Similar to Theorem 4.1, we have the following two corollaries:

Corollary 4.3. Let Λ be a Hausdorff topological vector space, $K : \Lambda \rightarrow 2^X$ a nonempty set-valued mapping, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f : X \times X \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) $K(\cdot)$ is continuous in Λ , and $K(x)$ is nonempty compactly convex for every $x \in \Lambda$;
- (ii) For every $\varepsilon \in \Lambda$, and for every $x \in K(\varepsilon)$, $\alpha \leq f(x, x) \leq \beta$;
- (iii) For each $\varepsilon \in \Lambda$, and for each $x \in K(\varepsilon)$,
 $\{y \in K(\varepsilon) : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ is convex;
- (iv) $f(\cdot, \cdot)$ is continuous in $X \times X$,

Then (a) For every $\varepsilon \in \Lambda$, $S(\varepsilon) \neq \emptyset$;

(b) The solution set mapping $S : \Lambda \rightarrow 2^X$ is upper semicontinuous in Λ .

Corollary 4.4. Let Λ be a Hausdorff topological vector space, K a nonempty compact convex subset of X , $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f : X \times X \times \Lambda \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) For every $\varepsilon \in \Lambda$, and for every $x \in K$, $\alpha \leq f(x, x, \varepsilon) \leq \beta$;
- (ii) For each $\varepsilon \in \Lambda$, and for each $x \in K$,
 $\{y \in K : f(x, y, \varepsilon) < \alpha \text{ or } f(x, y, \varepsilon) > \beta\}$ is convex;
- (iii) $f(\cdot, \cdot, \cdot)$ is continuous in $X \times X \times \Lambda$,

Then (a) For every $\varepsilon \in \Lambda$, $S(\varepsilon) \neq \emptyset$;

(b) The solution set mapping $S : \Lambda \rightarrow 2^X$ is upper semicontinuous in Λ .

Next, the lower semicontinuity and continuity of the solution set mapping $S(\cdot, \cdot)$ at $(\lambda_0, \mu_0) \in \Lambda_1 \times \Lambda_2$ will be studied. Suppose that $S(\cdot, \cdot)$ is nonempty-valued in a neighborhood of (λ_0, μ_0) , that is $S(\lambda, \mu) \neq \emptyset$ for every $\lambda \in U(\lambda_0)$ and for every $\mu \in V(\mu_0)$.

Theorem 4.3. Let $K : \Lambda_1 \rightarrow 2^X$ be a nonempty set-valued mapping, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$ and let $f : X \times X \times \Lambda_2 \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) $K(\cdot)$ is lower semicontinuous at λ_0 ;
- (ii) For every $x_0 \in S(\lambda_0, \mu_0)$, and for every neighborhood $W(x_0)$ of x_0 ,
 $K(\lambda) \cap W(x_0) \neq \emptyset$ implies $S(\lambda, \mu) \cap W(x_0) \neq \emptyset$ for every $\mu \in V(\mu_0)$,

Then $S(\cdot, \cdot)$ is lower semicontinuous at (λ_0, μ_0) .

Proposition 4.3. For a given $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, we have

$$S(\lambda, \mu) = \{\bar{x} \in K(\lambda) : \alpha \leq f(\bar{x}, y, \mu) \leq \beta, \forall y \in K(\lambda)\}.$$

Let $\{(\lambda_n, \mu_n)\} \subset \Lambda_1 \times \Lambda_2$ satisfying $(\lambda_n, \mu_n) \rightarrow (\lambda_0, \mu_0)$, for any $x_0 \in S(\lambda_0, \mu_0)$, by Lemma 2.3, we only need to prove: there exists $x_n \in S(\lambda_n, \mu_n)$, such that $x_n \rightarrow x_0$.

In fact, since $x_0 \in K(\lambda_0)$, and $K(\cdot)$ is lower semicontinuous at λ_0 , then, by definition 2.5, for every neighborhood $W(x_0)$ of x_0 with $W(x_0) \cap K(\lambda_0) \neq \emptyset$, there exists a neighborhood $I(\lambda_0)$ of λ_0 with $I(\lambda_0) \subset U(\lambda_0)$, such that $K(\lambda) \cap W(x_0) \neq \emptyset$ for every $\lambda \in I(\lambda_0)$. Since $\lambda_n \rightarrow \lambda_0$ and $\mu_n \rightarrow \mu_0$, so there exists a neighborhood $J(\mu_0)$ of μ_0 with $J(\mu_0) \subset V(\mu_0)$, and there exists N_0 , such that when $n \geq N_0$, we have $\lambda_n \in I(\lambda_0)$ and $\mu_n \in J(\mu_0)$. Hence, $K(\lambda_n) \cap W(x_0) \neq \emptyset$. Then, from Condition (ii), $S(\lambda_n, \mu_n) \cap W(x_0) \neq \emptyset$ which implies that there exists $x_n \in S(\lambda_n, \mu_n) \cap W(x_0)$. Since $W(x_0)$ is arbitrary and $x_n \in W(x_0)$ for $n \geq N_0$, we have $x_n \rightarrow x_0$. The proof is completed.

Theorem 4.4. Let $K : \Lambda_1 \rightarrow 2^X$ be a nonempty set-valued mapping, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$ and let $f : X \times X \times \Lambda_2 \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) $K(\cdot)$ is continuous at λ_0 , and $K(\lambda_0)$ is compact;
- (ii) $f(\cdot, \cdot, \cdot)$ is continuous in $X \times X \times \{\mu_0\}$;
- (iii) For every $x_0 \in S(\lambda_0, \mu_0)$, and for every neighborhood $W(x_0)$ of x_0 ,
 $K(\lambda) \cap W(x_0) \neq \emptyset$ implies $S(\lambda, \mu) \cap W(x_0) \neq \emptyset$ for every $\mu \in V(\mu_0)$,

Then $S(\cdot, \cdot)$ is continuous at (λ_0, μ_0) .

Proposition 4.4. First, similar to the proof of Theorem 4.1, Condition (i) and (ii) indicate that $S(\cdot, \cdot)$ is upper semicontinuous at (λ_0, μ_0) .

Second, similar to Theorem 4.3, the lower semicontinuity of $S(\cdot, \cdot)$ at (λ_0, μ_0) can be proved.

This completes the proof.

Theorem 4.5. Let $K : \Lambda_1 \rightarrow 2^X$ be a nonempty set-valued mapping, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$ and let $f : X \times X \times \Lambda_2 \rightarrow \mathbb{R}$ be a functional. If the following assumptions are satisfied:

- (i) $K(\cdot)$ is continuous at λ_0 , and $K(\lambda_0)$ is compact;
- (ii) $f(\cdot, \cdot, \cdot)$ is continuous in $X \times X \times \{\mu_0\}$;
- (iii) For every $x \in S(\lambda_0, \mu_0)$, and for every $y \in K(\lambda_0)$, $\alpha < f(x, y, \mu_0) < \beta$,

Then $S(\cdot, \cdot)$ is continuous at (λ_0, μ_0) .

Proposition 4.5. For a given $(\lambda, \mu) \in \Lambda_1 \times \Lambda_2$, we have

$$S(\lambda, \mu) = \{\bar{x} \in K(\lambda) : \alpha \leq f(\bar{x}, y, \mu) \leq \beta, \forall y \in K(\lambda)\}.$$

First, we claim that $S(\cdot, \cdot)$ is upper semicontinuous at (λ_0, μ_0) . This can be proved by Condition (i) and (ii), similar to the proof of Theorem 4.1.

Next, we prove the lower semicontinuity of $S(\cdot, \cdot)$ at (λ_0, μ_0) . Suppose that $S(\cdot, \cdot)$ is not lower semicontinuous at (λ_0, μ_0) which shows that there exists $x_0 \in S(\lambda_0, \mu_0)$, and there exists $\{(\lambda_n, \mu_n)\} \subset \Lambda_1 \times \Lambda_2$ with $(\lambda_n, \mu_n) \rightarrow (\lambda_0, \mu_0)$, for each $x_n \in S(\lambda_n, \mu_n)$, we have x_n doesn't converge to x_0 . Since $K(\cdot)$ is lower continuous at λ_0 , and then for $\{\lambda_n\} \subset \Lambda_1$ with $\lambda_n \rightarrow \lambda_0$, and for $x_0 \in S(\lambda_0, \mu_0) \subset K(\lambda_0)$, there exists $\hat{x}_n \in K(\lambda_n)$, such that $\hat{x}_n \rightarrow x_0$. By the hypothesis, we know that there exists $\{\hat{x}_{n_j}\} \subset \{\hat{x}_n\}$, such that $\hat{x}_{n_j} \notin S(\lambda_{n_j}, \mu_{n_j})$ for all $j \in \mathbb{N}$, which implies that there exists $y_{n_j} \in K(\lambda_{n_j}, \mu_{n_j})$, we have $f(\hat{x}_{n_j}, y_{n_j}, \mu_{n_j}) < \alpha$ or $f(\hat{x}_{n_j}, y_{n_j}, \mu_{n_j}) > \beta$. From the upper semicontinuity of $K(\cdot)$ at λ_0 and the compactness of $K(\lambda_0)$, it follows that there exists $y_0 \in K(\lambda_0)$, such that $y_{n_j} \rightarrow y_0$. By Condition (iii), $\alpha < f(x_0, y_0, \mu_0) < \beta$, and by Condition (ii), $f(\hat{x}_{n_j}, y_{n_j}, \mu_{n_j}) \rightarrow f(x_0, y_0, \mu_0)$. Contradiction! The proof is completed.

At Last, we will study the upper semicontinuity of the solution set mapping in a vector metric space, in the particular case where K is fixed, and f is perturbed by a parameter ε .

Let X, Y be metric spaces, K a nonempty compact convex subset of X , $U(\varepsilon_0) \subset Y$, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $W : K \times K \times U(\varepsilon_0) \rightarrow \mathbb{R}$ be a functional. For every $\varepsilon \in U(\varepsilon_0)$, consider the problem:

Find $\bar{x} \in K$, such that

$$\alpha \leq W(\bar{x}, y, \varepsilon) \leq \beta, \quad \forall y \in K \tag{4.4}$$

For every $\varepsilon \in U(\varepsilon_0)$, denote by $T_{\alpha\beta}(\varepsilon) \subset K$ the solution set of Problem (4.4), $T_{\alpha\beta} : U(\varepsilon_0) \rightarrow 2^K$ the solution set mapping of Problem (4.4). And we suppose that W satisfies: there exists $f, g : K \times K \times U(\varepsilon_0) \rightarrow \mathbb{R}$, such that

$$g(x, y, \varepsilon) \leq W(x, y, \varepsilon) \leq f(x, y, \varepsilon), \quad \forall x, y \in K, \forall \varepsilon \in U(\varepsilon_0)$$

Now, we consider the equilibrium problems related to f and g respectively:

Find $\bar{x} \in K$, such that

$$f(\bar{x}, y, \varepsilon) \leq \beta, \quad \forall y \in K \tag{4.5}$$

For every $\varepsilon \in U(\varepsilon_0)$, denote by $T_\beta(\varepsilon) \subset K$ the solution set of Problem (4.5), $T_\beta : U(\varepsilon_0) \rightarrow 2^K$ the solution set mapping of Problem (4.5).

$$T_\beta(\varepsilon) = \cap_{y \in K} \{x \in K : f(x, y, \varepsilon) \leq \beta\} = \{x \in K : \sup_{y \in K} f(x, y, \varepsilon) \leq \beta\}$$

$$\text{graph}(T_\beta) = \{(\varepsilon, x) \in U(\varepsilon_0) \times K : x \in T_\beta(\varepsilon)\}$$

Find $\bar{x} \in K$, such that

$$g(\bar{x}, y, \varepsilon) \geq \alpha, \quad \forall y \in K \tag{4.6}$$

For every $\varepsilon \in U(\varepsilon_0)$, denote by $T_\alpha(\varepsilon) \subset K$ the solution set of Problem (4.6), $T_\alpha : U(\varepsilon_0) \rightarrow 2^K$ the solution set mapping of Problem (4.6).

$$T_\alpha(\varepsilon) = \cap_{y \in K} \{x \in K : g(x, y, \varepsilon) \geq \alpha\} = \{x \in K : \inf_{y \in K} g(x, y, \varepsilon) \geq \alpha\}$$

$$\text{graph}(T_\alpha) = \{(\varepsilon, x) \in U(\varepsilon_0) \times K : x \in T_\alpha(\varepsilon)\}$$

Clearly, for every $\varepsilon \in U(\varepsilon_0)$, $T_\alpha(\varepsilon) \cap T_\beta(\varepsilon) \subset T_{\alpha\beta}(\varepsilon)$.

Define $T : U(\varepsilon_0) \rightarrow 2^K$ as follows:

$$T(\varepsilon) = T_\alpha(\varepsilon) \cap T_\beta(\varepsilon), \quad \forall \varepsilon \in U(\varepsilon_0).$$

Theorem 4.6. *Let X, Y be metric spaces, K a nonempty compact convex subset of X , $U(\varepsilon_0) \subset Y$, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f, g : K \times K \times U(\varepsilon_0) \rightarrow \mathbb{R}$ be functionals, $T_\alpha, T_\beta, T : U(\varepsilon_0) \rightarrow 2^K$ defined as above, satisfying $T_\alpha(\varepsilon) \cap T_\beta(\varepsilon) \neq \emptyset$ for every $\varepsilon \in U(\varepsilon_0)$. If the following conditions hold:*

- (i) *For each $y \in K$, $g(\cdot, y, \cdot)$ is upper semicontinuous;*
- (ii) *For every $y \in K$, and for every $\varepsilon \in U(\varepsilon_0)$, $g(\cdot, y, \varepsilon) : K \rightarrow \mathbb{R}$ is upper semicontinuous;*
- (iii) *For every $y \in K$, $f(\cdot, y, \cdot)$ is lower semicontinuous,*

Then $T = T_\alpha \cap T_\beta : \varepsilon \rightarrow T_\alpha(\varepsilon) \cap T_\beta(\varepsilon)$ is upper semicontinuous for every $\varepsilon \in U(\varepsilon_0)$.

Proposition 4.6. *By Condition (i) and Lemma 2.5, $\inf_{y \in K} g(x, y, \varepsilon)$ is upper semicontinuous for each $x \in K$ and for each $\varepsilon \in U(\varepsilon_0)$, then $\text{graph}(T_\alpha) = \{(\varepsilon, x) \in U(\varepsilon_0) \times K : \inf_{y \in K} g(x, y, \varepsilon) \geq \alpha\}$ is closed. Since K is compact, then from Lemma 2.1, T_α is upper semicontinuous. Condition (ii) and Lemma 2.4 imply that for all $\varepsilon \in U(\varepsilon_0)$, $T_\alpha(\varepsilon)$ is closed, and since K is compact, so $T_\alpha(\varepsilon)$ is compact for every $\varepsilon \in U(\varepsilon_0)$. As above, by Condition (iii), $\sup_{y \in K} f(x, y, \varepsilon)$ is lower semicontinuous for each $x \in K$ and for each $\varepsilon \in U(\varepsilon_0)$, then $\text{graph}(T_\beta) = \{(\varepsilon, x) \in U(\varepsilon_0) \times K : \sup_{y \in K} f(x, y, \varepsilon) \leq \beta\}$ is closed. Therefore, from Lemma 2.6, we have $T = T_\alpha \cap T_\beta : \varepsilon \rightarrow T_\alpha(\varepsilon) \cap T_\beta(\varepsilon)$ is upper semicontinuous for every $\varepsilon \in U(\varepsilon_0)$.*

Theorem 4.7. *Let X, Y be metric spaces, K a nonempty compact convex subset of X , $U(\varepsilon_0) \subset Y$, $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, and let $f, g : K \times K \times U(\varepsilon_0) \rightarrow \mathbb{R}$ be functionals, $T_\alpha, T_\beta, T : U(\varepsilon_0) \rightarrow 2^K$ defined as above, satisfying $T_\alpha(\varepsilon) \cap T_\beta(\varepsilon) \neq \emptyset$ for every $\varepsilon \in U(\varepsilon_0)$. If the following conditions hold:*

- (1) For every $\varepsilon \in U(\varepsilon_0)$, $g(\cdot, \cdot, \varepsilon) - \alpha$ is pseudomonotone;
- (2) For each $x \in K$, $g(x, \cdot, \cdot)$ is lower semicontinuous;
- (3) For every $y \in K$, and for every $\varepsilon \in U(\varepsilon_0)$, $g(\cdot, y, \varepsilon)$ is upper semicontinuous;
- (4) For each $x \in K$, and for each $\varepsilon \in U(\varepsilon_0)$, $g(x, \cdot, \varepsilon)$ is convex;
- (5) For all $t \in K$, $g(t, t, \varepsilon) = \alpha$;
- (6) For every $y \in K$, $f(\cdot, y, \cdot)$ is lower semicontinuous,

Then $T = T_\alpha \cap T_\beta : \varepsilon \rightarrow T_\alpha(\varepsilon) \cap T_\beta(\varepsilon)$ is upper semicontinuous for every $\varepsilon \in U(\varepsilon_0)$.

Proposition 4.7. First, we prove $T_\alpha : U(\varepsilon_0) \rightarrow 2^K$ is upper semicontinuous. Since K is compact, then from Lemma 2.1, we only need to prove T_α is closed. Let $\{(\varepsilon_n, x_n)\}$ satisfy $x_n \in T_\alpha(\varepsilon_n)$ and $(\varepsilon_n, x_n) \rightarrow (\varepsilon, \bar{x})$. Next, we prove $\bar{x} \in T(\varepsilon)$. $x_n \in T_\alpha(\varepsilon_n)$ implies $g(x_n, y, \varepsilon_n) \geq \alpha$ for each $y \in K$. By Condition (1), for every $y \in K$, $g(y, x_n, \varepsilon_n) \leq \alpha$, and by Condition (2), $g(y, \bar{x}, \varepsilon) \leq \liminf_{n \rightarrow \infty} g(y, x_n, \varepsilon_n) \leq \alpha$ for every $y \in K$. Define $y_t = ty + (1-t)\bar{x}$, then Condition (4) and (5) indicate that

$$\alpha = g(y_t, y_t, \varepsilon) = tg(y_t, y, \varepsilon) + (1-t)g(y_t, \bar{x}, \varepsilon) \leq \max\{g(y_t, y, \varepsilon), g(y_t, \bar{x}, \varepsilon)\}$$

Suppose $g(y_t, y, \varepsilon) < g(y_t, \bar{x}, \varepsilon) \leq \alpha$, then $g(y_t, y, \varepsilon) < \alpha$. $g(y_t, \bar{x}, \varepsilon) = \alpha$ implies $g(y_t, y_t, \varepsilon) < \alpha$, which clearly contradicts Condition (5). Thus, $g(y_t, y, \varepsilon) \geq g(y_t, \bar{x}, \varepsilon)$, and then $g(y_t, y, \varepsilon) \geq \alpha$. By Condition (3), we know that $\alpha \leq \limsup_{t \rightarrow 0^+} g(y_t, y, \varepsilon) \leq g(\bar{x}, y, \varepsilon)$, then $\bar{x} \in T_\alpha(\varepsilon)$. Hence, T_α is closed. Condition (3) and Lemma 2.4 show that for every $\varepsilon \in U(\varepsilon_0)$, $T_\alpha(\varepsilon)$ is closed, and since K is compact, so $T_\alpha(\varepsilon)$ is compact for each $\varepsilon \in U(\varepsilon_0)$. From Condition (6), it follows that

$$\text{graph}(T_\beta) = \{(\varepsilon, x) \in U(\varepsilon_0) \times K : \sup_{y \in K} f(x, y, \varepsilon) \leq \beta\}$$

is closed. Thus, by Lemma 2.6, we can prove $T = T_\alpha \cap T_\beta : \varepsilon \rightarrow T_\alpha(\varepsilon) \cap T_\beta(\varepsilon)$ is upper semicontinuous for every $\varepsilon \in U(\varepsilon_0)$.

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