Common Fixed Point Theorems in Non-archimedean Fuzzy Metric Spaces

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Abstract
The aim of this paper is to define the concept of weakly comparable multi-valued mappings. Also we obtain some common fixed point theorems for pairs of weakly comparable multi-valued mappings in ordered non-Archimedean fuzzy metric space.

Keywords: Partial order; Non-Archimedean fuzzy metric space; Weakly comparable mappings.

1 Introduction

In 1965, Zadeh [17] introduced the notion of fuzzy sets. Then some definitions of the fuzzy metric spaces are given by some authors [6, 10, 11], since then fixed point theory on these spaces has been developing (see e.g.[3, 4, 5, 7, 8, 9]). Generally this theory on fuzzy metric space is done for contractive or contractive type mappings. (See [10, 12, 13, 14, 15]).

In 2010, Altun [1] introduced a partial order on a non-Archimedean fuzzy metric space under the Lukasiewicz t-norm and proved some fixed point theorems for single and multi-valued mappings. In [2], Altun and Miheţ introduced the concept of fuzzy order ψ-contractive mappings and proved two fixed point theorems on ordered non-Archimedean fuzzy metric spaces for ψ-contractive type mappings. In the same paper they have given the concept of weakly comparable mappings and proved a common fixed point theorem for such mappings with a partial order induced by an appropriate function.

In this paper, we introduce the concept of weakly comparable multi-valued mappings and prove some common fixed point theorems using this concept with a partial order induced by a function on non-Archimedean fuzzy metric spaces.

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2 Preliminaries

Definition 2.1. [16], A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norms if $([0,1], *)$ is an abelian topological monoid with the unit 1 such that $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$ and $a,b,c,d \in [0,1]$.

Definition 2.2. [11], A triplet $(X,M,*)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following:

(KM-1) $M(x,y,0) = 0$ for all $x, y \in X$,

(KM-2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$,

(KM-3) $M(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t > 0$,

(KM-4) $M(x,y,\bullet) : [0,\infty) \rightarrow [0,1]$ is left continuous for all $x, y \in X$,

(KM-5) $M(x,y,t) \ast M(y,z,s) \leq M(x,z,t+s)$ for all $x, y, z \in X$ and $s,t > 0$,

Note that $M(x,y,t)$ can be thought of as the degree of nearness between $x$ and $y$ with respect to $t$. We will refer to such spaces as FM-spaces.

If in the above definition, the triangular inequality (KM-5) is replaced by

(NA) $M(x,y,t) \ast M(y,z,s) \leq M(x,z,\max \{t,s\})$ for all $x, y, z \in X$ and $s,t > 0$.

then the triplet $(X,M,*)$ is called a non-Archimedean fuzzy metric space. It is easy to note that the triangular inequality (NA) implies (KM-5) that is every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Definition 2.3. [3, 16], Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in $X$ is called an $M$- Cauchy sequence, if for each $\epsilon \in (0,1)$ and $t > 0$ there exist $n_0 \in N$ such that $M(x_n,x_m,t) > 1 - \epsilon$ for all $m,n \geq n_0$. A sequence $\{x_n\}$ in a fuzzy metric space $(X,M,*)$ is said to be convergent if there exist $x \in X$ such that $\lim_{n \rightarrow \infty} M(x_n,x,t) = 1$ for all $t > 0$. An FM space $(X,M,*)$ is called $M$- Complete if every $M$- Cauchy sequence is convergent.

Lemma 2.1. [1], Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with $a \ast b \geq \max \{a+b-1,0\}$ and $\varphi : X \times [0,\infty) \rightarrow R$. Define the relation $\preceq$ on $X$ as follows: $x \preceq y \iff M(x,y,t) \geq 1 + \varphi(x,t) - \varphi(y,t)$ for all $t > 0$. Then $\preceq$ is an (partial) order on $X$ called the partial order induced by $\varphi$.

Lemma 2.2. [2], Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with $a \ast b \geq \max \{a+b-1,0\}$ and $\varphi : X \times [0,\infty) \rightarrow R$. Define the relation $\succeq$ on $X$ as follows: $x \succeq y \iff \varphi(x,t) - \varphi(y,t) \geq \nu \{(1-M(x,y,t))\}$ for all $t > 0$. Then $\succeq$ is an (partial) order on $X$.

Definition 2.4. [5], Let $A$ and $B$ be two nonempty subsets of $X$ and $\preceq$ be a partial order on $X$. Then it is said that $A \preceq B$, if for every $a \in A$, there exists $b \in B$ such that $a \preceq b$. 

**Definition 2.5.** [5], If \( \{x_n\} \subset X \) satisfies \( x_1 \leq x_2 \leq x_3 \ldots \leq x_n \ldots \ or \ x_1 \geq x_2 \geq x_3 \ldots \geq x_n \ldots \) then \( \{x_n\} \) is called a monotone sequence.

**Definition 2.6.** [5], A multi-valued operator \( T : X \to 2^X \) is called order closed if, for monotone sequences \( \{u_n\} \) and \( \{v_n\} \) in \( X \), \( u_n \to u_0 \), \( v_n \to v_0 \) and \( v_n \in Tu_n \) imply \( v_0 \in Tu_0 \).

Altun and Mihet [2] give the concept of weakly comparable mappings on an ordered space as follows:

**Definition 2.7.** [5], Let \( (X, \preceq) \) be an ordered space. Two mappings \( f,g : X \to X \) are said to be weakly comparable if \( fx \preceq gx \) and \( gx \preceq fx \) for all \( x \in X \).

### 3 Main Results

We first define the concept of weakly comparable multi-valued mappings on an ordered space as follows:

**Definition 3.1.** Let \( (X, \preceq) \) be an ordered space. Two mappings \( F,G : X \to 2^X \) are said to be weakly comparable if \( Fx \preceq Gy \) for all \( x \in X \), \( y \in Fx \) and \( Gy \preceq Fy \) for all \( x \in X \), \( y \in Gx \).

**Example 3.1.** Let \( X = [0, \infty) \) and \( \leq \) be the usual ordering. Let \( F,G : X \to 2^X \) defined by

\[
F(x) = \begin{cases} 
\{x\} & \text{if } x \in [0, 1] \\
\{0\} & \text{if } x \in (1, \infty) 
\end{cases}
\] (3.1)

\[
G(x) = \begin{cases} 
\{\sqrt{x}\} & \text{if } x \in [0, 1] \\
\{0\} & \text{if } x \in (1, \infty) 
\end{cases}
\] (3.2)

Then it is obvious that \( Fx \preceq Gy \) for all \( x \in X, y \in Fx \) and \( Gy \preceq Fy \) for all \( x \in X, y \in Gx \). Thus \( F \) and \( G \) are weakly comparable.

Now we prove our main theorem based on the concept of weakly comparable mappings.

**Theorem 3.1.** Let \( (X, M, \ast) \) be an \( M \)-complete Non-Archimedean fuzzy metric space with \( a \ast b \geq \max \{a + b - 1, 0\} \) and \( \varphi : X \times [0, \infty) \to R \) be bounded from above function and \( ^{\prime}_{\sim}\ ) a partial order induced by \( \varphi \). Suppose \( F,G : X \to 2^X \) are two order closed operators and two weakly comparable mappings then \( F \) and \( G \) have a common fixed point.

**Proof.** Since \( Fx \) is non-empty for all \( x \in X \), there exists \( x_1 \in Fx_0 \) and since \( F \) and \( G \) are weakly comparable, we have \( Fx_0 \preceq Gx_1 \). And so by the definition of \( \preceq \) there exist \( x_2 \in Gx_1 \) such that \( x_1 \preceq x_2 \). Again since \( x_2 \in Gx_1 \), by weak comparability of \( F \) and \( G \) we have \( Gx_1 \preceq Fx_2 \). And so there exist \( x_3 \in Fx_2 \) such that \( x_2 \preceq x_3 \). Continuing in this way we get a sequence \( \{x_n\} \) which satisfies \( x_{2n+1} \in Fx_{2n} \) and \( x_{2n} \in Gx_{2n-1} \) such that

\[
x_1 \preceq x_2 \preceq x_3 \ldots x_n \preceq x_{n+1} \ldots
\] (3.3)

That is, the sequence \( \{x_n\} \) is nondecreasing . By definition of " \( \preceq \) " , we have

\[
\varphi(x_0, t) \leq \varphi(x_1, t) \leq \varphi(x_2, t) \leq \ldots
\] (3.4)
for all $t > 0$. In other words, the sequence $\{\varphi(x_n, t)\}$ is nondecreasing sequence of real numbers for all $t > 0$. Since $\varphi$ is bounded from above, $\{\varphi(x_n, t)\}$ is convergent and hence Cauchy. So, for all $\epsilon > 0$ there exist $n_0 \in N$ such that for all $m > n > n_0$ and $t > 0$. We have $|\varphi(x_m, t) - \varphi(x_n, t)| = \varphi(x_m, t) - \varphi(x_n, t) < \epsilon$.

Therefore, since $x_n \leq x_m$,

$$M(x_n, x_m, t) \geq 1 + \varphi(x_n, t) - \varphi(x_m, t) = 1 - |\varphi(x_m, t) - \varphi(x_n, t)| > 1 - \epsilon$$

This shows that the sequence $\{x_n\}$ is Cauchy in $X$ and since $X$ is $M$-complete, it converges to a point $z \in X$. Since the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$ therefore $x_{2n} \rightarrow z$ and $x_{2n+1} \rightarrow z$ with $x_{2n+1} \in Fx_{2n}$ and $x_{2n} \in Gx_{2n-1}$. Now since $F$ and $G$ are order closed, we have $z \in Fz$ and $z \in Gz$ i.e. $z \in Fz \cap Gz$. Hence $z$ is a common fixed point of $F$ and $G$.

We give an example to illustrate the above theorem:

**Example 3.2.** Let $X = N = \{1, 2, 3, \ldots\}$, $a * b = ab$ and

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x \end{cases} \quad (3.5)$$

for all $t > 0$. Then $(X, M*)$ is an $M$-complete non-Archimedean fuzzy metric space. Let $\varphi : X \times [0, \infty) \rightarrow R$ be defined as $\varphi(x, t) = 1 - \frac{3}{t}$. Define $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, \ldots\}$. Now if $x, y \in A$ and $x \leq y$, then $x \leq y$. If $x \in A$ and $y \in B$ then $x \leq y$. If $x, y \in B$, then $x$ and $y$ are not comparable. Now define $F, G : X \rightarrow 2^X$ as $Fx = \{6, x + 1\}$ and $Gx = \{6, x + 2\}$

It is clear that $F$ and $G$ are order closed and weakly comparable. And so all the conditions of Theorem (3.1) are satisfied. Therefore 6 is a common fixed point of $F$ and $G$.

**Theorem 3.2.** Let $(X, M, *)$ be an $M$-complete non-Archimedean fuzzy metric space with $a * b \geq \max\{a + b - 1, 0\}$ and $\varphi : X \times [0, \infty) \rightarrow R$ a bounded from above function and "$\leq$" a partial order induced by $\varphi$. Suppose $F, G, H : X \rightarrow 2^X$ are three order closed operators such that the pairs $\{F, G\}$ and $\{H, G\}$ are weakly comparable mappings then $F, G$ and $H$ have a common fixed point.

**Proof.** We construct a sequence $\{x_n\}$ in $X$ such that

$$x_{3n} \in Fx_{3n-1}, \quad x_{3n-1} \in Gx_{3n-2} \quad \text{and} \quad x_{3n-2} \in Hx_{3n-3} \quad \text{for all } n = 1, 2, 3, \ldots$$

We have $x_1 \in Hx_0$ and since the pair $\{H, G\}$ is weakly comparable, we have $Hx_0 \prec_1 Gx_1$. And so by the definition of $\prec_1$ there exist $x_2 \in Gx_1$ such that $x_1 \leq x_2$. Again since $x_2 \in Gx_1$ and the pair $\{F, G\}$ is weakly comparable, we have $Gx_1 \prec_1 Fx_2$. And so there exist $x_3 \in Fx_2$ such that $x_2 \leq x_3$. Continuing in this way, we get

$$x_1 \leq x_2 \leq x_3 \ldots x_n \leq x_{n+1} \ldots \quad (3.6)$$

By definition of "$\leq$", we have

$$\varphi(x_0, t) \leq \varphi(x_1, t) \leq \varphi(x_2, t) \leq \ldots \quad (3.7)$$
for all $t > 0$. In other words, the sequence $\{\varphi(x_n, t)\}$ is nondecreasing sequence of real numbers for all $t > 0$. Since $\varphi$ is bounded from above, $\{\varphi(x_n, t)\}$ is convergent and hence Cauchy. So, for all $\epsilon > 0$ there exist $n_0 \in N$ such that for all $m > n > n_0$ and $t > 0$. We have $|\varphi(x_m, t) - \varphi(x_n, t)| = \varphi(x_m, t) - \varphi(x_n, t) < \epsilon$.

Therefore, since $x_n \leq x_m$,

$$M(x_n, x_m, t) \geq 1 + \varphi(x_n, t) - \varphi(x_m, t)$$

$$= 1 - [\varphi(x_m, t) - \varphi(x_n, t)]$$

$$> 1 - \epsilon$$

This shows that the sequence $\{x_n\}$ is Cauchy in $X$ and since $X$ is $M$-complete, it converges to a point $z \in X$. Since the sequences $\{x_{3n}\}$, $\{x_{3n-1}\}$ and $\{x_{3n-2}\}$ are subsequences of $\{x_n\}$ therefore $x_{3n} \rightarrow z$, $x_{3n-1} \rightarrow z$ and $x_{3n-2} \rightarrow z$ with $x_{3n} \in Fx_{3n-1}$, $x_{3n-1} \in Gx_{3n-2}$ and $x_{3n-2} \in Hx_{3n-3}$. Now since $F$, $G$ and $H$ are order closed, we have $z \in Fz$, $z \in Gz$ and $z \in Hz$ i.e. $z \in Fz \cap Gz \cap Hz$. Hence $z$ is a common fixed point of $F, G$ and $H$.

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### References


http://dx.doi.org/10.1016/S0165-0114(00)00088-9.


http://dx.doi.org/10.1016/0165-0114(84)90069-1.


http://dx.doi.org/10.1016/j.fss.2006.11.012.

http://dx.doi.org/10.1016/j.fss.2007.07.006.


http://dx.doi.org/10.1155/S0161171294000372.


http://dx.doi.org/10.1016/S0019-9958(65)90241-X.