Approximation of reconstruction formula for continuous wavelet and Weyl-Heisenberg frames

Ali Akbar Arefijamaal 1*, Ghadir Sadeghi 1

(1) Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

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Abstract
As for an orthonormal basis, a frame allows each element in the underlying Hilbert space to be written as an unconditionally convergent infinite linear combination of the frame elements. The coefficients are called frame coefficients. Peter G. Casazza and Ole Christensen introduced some methods to approximate frame coefficients. In this article, we investigate some of these results for a continuous frame. As a consequence, approximation of the solution to a moment problem is also discussed. We also apply the results to wavelet frames and Weyl-Heisenberg frames.

Keywords: Frames; continuous frame; frame operator; approximation of the inverse frame operator.

1 Introduction and Preliminaries

A (discrete) frame for a separable Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ for which there are constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}).$$

The numbers $A, B$ are called frame bounds.

Suppose that $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$, the frame operator is defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i. \quad (1.1)$$

*Corresponding author. Email address: arefijamaal@sttu.ac.ir, arefijamaal@gmail.com
with the series in (1.1) converging unconditionally. It follows that
\[ \langle Sf, f \rangle = \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2. \]

So the frame operator $S$ is a positive, self-adjoint invertible operator on $\mathcal{H}$. This leads to the frame decomposition:
\[ f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle f_i, \quad (f \in \mathcal{H}). \]

The possibility of representing every $f \in \mathcal{H}$ in this way is the main feature of a frame. The coefficients $\{\langle f, S^{-1}f_i \rangle\}_{i=1}^{\infty}$ are called frame coefficients. But usually it is hard to compute the operator $S^{-1}$ if the underlying Hilbert space is infinite dimensional. Hence, we try to approximate $S^{-1}$ with the operators $S_n^{-1}$ which is defined on $\text{span}\{f_i\}_{i=1}^{n}$ by
\[ S_n f = \sum_{i=1}^{n} \langle f, f_i \rangle f_i. \]

In fact, the question is whether
\[ \langle f, S_n^{-1}f_i \rangle \to \langle f, S^{-1}f_i \rangle \text{ as } n \to \infty, \quad (f \in \mathcal{H}, i \in \mathbb{N}). \]

The answer is yes if $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis, but unfortunately it might be no if the frame consists of a Riesz basis plus just a single element, cf. [14]. Also, in [9] it is proven that the answer is no for Weyl-Heisenberg frames that are not Riesz bases. O. Christensen modifies this idea and turns out a way to approximate frame coefficients which apply to the important cases of Weyl-Heisenberg frames and wavelet frames [12]. These results can be applied for Fusion frames [18] and $g$-frames [2].

In the present paper we generalize some of these results to a continuous frame and introduce a method for approximation of the inverse continuous frame operator. Various consequences for approximation of the frame coefficients or approximation of the solution to a moment problem are discussed. We also apply the results to two well-known continuous frames.

For general references on frame theory, we refer to [13]. Recently, various generalization of frames have been proposed. For example, continuous frames [4, 5, 16], g-frames [1, 21], fusion frames [11], von Neumann-Schatten frames [20] and so on. Some applications of frame theory were give in [6, 7, 8, 17].

### 2 Approximation of continuous frame coefficients

Let $\mathcal{H}$ be a separable Hilbert space and $\Omega$ be a locally compact Hausdorff space endowed with a positive Radon measure $\mu$ with $\text{supp} \mu = \Omega$. A mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame with respect to $(\Omega, \mu)$ if
- the mapping $\omega \mapsto \langle f, F(\omega) \rangle$ is measurable for all $f \in \mathcal{H}$,
- there exist constants $0 < A, B < +\infty$ such that
\[ A\|f\|^2 \leq \int_{\Omega} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad (f \in \mathcal{H}). \]
A continuous frame is said to be tight when $A = B$. Note that if $\Omega$ is a countable set and $\mu$ the counting measure, then we obtain the usual definition of a (discrete) frame.

The frame operator $S$ associated to $F$ is defined in weak sense by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \int_\Omega \langle f, F(\omega) \rangle F(\omega) d\mu(\omega).$$

By (2.2), it follows that $\{F(\omega)\}_{\omega \in \Omega}$ is total in $\mathcal{H}$ and $S$ is a bounded, positive, and boundedly invertible operator. In particular

$$A \|f\|^2 \leq \langle Sf, f \rangle \leq B \|f\|^2, \quad (f \in \mathcal{H}).$$

Moreover, since $S$ is invertible and self-adjoint for all $f \in \mathcal{H}$ we have

$$f = \int_\Omega \langle f, F(\omega) \rangle S^{-1} F(\omega) \ d\mu(\omega) = \int_\Omega \langle f, S^{-1} F(\omega) \rangle F(\omega) \ d\mu(\omega). \quad (2.3)$$

Also, $S^{-1} F$ which is a continuous frame with bounds $B^{-1}, A^{-1}$ is called the standard dual frame of $F$. In particular

$$B^{-1} \|f\|^2 \leq \|S^{-1} f\|^2 \leq A^{-1} \|f\|^2, \quad (f \in \mathcal{H}). \quad (2.4)$$

The operator $S$ is a multiple of the identity if and only if $\{F(\omega)\}_{\omega \in \Omega}$ is a tight frame, for more details see [16, 19].

Given a continuous frame $F : \Omega \to \mathcal{H}$ where $(\Omega, \mu)$ is a $\sigma$-finite measure space. Choose finite measurable subsets $\Omega_n$ such that $\Omega_n \subseteq \Omega_{n+1}$ and $\Omega = \cup_{n=1}^{\infty} \Omega_n$. Take $\mathcal{H}_n = \text{span}\{F(\omega)\}_{\omega \in \Omega_n}$, and assume that $\mathcal{H}_n$ is finite dimensional. Then the mapping $F : \Omega_n \to \mathcal{H}_n$ is a continuous frame with the frame operator

$$S_n : \mathcal{H}_n \to \mathcal{H}_n, \quad S_nf = \int_{\Omega_n} \langle f, F(\omega) \rangle F(\omega) d\mu(\omega).$$

Observe that $S_n^{-1}$ can be found using finite dimensional methods. The basic idea is to approximate $S^{-1}$ by $S_n^{-1}$.

We begin with the following lemmas:

**Lemma 2.1.** Let $F : \Omega \to \mathcal{H}$ be a continuous frame with the lower frame bound $A$. Then for every $n \in \mathbb{N}$ there exists $m \geq n$ such that

$$\frac{A}{2} \|f\|^2 \leq \int_{\Omega_n} \|\langle f, F(\omega) \rangle\|^2 d\mu(\omega), \quad (f \in \mathcal{H}_n). \quad (2.5)$$

**Proof.** Consider the compact set $Y_n = \{f \in \mathcal{H}_n, \|f\| = 1\}$ and define

$$\tau_n : Y_n \to [0, +\infty), \quad \tau_n f = \int_{\Omega_n} \|\langle f, F(\omega) \rangle\|^2 d\mu(\omega).$$

Then $\{\tau_n\}_{n \in \mathbb{N}}$ is a sequence of continuous functions converges pointwise to continuous function

$$\tau : Y_n \to [0, +\infty), \quad \tau f = \int_{\Omega} \|\langle f, F(\omega) \rangle\|^2 d\mu(\omega).$$

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Now Dini’s theorem implies that $\tau_n \to \tau$ uniformly. So, there exists $m \geq n$ such that

$$|\tau_m f - \tau f| < \frac{A}{2}, \quad (f \in Y_n).$$

Combining this with (2.2) shows that

$$A \leq \int_{\Omega} |\langle f, \frac{1}{\| f \|^2} F(\omega) \rangle|^2 d\mu(\omega) \leq \frac{1}{\| f \|^2} \int_{\Omega_m} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) + |\tau_m f - \tau f| \leq \frac{1}{\| f \|^2} \int_{\Omega_m} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) + \frac{A}{2},$$

for all $f \in H_n$.

Lemma 2.2. Let $P_n$ be the orthogonal projection of $H$ on $H_n$. Then

$$P_n f = \int_{\Omega_n} \langle f, S_n^{-1} F(\omega) \rangle F(\omega) d\mu(\omega), \quad (f \in H).$$

Proof. Let $f \in H$ and choose $f_1 \in H_n$ and $f_2 \in H_n^\perp$ such that $f = f_1 + f_2$. Now by using (2.3) for continuous frame $F : \Omega_n \to H_n$ we obtain

$$P_n f = f_1 = \int_{\Omega_n} \langle f_1, S_n^{-1} F(\omega) \rangle F(\omega) d\mu(\omega) = \int_{\Omega_n} \langle f, S_n^{-1} F(\omega) \rangle F(\omega) d\mu(\omega).$$

Now assume that $f \in H_n$ and $m \geq n$ is given by (2.5), then

$$\int_{\Omega_m} |\langle f, P_n F(\omega) \rangle|^2 d\mu(\omega) = \int_{\Omega_m} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) \geq \frac{A}{2} \| f \|^2.$$

Moreover,

$$\int_{\Omega_m} |\langle f, P_n F(\omega) \rangle|^2 d\mu(\omega) \leq \int_{\Omega} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) \leq B \| f \|^2.$$

So $\{P_n F(\omega)\}_{\omega \in \Omega}$ is a frame for $H_n$ with bounds $\frac{A}{2}$ and $B$. The frame operator is given by $P_n S_m$. Hence, using (2.4),

$$\|P_n S_m\| \leq B, \quad \|P_n S_m^{-1}\| \leq \frac{2}{A}.$$

Thus we can approximate $S^{-1}$ in the strong operator topology:

**Theorem 2.1.** Let $F : \Omega \to H$ be a continuous frame with the frame bounds $A, B$. Then

$$(P_n S_m)^{-1} P_n f \to S^{-1} f \quad as \quad n \to \infty, \quad (f \in H)$$

where $m \geq n$ is given by (2.5).
Proof. Suppose that \( f \in \mathcal{H} \) and \( n \in \mathbb{N} \). Then
\[
\| (P_nS_n)^{-1}P_nf - S^{-1}f \| \leq \| (P_nS_n)^{-1}P_nf - P_nS^{-1}f \| + \| (I - P_n)S^{-1}f \|
\]
\[
\leq \| (P_nS_n)^{-1}P_nf - P_nS_mP_nS_n^{-1}f \| + \| (I - P_n)S^{-1}f \|
\]
\[
\leq \frac{2}{A} \| f - S_mP_nS_n^{-1}f \| + \| (I - P_n)S^{-1}f \| \to 0 \quad \text{as} \quad n \to \infty.
\]

In the following theorem we replace (2.5) by a similar condition.

**Theorem 2.2.** Let \( F : \Omega \to \mathcal{H} \) be a continuous frame with the frame bounds \( A, B \). Let \( \{a_n\} \subseteq (0, A) \) be a decreasing sequence converging to zero. For \( n \in \mathbb{N} \), let \( A_n \) denote a lower frame bound for \( \{F(\omega)\}_{\omega \in \Omega_n} \) and consider \( m \geq n \) such that
\[
\int_{\Omega \setminus \Omega_m} |\langle F(\omega), F(\omega) \rangle|^2 d\mu(\omega) \leq \frac{a_n}{\mu(\Omega_n)} A_n, \quad (\gamma \in \Omega_n).
\]
(2.7)

Let \( V_n : \mathcal{H}_n \to \mathcal{H}_n \) denote the frame operator for the family \( \{P_nF(\omega)\}_{\omega \in \Omega_m} \). Then
\[
\| S_n^{-1}f - V_n^{-1}P_nf \| \leq \frac{a_n}{A(A - a_n)} \| f \| + \left( \frac{B}{A - a_n} + 1 \right) \| (I - P_n)S^{-1}f \|, \quad (f \in \mathcal{H}).
\]

Proof. Since \( \{F(\omega)\}_{\omega \in \Omega_n} \) is a frame for \( \mathcal{H}_n \) by using (2.3) and (2.4) for every \( f \in \mathcal{H}_n \) we have;
\[
|\langle f, F(\omega) \rangle|^2 = \int_{\Omega_n} |\langle f, S^{-1}F(\gamma) \rangle |^2 |\langle F(\gamma), F(\omega) \rangle |^2 d\mu(\gamma)
\]
\[
\leq \int_{\Omega_n} |\langle f, S^{-1}F(\gamma) \rangle |^2 d\mu(\gamma) \int_{\Omega_n} |\langle F(\gamma), F(\omega) \rangle |^2 d\mu(\gamma)
\]
\[
\leq A_n^{-1} \| f \|^2 \int_{\Omega_n} |\langle F(\gamma), F(\omega) \rangle |^2 d\mu(\gamma).
\]

Thus, by assumption
\[
\int_{\Omega \setminus \Omega_m} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) \leq \int_{\Omega \setminus \Omega_m} |\langle f, F(\omega) \rangle|^2 d\mu(\omega)
\]
\[
\leq \int_{\Omega \setminus \Omega_m} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) \]
\[
= \int_{\Omega} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) - \int_{\Omega \setminus \Omega_m} |\langle f, F(\omega) \rangle|^2 d\mu(\omega)
\]
\[
\geq (A - a_n) \| f \|^2.
\]

Moreover, note that \( P_nS - V_n \) is a positive operator on \( \mathcal{H}_n \). Indeed, for every \( f \in \mathcal{H}_n \) we have;
\[
|\langle (P_nS - V_n)f, f \rangle| = |\langle P_nSf, f \rangle - \langle V_nf, f \rangle|
\]
\[
= \int_{\Omega} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) - \int_{\Omega} |\langle f, F(\omega) \rangle|^2 d\mu(\omega)
\]
\[
= \int_{\Omega \setminus \Omega_m} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) \geq 0.
\]
This follows that
\[
\|(P_nS - V_n)\| = \sup\{\langle (P_nS - V_n)f, f \rangle, \quad f \in H_n, \|f\| = 1 \} \leq a_n.
\]
As a consequence, \((A-a_n)\) is a lower frame bound for \(\{P_nF(\omega)\}_{\omega \in \Omega_m}\). Also \(\|V_n\|^{-1} \leq \frac{1}{A-a_n}\) by (2.4). Therefore, for all \(f \in H\) we obtain that
\[
\|S^{-1}f - V_n^{-1}P_nf\| \leq \|(I - P_n)S^{-1}f\| + \|P_nS^{-1}f - V_n^{-1}P_nf\|
\leq \|(I - P_n)S^{-1}f\| + \frac{1}{A-a_n}(\|V_nP_nS^{-1}f - P_nS^{-1}f\|
+ \|P_nSP_nS^{-1}f - P_nf\|)
\leq \|(I - P_n)S^{-1}f\| + \frac{1}{A-a_n}(\|V_n - P_nS\|\|P_nS^{-1}f\|
+ \|S\|\|SP_nS^{-1}f - f\|)
\leq \|(I - P_n)S^{-1}f\| + \frac{1}{A-a_n}\left(\frac{a_n}{A}\|f\| + B\|(I - P_n)S^{-1}f\|\right)
\leq \frac{a_n}{A(A-a_n)}\|f\| + \left(\frac{B}{A-a_n} + 1\right)\|(I - P_n)S^{-1}f\|.
\]
\[
\square
\]

3 Applications and Examples

In this section, we apply the results to a moment problem. Moreover, we present two examples to illustrate how our results can be applied. Wavelet frames and Weyl-Heisenberg frames, which are two important cases, can be considered as continuous frames.

Let \(F : \Omega \rightarrow H\) be a continuous frame and \(\phi \in L^2(\mu)\). We ask whether there exists \(f \in H\) such that
\[
\langle f, F(\omega) \rangle = \phi(\omega), \quad (\omega \in \Omega).
\]
A problem of this type is called a moment problem. For a more general theory of this fact see [22]. The moment problem has no solution in general, but we can find an unique element in \(H\) which minimizes the function
\[
f \mapsto \int_{\Omega} |\phi(\omega) - \langle f, F(\omega) \rangle|^2 d\mu(\omega).
\]
The answer is called the best approximation solution.

The next theorem shows that the best approximation solution to the moment problem can be approximated by using finite-dimensional methods.

**Theorem 3.1.** Let \(F : \Omega \rightarrow H\) be a continuous frame and \(\phi \in L^2(\mu)\). For \(n \in \mathbb{N}\) choose \(m \geq n\) by (2.5). Then
\[
\int_{\Omega_n} \phi(\omega)S(P_nS_m)^{-1}P_nF(\omega)d\mu(\omega) \rightarrow \int_{\Omega} \phi(\omega)F(\omega)d\mu(\omega).
\]
Proof. It is not difficult to see that if \( f \) is a solution of (3.8), then
\[
f = \int_\Omega \phi(\omega)F(\omega)d\mu(\omega).
\]
Observe that the above weak integral defines an element of \( \mathcal{H} \). Indeed, for every \( g \in \mathcal{H} \)
\[
|\langle \int_\Omega \phi(\omega)F(\omega)d\mu(\omega), g \rangle|^2 \leq \|\phi\|_2^2 \int_\Omega |\langle g, F(\omega) \rangle|^2 d\mu(\omega)
\]
\[
\leq B\|\phi\|_2^2 \|g\|^2.
\]
Now fix \( \phi \in L^2(\mu) \). By Theorem 2.1 for \( f = \int_\Omega \phi(\omega)F(\omega)d\mu(\omega) \) we have
\[
\int_\Omega \phi(\omega)S(P_nS_m)^{-1}P_nF(\omega)d\mu(\omega) = S(P_nS_m)^{-1}P_nf
\]
\[
\rightarrow SS^{-1}f = \int_\Omega \phi(\omega)F(\omega)d\mu(\omega).
\]
It is sufficient to show that
\[
\int_{\Omega \setminus \Omega_n} \phi(\omega)S(P_nS_m)^{-1}P_nF(\omega)d\mu(\omega) \to 0 \quad \text{as} \quad n \to \infty.
\]
To see this, choose \( g \in \mathcal{H} \) then
\[
|\langle \int_{\Omega \setminus \Omega_n} \phi(\omega)S(P_nS_m)^{-1}P_nF(\omega)d\mu(\omega), g \rangle|
\]
\[
\leq \int_{\Omega \setminus \Omega_n} |\langle g, S(P_nS_m)^{-1}P_nF(\omega) \rangle|^2 d\mu(\omega) \int_{\Omega \setminus \Omega_n} |\phi(\omega)|^2 d\mu(\omega)
\]
\[
\leq \int_{\Omega} |\langle P_nS_m)^{-1}Sg, F(\omega) \rangle|^2 d\mu(\omega) \int_{\Omega \setminus \Omega_n} |\phi(\omega)|^2 d\mu(\omega)
\]
\[
\leq B\|P_nS_m)^{-1}\|^2 \|S\|^2 \|g\|^2 \int_{\Omega \setminus \Omega_n} |\phi(\omega)|^2 d\mu(\omega)
\]
\[
\leq \frac{4B}{A^2} \|S\|^2 \|g\|^2 \int_{\Omega \setminus \Omega_n} |\phi(\omega)|^2 d\mu(\omega) \to 0 \quad \text{as} \quad n \to \infty.
\]
The last inequality follows from (2.6). \( \square \)

The Fourier transform of a function \( f \in L^1(\mathbb{R}) \) is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x \xi}dx,
\]
As usual the Fourier transform is extended to an isometry from \( L^2(\mathbb{R}) \) onto \( L^2(\mathbb{R}) \), see [15]. Moreover, for all \( a \in (0, +\infty) \) and \( b \in \mathbb{R} \) the unitary operators of translation \( T_a \), and dilation \( D_a \) for functions \( f \in L^2(\mathbb{R}) \) are defined by:
\[
T_a f(t) = f(t - a), \quad D_a f(t) = a^{-\frac{1}{2}} f\left(\frac{t}{a}\right).
\]
Example 3.1. Let \( \Omega = (0, +\infty) \times \mathbb{R} \) be the affine group, with group law \((a, b)(a', b') = (aa', b + ab')\). An element \( \psi \in L^2(\mathbb{R}) \) is said to be admissible if \( \|\psi\|_2 = 1 \) and
\[
C_\psi = \int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi} d\xi < \infty.
\]

For such admissible vector \( \psi \), the mapping
\[
W_\psi : L^2(\mathbb{R}) \to L^2(\Omega), \quad (W_\psi f)(a, b) = C_\psi^{\frac{1}{2}} \langle f, T_b D_a \psi \rangle,
\]
is a multiple of an isometry, the so-called continuous wavelet transform associated to \( \psi \). Moreover,
\[
\int_{-\infty}^{+\infty} \int_{0}^{+\infty} |\langle f, T_b D_a \psi \rangle|^2 \frac{dadb}{a^2} = C_\psi \|f\|^2, \quad (f \in L^2(\mathbb{R})), \quad (3.9)
\]
see [3]. That is, \( \{T_b D_a \psi\}_{a > 0, b \in \mathbb{R}} \) is a tight continuous frame with respect to \( (\Omega, \frac{dadb}{a^2}) \) with the frame operator \( S = C_\psi I \). Let \( \Omega_n = [\frac{1}{n}, n] \times [-n, n] \). It is obvious to see that the measure of \( \Omega_n \) is \( 2n^2 - 2 \) and \( \Omega = \cup_{n=1}^{\infty} \Omega_n \). Moreover, \( \mathcal{H}_n = \mathbb{R} \mathbb{R} \{T_b D_a \psi, (a, b) \in \Omega_n \} \) is finite dimensional since \((a, b) \mapsto T_b D_a \psi \) is continuous [15].

Now, it is natural to ask for \( n \in \mathbb{N} \) how to find \( m \geq n \) such that condition (2.5) in Lemma 2.1 is satisfied?

Proposition 3.1. Let \( \psi \in L^2(\mathbb{R}) \) be an admissible vector and \( m, n \in \mathbb{N} \) with \( m \geq n \). Then
\[
\int_{\Omega_m} |\langle T_b D_s \psi, T_b D_a \psi \rangle|^2 \frac{dadb}{a^2} \geq \int_{m/n}^{n/n} \int_{-m/n}^{m/n} |\langle \psi, T_b D_a \psi \rangle|^2 \frac{dadb}{a^2}, \quad (s, t) \in \Omega_n. \quad (3.10)
\]

Proof. It is obvious that \( D_a T_b = T_{ab} D_a, \ a > 0, b \in \mathbb{R} \). Therefore,
\[
\int_{\Omega_m} |\langle T_b D_s \psi, T_b D_a \psi \rangle|^2 \frac{dadb}{a^2} = \int_{\Omega_m} |\langle \psi, D_{\frac{1}{2}} T_{b-1} D_a \psi \rangle|^2 \frac{dadb}{a^2} = \int_{\Omega_m} |\langle \psi, T_{b^{-1}} D_{\frac{1}{2}} \psi \rangle|^2 \frac{dadb}{a^2} = \int_{-m}^{m} \int_{-m}^{m} |\langle \psi, T_b D_a \psi \rangle|^2 \frac{dadb}{a^2}.
\]
The last equality follows from the substitutions \( b \to (b + t)s \) and \( a \to as \). So the desired result follows the fact that \( \frac{1}{n} \leq s \leq n \) and \( -n \leq t \leq n \). \( \square \)

As \( m \to +\infty \) the right side of (3.10) converges to \( C_\psi \) by (3.9). Combining this with \( \dim \mathcal{H}_n < \infty \) shows that there exists sufficiently large \( m \geq n \) such that
\[
\int_{\Omega_m} |\langle f, T_b D_a \psi \rangle|^2 \frac{dadb}{a^2} \geq \frac{C_\psi}{2} \|f\|^2, \quad (f \in \mathcal{H}_n).
\]
As a matter of fact, (2.5) is satisfied.

The next example involves the unitary operator \( E_a, \ a \in \mathbb{R} \) on \( L^2(\mathbb{R}) \) defined by \( E_a f(t) = e^{2\pi i at} f(t) \).


Example 3.2. Let $\Omega = \mathbb{R}^2$ with the Lebesgue measure $dadb$. Given $g \in L^2(\mathbb{R})$, a continuous frame for $L^2(\mathbb{R})$ of the form

$$\{E_bT_a f\}_{a,b \in \mathbb{R}} = \{e^{2\pi ib \cdot a} f(\cdot - a)\}_{a,b \in \mathbb{R}}$$

is called a continuous Weyl-Heisenberg frame. For an excellent survey about discrete Weyl-Heisenberg frames we refer to [10]. For each $g \in L^2(\mathbb{R})$ the orthogonality relation (Theorem 8.2.1 of [3]) gives that

$$\int_{\mathbb{R}^2} |\langle f, E_bT_a g \rangle|^2 dadb = \|f\|^2 \|g\|^2_2, \quad (f \in L^2(\mathbb{R})).$$

Namely, in this case $\{E_bT_a g\}_{a,b \in \mathbb{R}}$ is a continuous tight Weyl-Heisenberg frame with the bound $\|g\|^2_2$ and the frame operator $Sf = \|g\|^2_2 f$.

To approximate the operator $S^{-1}$, in general case, we take $\Omega_n = [-n,n] \times [-n,n]$ and $H_n = \{E_bT_a g; \ (a,b) \in \Omega_n\}$.

The proof of the following lemma is similar to the discrete case ([8], Proposition 5.3.6) and we omit it.

Lemma 3.1. Let $n \in \mathbb{N}$ and $m > n$. For all $(a',b') \in \Omega_n$ we have

$$\int_{\Omega \setminus \Omega_m} |\langle E_bT_{a'}g, E_bT_{a}g \rangle|^2 dadb \leq \int_{\Omega \setminus \Omega_{m-n}} |\langle E_bT_{a}g, g \rangle|^2 dadb.$$ 

Thus, by combining Theorem 2.2 and Lemma 3.1 we have the following.

Theorem 3.2. Let $\{E_bT_a g\}_{a,b \in \mathbb{R}}$ be a continuous Weyl-Heisenberg frame with the bound $A,B$, and let $\{a_n\} \subseteq (0,A)$ be a decreasing sequence converging to zero. For $n \in \mathbb{N}$, let $A_n$ denote a lower frame bound for $\{E_bT_a g\}_{(a,b) \in \Omega_n}$ and consider $m \geq n$ such that

$$\int_{\Omega \setminus \Omega_{m-n}} |\langle E_bT_{a}g, g \rangle|^2 dadb \leq \frac{a_n}{\mu(\Omega_n)} A_n.$$ 

(3.11)

Let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ denote the frame operator for the family $\{P_n E_bT_a g\}_{(a,b) \in \Omega_m}$. Then

$$\|S^{-1}f - V_n^{-1}P_n f\| \leq \frac{a_n}{A(A - a_n)} \|f\| + \left(\frac{B}{A - a_n} + 1\right) \|(I - P_n)S^{-1}f\|, \quad (f \in \mathcal{H}).$$

Observe that by choosing $m \in \mathbb{N}$ sufficiently large (3.11) is satisfied since by the frame condition

$$\int_{\mathbb{R}^2} |\langle E_bT_{a}g, g \rangle|^2 dadb,$$

is finite.

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References


http://dx.doi.org/10.1006/acha.1993.1004


http://dx.doi.org/10.1023/A:1021312429186

http://dx.doi.org/10.1109/TSP.2002.803325


http://dx.doi.org/10.1016/j.jmaa.2005.09.039