On Trigonometric Series with Monotonic Coefficients in $L^1_\mu$

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Abstract
In this paper we consider the question of a representations of functions from weighed class $L^1_\mu[0, 2\pi]$ by series with monotonic coefficients concerning trigonometric systems.

Keywords: Trigonometric system; weighted spaces; monotonic coefficients.

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1 Introduction

In 1932 F. Riesz [1] proved that there exists a function $f_0(x) \in L^1[0, 2\pi]$ so that its Fourier series with respect to the trigonometric system does not converge in $L^1[0, 2\pi]$. Consequently, there exist functions in the space $L^1[0, 2\pi]$ that cannot be represented by trigonometric series in the metric of $L^1$.

In [5] it is proved that there is a weighted space $L^1_\mu[0, 2\pi] = \{f : \int_0^{2\pi} |f(x)|\mu(x)dx < \infty\}$, with $0 < \mu(x) \leq 1$, $x \in [0, 2\pi]$ so that for every function $f(x) \in L^1_\mu[0, 2\pi]$ one can find a series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}, \quad \sum_{k=-\infty}^{\infty} |C_k|^q < \infty, \forall q > 2,$$

which convergence to $f(x)$ in the metric $L^1_\mu[0, 1]$, i.e.,

$$\lim_{n \to \infty} \int_0^{2\pi} \left| \sum_{k=-n}^{n} c_k e^{ikx} - f(x) \right| \mu(x)dx = 0.$$

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In this paper we consider the question of a representations of functions from weighed class $L^1_\mu$, $0 < \mu \leq 1$ by series with monotonic coefficients concerning trigonometric system.

The importance of these questions particulary follows from this known fact, the sequence of partial sums of representation of elements from Hilbert spaces, which representations coefficients by normalized basis monotonically decreasing is non linear and gives the best approximations of this element.

Note, that many papers are devoted to the question on existence of various types of representation by different systems in the sense of convergence almost everywhere, on a measure, in $L^p$ metric [4, 6, 7, 8, 9].

In this paper we’ll prove the following result.

**Theorem 1.1.** There exists a rearrangement $\{\sigma(k)\}$ of integer numbers so that the rearranged trigonometric system $\{e^{i\sigma(k)x}\}$ have following property: for any number $\varepsilon > 0$ here exists a weighted function $\mu(x)$, with $0 < \mu(x) \leq 1$, $\{|\{x \in [0, 2\pi] : \mu(x) \neq 1\}| < \varepsilon$ so that for every function $f(x) \in L^1_\mu[0, 2\pi]$ there exits a series of the form $\sum_{k=-\infty}^{\infty} C_k e^{i\sigma(k)x}$, with satisfies:

(i) The series convergence to $f(x)$ in the metric $L^1_\mu[0, 2\pi]$,

(ii) $\sum_{k=-\infty}^{\infty} |C_k|^q < \infty$, for all $q > 2$,

(iii) $|C_k| > |C_{k+1}|$, $\forall k \geq 1$.

**Remark 1.1.** Note, that to construct a series by trigonometric system $\{e^{ikx}\}$ (non rearranged) which satisfy the Theorem 1.1, still is an open problem.

2 Basic Lemmas

In [5] it is proved the following:

**Lemma 2.1.** For any given numbers $\gamma \neq 0$, $N_0 > 1$, $\delta \in (0, 1)$, $\varepsilon_0 \in (0, 1)$ and interval $\Delta \subset [0, 2\pi]$ there exists a measurable set $E \subset \Delta$, a rearranged $\{\sigma(k)\}_{k=N_0}^{N}$ of integer numbers $N_0, ..., N$ and a polynomial $P(x)$ of the form

$$P(x) = \sum_{N_0 \leq |k| < N} C_k e^{ikx}, \quad C_{-k} = \overline{C_k}$$

which satisfy the conditions:

(i) $g(x) = \gamma$ if $x \in E$ and 0 if $x \notin \Delta$,

(ii) $|E| > (2\pi - \delta) \cdot |\Delta|,$

(iii) $\int_E |P(x) - f(x)|dx < \varepsilon$,

(iv) $\int_0^{2\pi} |g(x)|^2 dx < \frac{2}{3} \cdot \gamma^2 \cdot |\Delta|,$

(v) $\sum_{N_0 \leq |k| < N} |C_k|^{2+\delta} < \varepsilon_0$ where $|C_{\sigma(k)}| > |C_{\sigma(k+1)}| > 0$,

(vi) $\left[\sum_{N_0 \leq |k| < N} |C_{\sigma(k)}|^2 \right]^\frac{1}{2} < 2 \cdot |\gamma| \cdot \sqrt{\frac{|\Delta|}{\delta}}$ where $\sigma(-k) = -\sigma(k)$.
Applying this lemma we can proof next lemma.

**Lemma 2.2.** For any given numbers \(0 < \varepsilon < \frac{1}{2}, N_0 > 2\) and a step function

\[
  f(x) = \sum_{s=1}^{q} \gamma_s \cdot \chi_{\Delta_s}(x),
\]

(2.1)

where \(\Delta_s\) is an interval of the form \(\Delta_m^{(i)} = \left[\frac{i-1}{2^m}, \frac{i}{2^m}\right], 1 \leq i \leq 2^m\) with

\[
  |\gamma_s| \cdot \sqrt{|\Delta_s|} < \frac{\varepsilon^3}{16 \cdot q} \cdot \left(\int_0^{2\pi} |f(x)|dx\right)^2, \quad s = 1, 2, ..., q,
\]

(2.2)

there exists a measurable set \(E \subset [0, 2\pi]\), a rearranged \(\{\sigma(k)\}_{k=N_0}^{N_1}\) of integer numbers \(N_0, ..., N\) and a polynomial \(P(x)\) of the form

\[
  P(x) = \sum_{N_0 \leq |k| < N} C_k e^{ikx}, \quad C_{-k} = \overline{C_k}
\]

which satisfy the conditions:

(i) \(|E| > 2\pi - \varepsilon, \)

(ii) \(\int_E |P(x) - f(x)|dx < \varepsilon, \)

(iii) \(\sum_{N_0 \leq |k| < N} |C_k|^{2+\varepsilon} < \varepsilon, \)

(iv) \(|C_{\sigma(k)}| > |C_{\sigma(k+1)}| > 0\) where \(N_0 \leq |k| < N, \)

(v) \(\max_{N_0 \leq m < N} \left[\int_E \sum_{N_0 \leq |k| \leq m} C_{\sigma(k)} e^{i\sigma(k)x} |dx| < \varepsilon + \int_E |f(x)|dx, \text{for every measurable } \delta \text{ of subset } E \right]. \)

**Proof.** Let \(0 < \varepsilon < \frac{1}{2}\) be an arbitrary number. We apply lemma (2.1), setting

\[
  \Delta = \Delta_1, \quad \gamma = \gamma_1, \quad N_0 = N_0, \quad \delta = \frac{\varepsilon}{4 \cdot q} = \alpha_0.
\]

Then there exists a measurable set \(E_1 \subset \Delta_1, \) a rearranged \(\{\sigma_1(k)\}_{k=N_0}^{N_1}\) of integer numbers \(N_0, ..., N_1, \) a function \(g_1(x)\) and a polynomial \(P_1(x)\) of the form

\[
  P_1(x) = \sum_{N_0 \leq |k| < N_1} C_k^{(1)} e^{ikx}, \quad C_{-k}^{(1)} = \overline{C_k^{(1)}}
\]

which satisfy the conditions:

\[
  g_1(x) = \begin{cases} 
    \gamma_1, & x \in E_1 \\
    0, & x \notin \Delta_1,
  \end{cases} \quad |E_1| > (2\pi - \varepsilon) \cdot |\Delta_1|
\]

\[
  \int_{E_1} |P_1(x) - g_1(x)|dx < \alpha_0,
\]

\[
  \int_0^{2\pi} |g_1(x)|^2dx < \frac{2}{\varepsilon} \cdot |\gamma_1|^2 \cdot |\Delta_1|,
\]

\[
  \sum_{N_0 \leq |k| < N_1} |C_k^{(1)}|^2e^{\varepsilon} < \alpha_0,
\]

\[
  \alpha_0 > |C_{\sigma_1(N_0)}^{(1)}| > \cdots > |C_{\sigma_1(N_1-1)}^{(1)}| > 0, \quad \sigma_1(-k) = -\sigma_1(k).
\]

\[
  \left[\sum_{N_0 \leq |k| < N_1} |C_{\sigma_1(k)}^{(1)}|^2\right]^\frac{1}{2} \leq 2 \cdot |\gamma_1| \cdot \sqrt{|\Delta_1| \varepsilon}.\]
We set
\[ \alpha_1 = \min \left[ \alpha_0; \min_{N_0 \leq |k| < N_1} |C_k^{(1)}| \right]. \]

Again applying lemma (2.1), setting
\[ \Delta = \Delta_2, \quad \gamma = \gamma_2, \quad N_0 = N_1, \quad \delta = \alpha_1. \]

Then there exists a measurable set \( E_2 \subset \Delta_2 \), a rearranged \( \{\sigma_2(k)\}_{|k|=N_2}^{N_2} \) of integer numbers \( N_1, \ldots, N_2 \), a function \( g_2(x) \) and a polynomial \( P_2(x) \) of the form
\[ P_2(x) = \sum_{N_1 \leq |k| < N_2} C_k^{(2)} e^{ikx}, \quad C_{-k}^{(2)} = \overline{C_k^{(2)}} \]
which satisfy the conditions:
\[ g_2(x) = \begin{cases} \gamma_2, & x \in E_2 \\ 0, & x \notin \Delta_2. \end{cases}, \quad |E_2| > (2\pi - \varepsilon) \cdot |\Delta| \]
\[ \int_{E_2} |P_2(x) - g_2(x)| dx < \alpha_1, \]
\[ \int_0^{2\pi} |g_2(x)|^2 dx < \frac{2}{\varepsilon} \cdot |\gamma_2|^2 \cdot |\Delta|, \]
\[ \sum_{N_1 \leq |k| < N_2} |C_k^{(2)}|^{2+\varepsilon} < \alpha_1, \]
\[ \alpha_1 > |C_{\sigma_2(N_1)}^{(2)}| > \cdots > |C_{\sigma_2(N_2-1)}^{(2)}| > 0, \quad \sigma_2(-k) = -\sigma_2(k). \]

Continuing reasonings, we can define numbers \( \alpha_0 > \alpha_1 > \cdots > \alpha_{q-1} \), measurable sets \( E_1, \ldots, E_q \), functions \( g_1(x), \ldots, g_q(x) \) and polynomials \( P_1(x), \ldots, P_q(x) \) of the form
\[ P_s(x) = \sum_{N_{s-1} \leq |k| < N_s} C_k^{(s)} e^{ikx}, \quad C_{-k}^{(s)} = \overline{C_k^{(s)}}, \quad 1 \leq s \leq q, \quad (2.3) \]
rearrangement \( \{\sigma_s(k)\}_{|k|=N_{s-1}}^{N_s-1} \) of integer numbers \( N_{s-1}, \ldots, N_s - 1 \), which satisfy the conditions:
\[ \alpha_0 = \frac{\varepsilon}{4 \cdot q}, \quad \alpha_s = \min \left[ \alpha_0; \min_{N_{s-1} \leq |k| < N_s} |C_k^{(s)}| \right], \quad (2.4) \]
\[ g_s(x) = \begin{cases} \gamma_s, & x \in E_s \\ 0, & x \notin \Delta_s. \end{cases}, \quad |E_s| > (2\pi - \varepsilon) \cdot |\Delta_s|, \quad (2.5) \]
\[ \int_0^{2\pi} |g_s(x)|^2 dx < \frac{2}{\varepsilon} \cdot |\gamma_s|^2 \cdot |\Delta_s|, \quad (2.6) \]
\[ \int_{E_s} |P_s(x) - g_s(x)| dx < \alpha_{s-1}, \quad (2.7) \]
\[ \sum_{N_{s-1} \leq |k| < N_s} |C_k^{(s)}|^{2+\varepsilon} < \alpha_{s-1}, \quad (2.8) \]
\[ \alpha_{s-1} > |C_{\sigma_s(N_{s-1})}^{(s)}| > \ldots > |C_{\sigma_s(N_{s-1})}^{(s)}| > 0, \ \sigma_s(-k) = -\sigma_s(k). \quad (2.9) \]

\[
\left[ \sum_{N_{s-1} \leq |k| < N_s} \left| C_{\sigma_s(k)}^{(s)} \right|^2 \right]^{1/2} \leq 2 \cdot |\gamma_s| \cdot \sqrt{\frac{|\Delta_s|}{\varepsilon}}. \quad (2.10)
\]

We define a set \( E \), a rearrangement \( \{\sigma(k)\} \) and a polynomial \( P(x) \) as follows:

\[ E = \bigcup_{s=1}^{q} E_s, \quad (2.11) \]

\[ P(x) = \sum_{s=1}^{q} P_s(x) = \sum_{s=1}^{q} \left[ \sum_{N_{s-1} \leq |k| < N_s} C_{\sigma_s(k)}^{(s)} e^{ikx} \right] = \sum_{N_0 \leq |k| < N} C_k e^{ikx}, \quad (2.12) \]

where

\[ C_k = C_{\sigma(k)}^{(s)} \text{ for } N_{s-1} \leq |k| < N_s, \ \ s = 1, 2, \ldots, q, \ \ N = N_q - 1. \quad (2.13) \]

and

\[ \sigma(k) = \sigma_s(k) \text{ for } k \in [N_{s-1}, N_s), \ \ s = 1, 2, \ldots, q. \quad (2.14) \]

From Eqs. (2.1), (2.3)-(2.5), (2.7) and (2.11)-(2.13), it follows that

\[ |E| > 2\pi - \varepsilon. \]

and

\[ \int_{E} |P(x) - f(x)|dx \leq \sum_{s=1}^{q} \left[ \int_{E} |P_s(x) - g_s(x)|dx \right] \leq \sum_{s=1}^{q} \alpha_{s-1} \]

\[ < \varepsilon. \]

Thus, the statements (i), (ii) and (iii) of Lemma (2.2) are satisfied.

By (2.4) and (2.9) for any \( s = 1, 2, \ldots, q \) and for all \( |k| \in [N_0, N] \) we have

\[ |C_k^{(s)}| \leq \alpha_{s-1} \leq \min_{N_{s-2} \leq |k| < N_{s-1}} \left( |C_k^{(s)}| \right). \]

Consequently, from (2.9), (2.13) and (2.14) we obtain statements (iv)

\[ |C_{\sigma(k)}| > |C_{\sigma(k+1)}| > 0, \ \ N_0 \leq |k| < N. \]

Now we will check the fulfillment of statement (v) of Lemma (2.2).

Let \( N_0 \leq m < N \), then for some \( s_0, 1 \leq s_0 \leq q, (N_{s_0} \leq m < N_{s_0+1}) \) we will have (see (2.3), (2.12) - (2.14))

\[ \sum_{N_0 \leq |k| \leq m} C_{\sigma(k)} e^{i\sigma(k)x} = \sum_{s=1}^{s_0} P_s(x) + \sum_{N_{s_0-1} \leq |k| \leq m} C_{\sigma_s(k)}^{(s_0+1)} e^{i\sigma_s(k)x}. \quad (2.15) \]
From (2.1), (2.5) and (2.11) we get
\[
\left| \sum_{s=1}^{s_0} g_s(x) \right| = |f(x)|, \quad \forall x \in E, \quad s_0 = 1, 2, \ldots, q
\]
Hence and from (2.2), (2.4), (2.7), (2.10), (2.11), (2.14) and (2.15) for any measurable set \( e \subset E \) we obtain
\[
\int_e \left| \sum_{N_{s-1} \leq |k| \leq m} C_{\sigma(k)} e^{i\sigma(k)x} \right| dx \leq \sum_{s=1}^{s_0} \left[ \int_e |P_s(x) - g_s(x)| dx \right] + \sum_{s=1}^{s_0} \int_e |g_s(x)| dx
+ \int_e \left| \sum_{N_{s_0-1} \leq |k| \leq m} C_{\sigma_s(k)} e^{i\sigma_s(k)x} \right| dx < \sum_{s=1}^{s_0} \alpha_{s-1}
+ \int_e |f(x)| dx + \frac{2}{\sqrt{\varepsilon}} \cdot |\gamma_{s_0+1}| \cdot \sqrt{|A_{s_0+1}|} < \int_e |f(x)| dx + \varepsilon.
\]

\[\blacksquare\]

### 3 Proof of Theorem 1.1

**Proof.** Let
\[
f_1(x), f_2(x), \ldots, f_n(x), \quad x \in [0, 2\pi]
\]
be a sequence of all step functions, values and constancy interval endpoints of which are rational numbers. Applying Lemma (2.2) consecutively, we can find a sequence \( \{E_s\}_{s=1}^{\infty} \) of sets and a sequence of polynomials
\[
P_s(x) = \sum_{N_{s-1} \leq |k| < N_s} A_k^{(s)} e^{i\sigma_s(k)x}
\]
where \( \{\sigma_s(k)\}_{N_s-1 \leq |k| = N_s-1}^{N_s-1} \), \( \sigma_s(-k) = -\sigma_s(k) \) a some rearrangement of integer numbers \( N_{s-1}, \ldots, N_s - 1 \), which satisfy the conditions:
\[
|E_s| > 2\pi - 2^{-2(s+1)}, \quad E_s \subset [0, 2\pi],
\]
\[
\int_{E_s} |P_s(x) - f_s(x)| dx < 2^{-2(s+1)},
\]
\[
\sum_{N_{s-1} \leq |k| < N_s} \left| A_k^{(s)} \right|^2 + 2^{-2s} < 2^{-2s}, \quad A_{-k} = \overline{A_k^{(s)}}
\]
\[
|A_k^{(s)}| > |A_{k+1}^{(s)}| > |A_{N_s}^{(s)}| > 0, \quad \forall s \geq 1, \quad |k| \in [N_{s-1}, N_s).
\]
\[
\max_{N_{s-1} \leq m < N_s} \left[ \int_e \left| \sum_{N_{s-1} \leq |k| \leq m} A_k^{(s)} e^{i\sigma_s(k)x} \right| dx \right] < 2^{-2(s+1)} + \int_e |f_s(x)| dx,
\]
for every measurable subset \( e \) of \( E_s \).
Denote the rearrangement \( \{\sigma(k)\} \) and the series by rearranged trigonometric systems \( \{e^{i\sigma_k x}\}_{k=-\infty}^{\infty} \) the following way

\[
\sigma(k) = \sigma_s(k), \quad \text{if} \quad k \in [N_{s-1}, N_s), \quad \forall s \geq 1 \tag{3.23}
\]

and

\[
\sum_{k=-\infty}^{\infty} C_k e^{i\sigma(k)x} = \sum_{s=1}^{\infty} P_s(x) = \sum_{s=1}^{\infty} \left[ \sum_{|k| < N_s} A_k^{(s)} e^{i\sigma_s(k)x} \right],
\tag{3.24}
\]

where \( C_k = A_k^{(s)} \) for \( N_{s-1} \leq |k| < N_s, \ s = 1, 2, \ldots \)

Let \( \varepsilon \) be an arbitrary positive number. Setting

\[
\begin{cases}
\Omega_n = \bigcap_{s=n}^{\infty} E_s, & n = 1, 2, \ldots,
\Omega_n = \bigcap_{s=n_0}^{\infty} E_s, & n_0 = \lfloor \log_{1/2} \varepsilon \rfloor + 1 \tag{3.25}
\end{cases}
\]

\[
B = \bigcup_{n=n_0}^{\infty} \Omega_n = \Omega_{n_0} \cup \left( \bigcup_{n=n_0+1}^{\infty} \Omega_n \setminus \Omega_{n-1} \right).
\]

It is clear (see (3.18)) that \( B = 2\pi \) and \( E > 2\pi - \varepsilon \).

We define a function \( \mu(x) \) in the following way:

\[
\mu(x) = \begin{cases} 
1, & x \in E \cup \left( [0, 2\pi] \setminus B \right), \\
\mu_n, & x \in \Omega_n \setminus \Omega_{n-1}, \quad n \geq n_0 + 1, 
\end{cases} \tag{3.26}
\]

where

\[
\begin{cases}
\mu_n = \left[ 2^{4n} \cdot \prod_{i=1}^{n} h_s \right]^{-1}; \\
h_s = \|f_s(x)\|_C + \max_{N_{s-1} \leq m < N_s} \left\| \sum_{|k| \leq m} A_k^{(s)} e^{i\sigma_s(k)x} \right\|_C + 1, \tag{3.27}
\end{cases}
\]

where \( g(x) \) is a continuous function on \( [0, 2\pi] \) and \( \|g(x)\|_C = \max_{x \in [0, 2\pi]} |g(x)| \).

From (3.20) and (3.24)-(3.27), we obtain

(a) \( 0 < \mu(x) \leq 1, \ \mu(x) \) is a measurable function and \( \{x \in [0, 2\pi] : \mu(x) \neq 1\} < \varepsilon \).

(b) \( \sum_{k=1}^{\infty} |C_k|^q < \infty, \quad \forall q > 2, \quad |C_k| > |C_{k+1}|, \quad \forall k \geq 1. \)

It follows from (3.23)-(3.25) that for all \( s \geq n_0 \) and \( m \in [N_{s-1}, N_s] \)

\[
\int_{[0,2\pi] \setminus \Omega_s} \left| \sum_{N_{s-1} \leq |k| \leq n_s} A_k^{(s)} e^{i\sigma_s(k)x} \right| \mu(x) dx
\]

\[
= \sum_{n=N_{s+1}}^{\infty} \left[ \int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{N_{s-1} \leq |k| \leq m} A_k^{(s)} e^{i\sigma_s(k)x} \right| \mu_n dx \right]
\]

\[
\leq \sum_{n=N_{s+1}}^{\infty} 2^{-4n} \left[ \int_0^{2\pi} \left| \sum_{N_{s-1} \leq |k| \leq m} A_k^{(s)} e^{i\sigma_s(k)x} \right| h_s^{-1} dx \right]
\leq 2^{-4s}.
\tag{3.28}
\]
By (3.19), (3.23)-(3.25) for all $s \geq n_0$ we have

$$
\int_0^{2\pi} |P_s(x) - f_s(x)| \mu(x) dx = \int_{\Omega_s} |P_s(x) - f_s(x)| \mu(x) dx + \int_{[0,2\pi] \setminus \Omega_s} |P_s(x) - f_s(x)| \mu(x) dx
$$

$$
= 2^{-2(s+1)} + \sum_{n=n_0+1}^{\infty} \int_{\Omega_n \setminus \Omega_{n-1}} |P_s(x) - f_s(x)| \mu_n dx
$$

$$
\leq 2^{-2(s+1)} + \sum_{n=n_0+1}^{\infty} 2^{-4s} \left[ \int_0^{2\pi} |f_s(x)| \right.
$$

$$
+ \sum_{N_{s-1} \leq |k| \leq N_s} A_k(s) e^{i\sigma(s)k} |h_s| dx \bigg] < 2^{-2(s+1)} + 2^{-4s}
$$

$$
< 2^{-2s}.
$$

(3.29)

Taking relations (3.21), (3.23)-(3.25) and (3.27) into account we obtain that for all $m \in [N_{s-1}, N_s]$ and $s \geq n_0 + 1$

$$
\int_0^{2\pi} \left| \sum_{N_{s-1} \leq |k| \leq m} A_k(s) e^{i\sigma(s)k} x \right| \mu(x) dx = \int_{\Omega_s} \left[ \sum_{N_{s-1} \leq |k| \leq m} A_k(s) e^{i\sigma(s)k} x \right] \mu(x) dx
$$

$$
+ \int_{[0,2\pi] \setminus \Omega_s} \left[ \sum_{N_{s-1} \leq |k| \leq m} A_k(s) e^{i\sigma(s)k} x \right] \mu(x) dx
$$

$$
< \sum_{n=n_0+1}^{s} \int_{\Omega_n \setminus \Omega_{n-1}} \sum_{N_{s-1} \leq |k| \leq m} A_k(s) e^{i\sigma(s)k} x |dx|
$$

$$
\times \mu_n + 2^{-4s}
$$

$$
< \sum_{n=n_0+1}^{s} \left( 2^{-2(s+1)} + \int_{\Omega_n \setminus \Omega_{n-1}} |f_s(x)| dx \right) \mu_n
$$

$$
+ 2^{-4s}
$$

$$
= 2^{-2(s+1)} \cdot \sum_{n=n_0+1}^{s} \mu_n + \int_{\Omega_s} |f_s(x)| \mu(x) dx + 2^{-4s}
$$

$$
< \int_0^{2\pi} |f_s(x)| \mu(x) dx + 2^{-4s}.
$$

(3.30)

Let $f(x) \in L^1_{\mu}[0, 2\pi]$, i.e. $\int_0^{2\pi} |f(x)| \mu(x) dx < \infty$.

It is easy to see that we can choose a function $f_{r_1}(x)$ from the sequence (3.16) such that

$$
\int_0^{2\pi} |f(x) - f_{r_1}(x)| \mu(x) dx < 2^{-2}, \quad r_1 > n_0 + 1.
$$

(3.31)

Hence, we have

$$
\int_0^{2\pi} |f_{r_1}(x)| \mu(x) dx < 2^{-2} + \int_0^{2\pi} |f(x)| \mu(x) dx.
$$

(3.32)

From (3.29) and (3.31), we obtain

$$
\int_0^{2\pi} |f(x) - P_{r_1}(x)| \mu(x) dx \leq \int_0^{2\pi} |f(x) - f_{r_1}(x)| \mu(x) dx
$$

$$
+ \int_0^{2\pi} |f_{r_1}(x) - P_{r_1}(x)| \mu(x) dx
$$

$$
< 2 \cdot 2^{-2}.
$$

(3.33)
Assume that numbers $r_1 < r_2 < \cdots < r_{q-1}$ are chosen in such a way that the following condition is satisfied:

$$\int_0^{2\pi} \left| f(x) - \sum_{s=1}^{j} P_s(x) \right| \mu(x) dx < 2 \cdot 2^{-2j}, \quad 1 \leq j \leq q - 1. \quad (3.34)$$

We choose a function $f_{r_q}(x)$ from the sequence (3.16) such that

$$\int_0^{2\pi} \left| f(x) - \sum_{s=1}^{q-1} P_s(x) \right| - f_{r_q}(x) \right| \mu(x) dx < 2^{-2q}, \quad (3.35)$$

where $\nu_q > \nu_{q-1} > \nu_q > m_{q-1}$

This with (3.34) imply

$$\int_0^{2\pi} \left| f_{r_q}(x) \right| \mu(x) dx < 2^{-2q} + 2 \cdot 2^{-2(q-1)} = 9 \cdot 2^{-2q}. \quad (3.36)$$

By (3.29), (3.30) and (3.36), we obtain

$$\int_0^{2\pi} \left| f_{r_q}(x) - P_{r_q}(x) \right| \mu(x) dx < 2^{-2r_q}, \quad (3.37)$$

$$P_{r_q}(x) = \sum_{N_{r_{q-1}} \leq |k| < N_{r_q}} A_k^{(r_q)} e^{i\sigma_{r_q}(k)x}. \quad (3.38)$$

We define a series as follows

$$\sum_{k=-\infty}^{\infty} C_k e^{i\sigma(k)x} = \sum_{q=1}^{\infty} \left[ \sum_{N_{r_{q-1}} \leq |k| < N_{r_q}} A_k^{(\nu_q)} e^{i\sigma_{\nu_q}(k)x} \right], \quad (3.39)$$

where $C_k = A_k^{(\nu_q)}$ for $N_{r_{q-1}} \leq |k| < N_{r_q}$, $q = 1, 2, \ldots$.

Hence and from (3.20), (3.21), we obtain statements (ii) and (iii) of Theorem (1.1). Now taking into account (3.35) and (3.37), we have

$$\int_0^{2\pi} \left| f(x) - \sum_{s=1}^{q} P_s(x) \right| \mu(x) dx \leq \int_0^{2\pi} \left| \left( f(x) - \sum_{s=1}^{q-1} P_s(x) \right) - f_{r_q}(x) \right| \mu(x) dx$$

$$+ \int_0^{2\pi} \left| f_{r_q}(x) - P_{r_q}(x) \right| \mu(x) dx$$

$$< 2 \cdot 2^{-2q}$$

Hence it follows that the series (3.39) converges to $f(x)$ in the metric $L^1_\mu[0, 1]$. \hfill \Box

## 4 Conclusion

In this paper, we consider a rearrangement of the trigonometric system $\{e^{i\sigma(k)x}\}$ and weighted spaces $L^1_\mu[0, 1]$ so that every function $f(x) \in L^1_\mu[0, 2\pi]$ can be represented by the series of the form

$$\sum_{k=-\infty}^{\infty} C_k e^{i\sigma(k)x} \text{ with } \sum_{k=-\infty}^{\infty} |C_k|^q < \infty, \quad \forall q > 2, \quad |C_k| > |C_{k+1}|, \quad \forall k \geq 1.$$
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