Existence of Three Solutions for $p$-biharmonic Equation

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Abstract
In this paper, the existence of at least three solutions to a Navier boundary problem involving the $p$-biharmonic equation, will be established. The technical approach is mainly based on the three critical points theorem of B. Ricceri.

Keywords: $p$-biharmonic, Navier condition; Multiple solutions, Three critical points theorem

1 Introduction and main results

Consider the Navier boundary value problem involving the $p$-biharmonic equation

\[
\begin{cases}
-\Delta (|\Delta u|^{p-2} \Delta u) = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\
u = \Delta u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where $\lambda, \mu \in [0, +\infty)$, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a non-empty bounded open set with a sufficient smooth boundary $\partial \Omega$, $p > \max \{1, N/2\}$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are the Carathéodory functions.

We recall that a function $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if

- $t \to f(t, x)$ is measurable for every $x \in \mathbb{R}$;
- $x \to f(t, x)$ is continuous for a.e. $t \in \Omega$.

Here in the sequel, $X$ will be denoted as the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. The space $X$ will be endowed with the norm

$$||u|| = \left( \int_\Omega |\Delta u|^p \, dx \right)^{\frac{1}{p}}.$$

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Let
\[ K = \sup_{u \in W^{2, p}(\Omega) \cap W_0^{1, p}(\Omega) \setminus \{0\}} \frac{\sup_{x \in \Omega} |u(x)|}{\|u\|}. \]  
(1.1)

Since \( p > \max \{1, \frac{N}{2}\} \), \( W^{2, p}(\Omega) \cap W_0^{1, p}(\Omega) \hookrightarrow C^0(\Omega) \) is compact, and one has \( K < +\infty \).

As always, a weak solution of problem \((P)\) is any \( u \in X \) such that
\[ -\int_\Omega |\Delta u|^{p-2} \Delta u \Delta \xi dx = \lambda \int_\Omega f(x, u, v)\xi dx + \mu \int_\Omega g(x, u, v)\xi dx \]  
(1.2)

for every \( \xi \in X \).

In recent years, the three critical points theorem of B. Ricceri has been widely used to solve differential equations, see [1, 2, 3, 5, 6, 7, 9, 12, 13] and reference therein.

The fourth-order equation of nonlinearity furnishes a model to study traveling waves in suspension bridges; therefore this becomes very significant in Physics. Many authors consider this type of equation, we refer to [4, 8, 10, 11] and there reference therein.

To the best of our knowledge, there are only very few results regarding multiple solutions to the \( p \)-biharmonic equation. In this paper, the existence of at least three solutions of problem \((P)\) will be proved. The technical approach is based on the three critical points theorem of B. Ricceri [14]. Our theorem, under the new assumptions, ensures the existence of an open interval \( \Lambda \subseteq [0, +\infty) \) and a positive real number \( \rho \) such that, for each \( \lambda \in \Lambda \), problem \((P)\) admits at least three weak solutions whose norms in \( X \) are less than \( \rho \).

Now, for every \( x^0 \in \Omega \) and selected \( r_1, r_2 \) with \( r_2 > r_1 > 0 \), such that \( B(x^0, r_2) \subseteq B(x^0, r_1) \subseteq \Omega \), where \( B(x^0, r_1) \) denotes the ball with the center at \( x^0 \) and radius of \( r_1 \). Put
\[ \theta = \begin{cases} 
\frac{3KN}{(r_2-r_1)(r_2+r_1)} \left( \frac{N}{2} \left( \frac{(r_2+r_1)^{N-2(r_2)N}}{2^{N(1+\frac{N}{2})}} \right)^{\frac{1}{p}} \right), & N < \frac{4r_1}{r_2-r_1}, \\
\frac{12Kr_1}{(r_2-r_1)^2(r_2+r_1)} \left( \frac{N}{2} \left( \frac{(r_2+r_1)^{N-2(r_2)N}}{2^{N(1+\frac{N}{2})}} \right)^{\frac{1}{p}} \right), & N \geq \frac{4r_1}{r_2-r_1},
\end{cases} \]  
(1.3)

where \( \Gamma(\cdot) \) is the Gamma function. Also, let \( F(x, t) = \int_0^1 f(x, \xi) d\xi \). Our main results are the following theorems.

**Theorem 1.1.** Suppose that \( r_2 > r_1 > 0 \), such that \( B(x^0, r_2) \subseteq \Omega \) and assuming there exist three positive constants \( c, d \) and \( \gamma \) with \( \gamma < p, c < \theta d \), and a negative function \( \alpha \in L^1(\Omega) \) such that

1. \((j_1)\) \( F(x, s) \leq 0 \) for a.e. \( x \in \Omega \setminus B(x^0, r_1) \) and all \( s \in [0, d] \);
2. \((j_2)\) \( m(\Omega) \inf_{(x,s) \in \Omega \times [-c, c]} F(x, s) > \left( \frac{c}{d\theta} \right)^p \int_{B(x^0, r_1)} F(x, d) dx \);
3. \((j_3)\) \( F(x, s) \geq \alpha(x)(1 + |s|^\gamma) \) for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R} \).

Then, there exist an open interval \( \Lambda \subseteq [0, +\infty) \) and a positive real number \( \rho \) with the following property: for each \( \lambda \in \Lambda \) and for each Carathéodory function \( g: \Omega \times \mathbb{R} \mapsto \mathbb{R} \), thus satisfying

\[ (j_4) \sup_{(|\xi| \leq \xi)} |g(\cdot, s)| \in L^1(\Omega), \text{ for all } \xi > 0, \]

**Significantly,** there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), problem \((P)\) has at least three solutions whose norms in \( X \) are less than \( \rho \).
Let $h_1 \in C(\Omega)$ be a positive function and $h_2 \in C(\mathbb{R})$ be a negative function. Let

$$f(x, u) = h_1(x)h_2(u)$$

for each $(x, u) \in \Omega \times \mathbb{R}$,

$$H(t) = \int_0^t h_2(\zeta)d\zeta$$

for all $t \in \mathbb{R}$ and $\alpha_1(x) = \frac{\alpha(x)}{h_1(x)}$. Then, using Theorem 1.1, the following result is obtained:

**Corollary 1.1.** Assume that there exist three positive constants, $c$, $d$, and $\gamma$ with $\gamma < p$, $c < \theta d$, and a negative function $\alpha \in L^1(\Omega)$ such that

1. $F(x, s) \leq 0$ for a.e. $x \in \Omega \setminus B(x_0, r_1)$ and all $s \in [0, d]$;
2. $m(\Omega) \inf_{(x, s) \in \Omega \times [-c, c]} F(x, s) > \left( \frac{c}{md} \right)^p \frac{H(d)}{H(c)} \int_{B(x_0, r_1)} h_1(x) dx$;
3. $F(x, s) \geq \alpha_1(x)(1 + |s|^{\gamma})$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

Then there exist an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number $\rho$ with the following property: for each $\lambda \in \Lambda$ and for two Carathéodory functions $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$, satisfying

1. $\sup_{|s| \leq \zeta} |g(\cdot, s)| \in L^1(\Omega)$, for all $\zeta > 0$,

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, problem

$$\begin{cases}
-\Delta(\Delta u)^{p-2}\Delta u = \lambda h_1(x)h_2(u) + \mu g(x, u), & \text{in } \Omega, \\
u = \Delta u = 0, & \text{on } \partial\Omega,
\end{cases}$$

has at least three solutions whose norms in $X$ are less than $\rho$.

A simple consequence of Theorem 1.1 is as follows:

**Theorem 1.2.** Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Put $F(s) = \int_\Omega f(t) dt$ for each $s \in \mathbb{R}$ and assume that there exist three positive constants $c$, $d$, $\gamma$ with $c < \theta d$, $\gamma < p$ and a negative constant $\alpha$ such that

1. $F(s) \leq 0$ for all $s \in [0, d]$;
2. $m(\Omega) \inf_{[-c, c]} F(s) > \left( \frac{c}{md} \right)^p \frac{r_1^{p-\frac{\gamma}{2}}}{\Gamma(1+\frac{\gamma}{2})} F(d);
3. $F(s) \geq \alpha(1 + |s|^{\gamma})$ for all $s \in \mathbb{R}$.

Then there exist an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number $\rho$ with the following property: for each $\lambda \in \Lambda$ and for each Carathéodory function $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$, satisfying

1. $\sup_{|s| \leq \zeta} |g(\cdot, s)| \in L^1(\Omega)$, for all $\zeta > 0$,

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, problem $(P)$ has at least three solutions whose norms in $X$ are less than $\rho$. 


2 Proof of theorems

For the reader’s convenience, the revised form of Ricceri’s three critical points theorem
Theorem 1 in [14] and Proposition 3.1 of [12] are recalled.

\textbf{Theorem 2.1} ([14], Theorem 1). Let \( X \) be a reflexive real Banach space. \( \Phi: X \rightarrow \mathbb{R} \) is a continuously Gâteaux differentiable and a sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \( X^* \) and \( \Phi \) is bounded on each bounded subset of \( X \); \( \Psi: X \rightarrow \mathbb{R} \) is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; \( I \subseteq \mathbb{R} \) an interval. Assuming that

\[
\lim_{\|x\| \rightarrow +\infty} (\Phi(x) + \lambda \Psi(x)) = +\infty
\]

for all \( \lambda \in I \), and that there exists \( h \in \mathbb{R} \) such that

\[
\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda (\Psi(x) + h)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda (\Psi(x) + h)). \tag{2.4}
\]

Then, there exists an open interval \( \Lambda \subseteq I \) and a positive real number \( \rho \) with the following property: for every \( \lambda \in \Lambda \) and every \( C^1 \) functional \( J: X \rightarrow \mathbb{R} \) with compact derivative, there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), the equation

\[
\Phi'(x) + \Psi'(x) + \mu J'(x) = 0
\]

has at least three solutions in \( X \) whose norms are less than \( \rho \).

In Proposition 3.1 of [12], if a minus is added to \( \Psi \), then the result is as follows:

\textbf{Proposition 2.1} ([12], Proposition 3.1). Let \( X \) be a non-empty set and \( \Phi, \Psi \) two real functions on \( X \). Assuming that there are \( r > 0 \) and \( x_0, x_1 \in X \) such that

\[
\Phi(x_0) = \Psi(x_0) = 0, \quad \Phi(x_1) > r, \quad \inf_{x \in \Phi^{-1}([-\infty, r])} \Psi(x) > r \frac{\Psi(x_1)}{\Phi(x_1)}.
\]

Then, for each \( h \) satisfying

\[
\sup_{x \in \Phi^{-1}([-\infty, r])} -\Psi(x) < h < r \frac{\Psi(x_1)}{\Phi(x_1)},
\]

one has

\[
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda (h + \Psi(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda (h + \Psi(x))).
\]

Before giving the proof for Theorem 1.1, consider the following two lemmas.

\textbf{Lemma 2.1}. Assume that there exist two positive constants \( c, d \) with \( c < \theta d \), such that

\( (j_1) \) \( F(x, s) \leq 0 \) for a.e. \( x \in \Omega \setminus B(x^0, r_1) \) and all \( s \in [0, d] \);

\( (j_2) \) \( m(\Omega) \inf_{(x,s) \in \Omega \times [-c, c]} F(x, s) > \left( \frac{c}{\theta d} \right) \int_{B(x^0, r_1)} F(x, d) dx. \)
Then there exists $u^* \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ such that
\[ \|u^*\|^p > pr, \]
and
\[ m(\Omega) \inf_{(x,s) \in \Omega \times [-c,c]} F(x,s) > \left( \frac{c}{K\|u^*\|} \right) \int_{\Omega} F(x,u^*(x))\,dx. \]

Proof. Let
\[ w(x) = \begin{cases} 
0, & x \in \Omega \setminus B(x_0, r_2), \\
\frac{d(3(t^4-r_2^4)-4(r_1+r_2)(t^2-r_2^2)+6r_1r_2(t^2-r_1^2))}{(r_2-r_1)^4(r_1+r_2)}, & x \in B(x_0, r_2) \setminus B(x_0, r_1), \\
d, & x \in B(x_0, r_1),
\end{cases} \quad (2.5) \]
where $u^*(x) = w(x)$ and $l = \text{dist}(x,x^0) = \sqrt{\sum_{i=1}^N(x_i-x_i^0)^2}$. We then have
\[ \frac{\partial u^*(x)}{\partial x_i} = \begin{cases} 
0, & x \in \Omega \setminus B(x_0, r_2) \cup B(x_0, r_1), \\
\frac{12d((t^4-r_1^4)-r_1^4+r_1^2(t^2-r_1^2))}{(r_2-r_1)^4(r_1+r_2)}, & x \in B(x_0, r_2) \setminus B(x_0, r_1),
\end{cases} \quad (2.5) \]
\[ \frac{\partial^2 u^*(x)}{\partial^2 x_i} = \begin{cases} 
0, & x \in \Omega \setminus B(x_0, r_2) \cup B(x_0, r_1), \\
\frac{12d((r_2-r_1)^2-(r_1+r_2)l)}{(r_2-r_1)^4(r_1+r_2)}, & x \in B(x_0, r_2) \setminus B(x_0, r_1),
\end{cases} \quad (2.5) \]
\[ \sum_{i=1}^N \frac{\partial^2 u^*(x)}{\partial^2 x_i} = \begin{cases} 
0, & x \in \Omega \setminus B(x_0, r_2) \cup B(x_0, r_1), \\
\frac{12d((N+2)(N+1)(r_1+r_2)l)}{(r_2-r_1)^4(r_1+r_2)}, & x \in B(x_0, r_2) \setminus B(x_0, r_1),
\end{cases} \quad (2.5) \]
It is easy to verify that $u^* \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, and particularly, one has
\[ \|u^*\|^p = \frac{(2d)^p 2\pi^2}{(r_2-r_1)^3(r_1+r_2)^p} \int_{r_1}^{r_2} (N+2)^2 - (N+1)(r_1+r_2)r + Nr_1r_2|\Omega| \, dr. \quad (2.6) \]
Here, we obtain from (1.3) and (2.6) that
\[ \frac{\theta d}{K} < \|u^*\|. \quad (2.7) \]
By the assumption
\[ c < \theta d, \]
it follows from (2.7) that
\[ \|u^*\|^p > \frac{1}{p} \left( \frac{\theta d}{K} \right)^p > \frac{1}{p} \left( \frac{c}{K} \right)^p. \]
Since, $0 \leq u^* \leq d$, for each $x \in \Omega$, the condition $(j_1)$ ensures that
\[ \int_{\Omega \setminus B(x_0, r_2)} F(x,u^*(x))\,dx + \int_{B(x_0, r_2) \setminus B(x_0, r_1)} F(x,u^*(x))\,dx \leq 0. \]
Hence, by condition (j2) and (2.7), we have

\[
m(\Omega) \inf_{(x,s) \in \Omega \times [-c,c]} F(x,s) > \left( \frac{c}{\theta d} \right)^p \int_{B(x^0,r_1)} F(x,d)dx \\
> \left( \frac{c}{K\|u^*\|} \right)^p \int_{B(x^0,r_1)} F(x,d)dx \\
\geq \left( \frac{c}{K\|u^*\|} \right)^p \int_{\Omega} F(x,u^*)dx.
\]

Lemma 2.2. Let \( T: X \mapsto X^* \) be the operator defined by

\[
T(u)(\xi) = \int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta \xi(x)dx
\]

for all \( u, \xi \in X \), where \( X^* \) denotes the dual of \( X \). Then \( T \) admits a continuous inverse on \( X^* \).

Proof. If \( x, y \in \mathbb{R}^N \) and \( \langle \cdot, \cdot \rangle \) denotes the usual inner product in \( \mathbb{R}^N \), owing to (2.2) of [15], there exists a positive constant \( C_p \) such that

\[
\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq C_p|x - y|^p.
\]

Thus, it becomes easy to verify that

\[
(T(u) - T(v))(u - v) \geq \|x - y\|^p.
\]

for every \( u \) and \( v \) belonging to \( X \). This actually implies that \( T \) is a uniformly monotone in \( X \). In addition, standard arguments ensure that \( T \) also appears to be coercive and hemicontinuous in \( X \). Therefore, the conclusion follows immediately by applying Theorem 26. A. of [16].

The proof of our main results follows:

Proof of Theorem 1.1. For each \( u \in X \), let

\[
\Phi(u) = \frac{\|u\|^p}{p}, \quad \Psi(u) = -\int_{\Omega} F(x,u)dx, \quad J(u) = -\int_{\Omega} \int_{0}^{u(x)} g(x,\xi)d\xi dx.
\]

Under the condition of Theorem 1.1, \( \Phi \) is a continuously Gâteaux differentiable and a sequentially weakly lower semicontinuous functional. Moreover, from Lemma 2 the Gâteaux derivative of \( \Phi \) admits a continuous inverse on \( X^* \). \( \Psi \) and \( J \) are continuously Gâteaux differential functionals whose Gâteaux derivative is compact. Obviously, \( \Phi \) is bounded on each bounded subset of \( X \). Particularly, for each \( u, \xi \in X \),

\[
\langle \Phi'(u), (\xi) \rangle = \int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta \xi(x)dx,
\]

\[
\langle \Psi'(u), (\xi) \rangle = \int_{\Omega} f(x,u)\xi(x)dx,
\]

\[
\langle J'(u), (\xi) \rangle = \int_{\Omega} g(x,u)\xi(x)dx.
\]
\( \langle J'(u), (\xi) \rangle = \int_\Omega g(x, u)\xi(x)dx. \)

Hence, it follows from (1.2) that the weak solutions of equation (P) are exactly the solutions of the equation
\[ \Phi'(u) + \lambda \Psi'(u) + \mu J'(u) = 0. \]

Due to \((j_3)\), for each \( \lambda > 0 \), it appears that \( \lim_{\|u\| \to +\infty} (\Phi(u) + \lambda \Psi(u)) = +\infty \) \hspace{1cm} (2.8)

and therefore the first assumption of Theorem 2.1 holds true.

Due to Lemma 2.1, there exists \( u^* \in X \) such that
\[ \Phi(u^*) = \frac{\|u^*\|^p}{p} > \frac{1}{p} \left( \frac{c}{K} \right)^p > 0 = \Phi(0) \] \hspace{1cm} (2.9)

and
\[ m(\Omega) \inf_{(x, s) \in \Omega \times [-c, c]} F(x, s) > \left( \frac{c}{K\|u^*\|} \right) \int_\Omega F(x, u^*(x))dx. \] \hspace{1cm} (2.10)

Now, we can obtain from (1.1) that
\[ \sup_{x \in \Omega} |u(x)| \leq K\|u\| \] \hspace{1cm} (2.11)

for each \( u \in X \). Let \( r = \frac{1}{p} \left( \frac{c}{K} \right)^p \), for each \( u \in X \) such that
\[ \Phi(u) = \frac{\|u\|^p}{p} \leq r. \]

So, it follows from (2.9) and (2.10) that
\[ \inf_{\{u : \Phi(u) \leq r\}} (\Psi(u)) = \inf_{\left\{ u : \frac{\|u\|^p}{p} \leq r \right\}} \int_\Omega F(x, u)dx \]
\[ \geq m(\Omega) \inf_{(x, s) \in \Omega \times [-c, c]} F(x, s) \]
\[ > \left( \frac{c}{K\|u^*\|} \right) \int_\Omega F(x, u^*(x))dx. \]

Therefore, using Proposition 1, with \( u_0 = 0 \) and \( u_1 = u^* \), we obtain
\[ \sup_{\lambda \geq 0 \ x \in X} \inf_{x \in X} (\Phi(x) + \lambda (h + \Psi(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda (h + \Psi(x))) \]
\hspace{1cm} (2.12)

and therefore the assumption (2.4) of Theorem 2.1 holds true.

Now, set \( I = [0, +\infty) \), by (2.8), (2.12), all the assumptions of Theorem 2.1 are satisfied. Hence, from Theorem 2.1 we conclude. \( \square \)

**Proof of Theorem 1.2.** From \((l_2)\) and since \( \int_{B(x^0, r_1)} F(d)dx = r_1^N \frac{\pi \theta}{1 + \theta} \int F(d)\), we have
\[ m(\Omega) \inf_{s \in \Omega \times [-c, c]} F(s) > \left( \frac{c}{\theta d} \right)^p \int_{B(x^0, r_1)} F(d)dx. \]

Therefore, the conclusion is drawn using Theorem 1.1. \( \square \)
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References


