An Accurate Method for Solving Riccati Equation with Fractional Variable-Order

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Abstract
In this article, the variable fractional order Riccati equation is investigated. The modified reproducing kernel Hilbert space method (RKHSM) is employed for computing an approximation to the proposed problem. The construction of the reproducing kernel using the orthonormal shifted Legendre polynomials is presented. Validity of RKHSM is ascertained by presenting three examples. The existence of the solution is proved. The convergence analysis is investigated. Error estimation for the proposed method is proven. The numerical examples show the efficacy of the proposed method. Theoretical and numerical results are presented.

Keywords: Riccati Equation, Fractional Variable-Order, RKHM.

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1 Introduction

Fractional calculus has several applications in physics, engineering, and economics, see [1]. Several researchers studied the theory of the fractional calculus such as [2, 3]. Since it is difficult to find the exact solution of such problems, more attention is given to the numerical schemes, see [4]-[10]. However, only few researchers have numerically analyze the variable-order fractional differential equations. Some of these methods can be found in [11, 12]. Chen et al. [13] solved the electronic energy transfer equation. Riccati differential equations appear in several applications such as [14]-[16]. However, few researches discussed the variable fraction order Riccati equation, see [16]-[18]. RKHSM is an efficient method for solving fractional differential equations. It is used to study various scientific models. The RKHSM presents the solution in a series solution form which is useful in several models. The approximate solution produced using this method is convergent to the exact solution rapidly. This method is used to study several scientific applications, see [19]-[27].

In this paper, the RKHSM is used to solve a class of variable-order fractional Riccati equation of the form

\[ D^{\alpha(x)}u(x) + b(x)u(x) + c(x)u^2(x) = f(x), \quad x \in (0, 1), \quad 0 < \alpha(x) < 1, \]  

subject to

\[ u(0) = \theta \]  

(1.1)  

(1.2)

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where \( b, c, f \in C^1 [0, 1] \), and \( \theta \) is a constant. We use a modified version of RKHSM to solve it. We derive the reproducing kernel and the modified version of RKHSM. In addition, we prove the existence and the uniformly convergence of the approximate solution using this approach. Error estimation for our approximation is proved.

This paper is organized as follows. In Section 2, some preliminaries which we use in this paper are presented. We contract the reproducing kernel with shifted Legendre polynomial form in Section 3. In Section 4, we present the modified RKHSM for solving variable-order fractional linear initial value problem. Convergence and error estimate are presented in this section. We present an iterative method to deal with Problem (1.1)-(1.2) in Section 5. This iterative method is a combination between the Homotopy perturbation method and the modified RKHSM. Some numerical results are presented in Section 6. Finally, conclusions are presented in Section 7.

2 Preliminaries

First, we write the definition of the Caputo fractional derivative and some of its properties.

**Definition 2.1.** The Riemann-Liouville fractional integral operator \( I^\alpha \) of order \( \alpha > 0 \) on \( L_1 [0, 1] \) is defined as

\[
I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds, \\
I^0 g(s) = g(s),
\]

where \( \Gamma(\xi) = \int_0^m t^{\xi-1} e^{-t} dt \), see [1].

Let \( A = [0, 1] \). If \( g \in L_1 (A), \alpha, \beta \geq 0 \), and \( \eta > -1 \), \( I^\alpha \) exists for any \( t \in A \) and

\[
I^\alpha t^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)} t^{\alpha+\eta}, \tag{2.3}
\]

**Definition 2.2.** The Caputo fractional derivative of order \( \beta \) is defined as

\[
D^\beta g(t) = I^{m-\beta} D^m g(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{g^{(m)}(s)}{(t-s)^{\beta-m+1}} ds,
\]

provided that the integral exists, where \( m = [\beta] + 1 \), \( |\beta| \) is the smallest integer greater than or equal to \( \beta, t > 0 \).

Some of the properties of this derivative are given as follows. For \( g \in L_1 [0, 1] \) and \( \eta, \zeta \geq 0 \):

\[
I^\beta D^\beta g(t) = g(t) - \sum_{l=0}^{m-1} g^{(l)}(0^+) \frac{t^l}{l!}, \tag{2.4}
\]

and

\[
D^\eta t^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\eta+1)} t^{\zeta-\eta}. \tag{2.5}
\]

Next, we define the Legendre polynomials and we write some of their properties which we will use in this paper.

**Definition 2.3.** The Legendre polynomials \( \{L_n(t) : i \text{ is nonnegative integer}\} \) are the eigenfunctions of

\[
((-1-t^2)L'_n(t))' + i(i+1)L_n(t) = 0, \quad t \in [-1, 1].
\]

Let \( SL_i(w) = L_i(2w - 1) \). It is easy to see that

\[
\int_0^1 SL_i(w) SL_j(w) dz = \frac{1}{2i+1} \delta_{ij},
\]
where $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. In this paper, we use the orthonormal shifted Legendre polynomials. For this reason, we define the orthonormal shifted Legendre polynomials to be

$$S_i(w) = \sqrt{\frac{2i+1}{2\pi}} L_i(w),$$

where $i = 0, 1, 2, \ldots$. Then,

$$\int_0^1 S_i(w) S_j(w) dz = \delta_{ij}.$$

RKHSM is a useful numerical method to solve nonlinear problems. The reproducing kernel is given by this definition.

**Definition 2.4.** Let $\Omega \neq \emptyset$. A function $M : \Omega \times \Omega \to \mathbb{C}$ is a reproducing kernel of the Hilbert space $H$ if and only if

- $M(\cdot, x) \in H$ for all $x \in \Omega$,
- $(f(\cdot), M(\cdot, t)) = f(t)$ for all $t \in \Omega$ and $f \in H$.

A Hilbert space with a reproducing kernel is called RKHS.

3 Construction of reproducing kernel with shifted Legendre polynomial form

Let

$$\Omega = \text{Span}\{S_0(x), S_1(x), \ldots, S_n(x)\}.$$

Define an inner product

$$(f(t), g(t)) = \int_0^1 f(t)g(t) dt$$

and the norm

$$\|f\| = \sqrt{(f(t), f(t))}.$$

It is easy to see that $\Omega$ is a Hilbert space. In the next theorem, we show that $\Omega$ is a RKS.

**Theorem 3.1.** $\Omega$ is a RKS and its kernel can be written as

$$K_n(t, s) = \sum_{i=0}^n S_i(t) S_i(s). \quad (3.6)$$

**Proof.** See [12].

4 Existence and convergence of the approximate solution of first-order variable fractional linear initial value problem

Now, we present the solution of the first-order variable fractional linear initial value problem using RKHSM:

$$D^{\alpha(x)} u(x) + b(x)u(x) = f(x), \quad x \in (0, 1), \quad 0 < \alpha(x) < 1, \quad (4.7)$$

subject to

$$u(0) = \theta, \quad (4.8)$$

where $b, f \in C^1 [0, 1]$, and $\theta$ is a constant.

Let

$$Lu(x) = D^{\alpha(x)} u(x) + b(x)u(x).$$

Then, $L$ is a linear operator. Let $\{x_1, x_2, \ldots, x_n\}$ be $n$ nodes in the interval $[0, 1]$. In this paper, we choose

$$x_j = \frac{1 + \cos\left(\frac{\pi j}{n}\right)}{2}, \quad j = 1, 2, \ldots, n.$$

Let
Next, we want to prove that $\psi_{i}(x) = L_{\eta}K_{n}(x, \eta)_{q-x}$ for $i = 1, 2, \ldots, n$, and $\psi_{n+1}(x) = S_{n+1}(x)$. Using Gram-Schmidt orthonormalization to generate orthonormal basis $\{ \tilde{\psi}_{i}(x) \}_{i=1}^{n+1}$ of $\tilde{\Omega} = \text{Span}\{\psi_{0}(x), \psi_{1}(x), \ldots, \psi_{n+1}(x)\}$

$$\psi_{i}(x) = \sum_{j=1}^{i} \alpha_{ij} \tilde{\psi}_{j}(x)$$

(4.9)

where $\alpha_{ij}$ are coefficients of Gram-Schmidt orthonormalization, $\alpha_{ii} > 0$, $i = 1, 2, \ldots, n+1$.

**Theorem 4.1.** (Existence). The approximate solution of Equations (4.7)-(4.8) is given by

$$u_{n}(x) = \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_{j}) \tilde{\psi}_{j}(x) + \gamma \psi_{n+1}(x).$$

(4.10)

**Proof.** Since $\{ \tilde{\psi}_{i}(x) \}_{i=1}^{n+1}$ is orthonormal basis, $K_{n}(x, \eta)$ is a reproducing kernel, and $L_{\eta}(\eta) = f(\eta)$, we get

$$u_{n}(x) = \sum_{i=1}^{n+1} (u(x), \tilde{\psi}_{i}(x)) \tilde{\psi}_{i}(x)$$

$$= \sum_{i=1}^{n} (u(x), \tilde{\psi}_{i}(x)) \tilde{\psi}_{i}(x) + (u(x), \tilde{\psi}_{n+1}(x)) \psi_{n+1}(x)$$

$$= \sum_{i=1}^{n} (u(x), \sum_{j=1}^{i} \alpha_{ij} \tilde{\psi}_{j}(x)) \tilde{\psi}_{j}(x) + (u(x), \tilde{\psi}_{n+1}(x)) \psi_{n+1}(x)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} (u(x), K_{n}(x, \eta)_{q-x}) \tilde{\psi}_{j}(x) + (u(x), \tilde{\psi}_{n+1}(x)) \psi_{n+1}(x)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} (u(x), K_{n}(x, \eta)_{q-x}) \tilde{\psi}_{j}(x) + (u(x), \tilde{\psi}_{n+1}(x)) \psi_{n+1}(x)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_{j}) \tilde{\psi}_{j}(x) + \gamma \psi_{n+1}(x)$$

where $\gamma = (u(x), \psi_{n+1}(x))$. To find $\gamma$, we set $u_{n}(0) = 0$.

Next, we want to prove that $u_{n}(x)$ is a solution to Equation at $x_{i}$ for $i = 1, 2, \ldots, n$. For any $k \in \{1, 2, \ldots, n\}$,

$$L_{\eta}u_{n}(x_{k}) = \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_{j}) L_{\eta} \tilde{\psi}_{j}(x)_{q-x_{k}} + \gamma L_{\eta} \psi_{n+1}(x)_{q-x_{k}}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_{j}) L_{\eta} (\tilde{\psi}_{j}(\eta), K_{n}(x, \eta)_{q-x_{k}}) + \gamma L_{\eta} (\psi_{n+1}(\eta), K_{n}(x, \eta)_{q-x_{k}})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_{j}) (\tilde{\psi}_{j}(\eta), L_{\eta} K_{n}(x, \eta)_{q-x_{k}}) + \gamma (\psi_{n+1}(\eta), L_{\eta} K_{n}(x, \eta)_{q-x_{k}})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_{j}) (\tilde{\psi}_{j}(\eta), \psi_{k}(\eta))_{q-x_{k}} + \gamma (\psi_{n+1}(\eta), \psi_{k}(\eta))_{q-x_{k}}.$$
Thus, for any \( m \in \{1, 2, \ldots, n\} \)

\[
\sum_{k=1}^{m} \alpha_m k u_n(x_k) = \sum_{k=1}^{m} \alpha_m \left[ \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_k)(\tilde{\psi}_j(\eta), \psi_k(\eta))_{\eta=x_k} + \gamma(\tilde{\psi}_{n+1}(\eta), \psi_k(\eta))_{\eta=x_k} \right]
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_k)(\tilde{\psi}_j(\eta), \psi_k(\eta))_{\eta=x_k} + \gamma(\tilde{\psi}_{n+1}(\eta), \psi_k(\eta))_{\eta=x_k}
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f(x_k)(\tilde{\psi}_j(\eta), \psi_m(\eta))_{\eta=x_k} + \gamma(\tilde{\psi}_{n+1}(\eta), \psi_m(\eta))_{\eta=x_k}
\]

\[
= \sum_{i=1}^{m} \alpha_i f(x_k) \delta_{im}
\]

where \( \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \).

This implies that

\[
u_n(x_k) = f(x_k)
\]

for \( k = 1, 2, \ldots, n \).

\[\square\]

**Theorem 4.2.** (Convergence and error estimation). Let \( u_n \) be the approximate solution of the Equation in space \( \tilde{\Omega} \) and \( u(x) \) be the exact solution of Equation. If \( \{x_i\}_{i=1}^{n} \) be \( n \) nodes in \( [0, 1] \) and \( b, f \in C^4[0, 1] \), then

\[
\sup_{x \in [0, 1]} ||u(x) - u_n(x)|| \leq Ch^3
\]

where \( C \) is constant and

\[
h = \max_{i \in \{1, 2, \ldots, n-1\}} |x_{i+1} - x_i|.
\]

**Proof.** Let

\[R_n(x) = ku_n(x) - f(x).
\]

From Theorem 4.1, we see that

\[R_n(x_i) = 0
\]

for \( i = 1, 2, \ldots, n \). Using Roll’s theorem, we get

\[R_n'(y_i) = 0, y_i \in (x_i, x_{i+1}), i = 1, 2, \ldots, n - 1
\]

and

\[R_n'(z_i) = 0, z_i \in (y_i, y_{i+1}), i = 1, 2, \ldots, n - 2
\]

\[\square\]

Let \( k_i(x) \) be interpolating polynomial of degree 1 of \( R_n'(x) \) at \( z_i \) and \( z_{i+1} \) where \( i \in \{1, 2, \ldots, n - 2\} \). Then, \( k \equiv 0 \) on \( [z_i, z_{i+1}] \). Then, there exists \( \zeta \in [z_i, z_{i+1}] \) and a constant \( \omega \) such that

\[R_n''(x) = R_n'(x) - k_i(x) = \frac{R^{(4)}(\zeta)}{2} (x - z_i)(x - z_{i+1}) \leq \omega h^2, \quad x \in [z_i, z_{i+1}].
\]

Thus,

\[
\sup_{x \in [0, 1]} |R_n''(x)| \leq \omega h^2
\]

where \( \omega = \max \{\omega_1, \omega_2, \ldots, \omega_{n-2}\} \). Hence,

\[R_n'(x) = \int_{y_i}^{x} R_n''(t)dt, \quad x \in [x_i, x_{i+1}], i = 1, 2, \ldots, n - 1
\]
which implies
\[ |R_n'(x)| \leq \sup_{x \in [0,1]} |R_n(x)| |x - y_i| \leq \omega h^2 h = \omega h^3. \]

Moreover,
\[ R_n(x) = \int_{x_i}^{x} R_n'(v) dv, \quad x \in [x_i, x_{i+1}], i = 1, 2, ..., n - 1 \]
which implies that
\[ |R_n(x)| \leq \sup_{x \in [0,1]} |R_n'(x)| |x - x_i| \leq \omega h^3 h = \omega h^4. \]

Thus,
\[ \sup_{x \in [0,1]} |R_n''(x)| \leq \omega h^4. \]

Now,
\[ \|R_n\|_1^2 = R_n(0) + \int_0^1 R_n^2(x) dx \leq \omega h^8 + \omega^2 h^6 \leq \kappa^2 h^6 \]
for some positive constant \( \kappa \). Thus,
\[ \|u - u_n\|_{\Omega} = \|k^{-1}R_n(x)\|_{\Omega} \leq \|k^{-1}\|_1 \|R_n\|_1 \leq \|k^{-1}\|_{\Omega} \kappa h^3 = \rho h^3 \]
where \( \rho = \|k^{-1}\|_{\Omega} \kappa \). Then,
\[ \sup_{x \in [0,1]} |u(x) - u_n(x)| \leq \rho h^3. \]

5 Analysis of RKHSM for first-order variable fractional nonlinear initial value problem

Now, we want to present the solution of the Riccati Problem using RKHSM:
\[ D^{\alpha(x)}y(x) + b_1(x)y(x) + c_1(x)y^2(x) = g(x), \quad x \in (0,1), \quad 0 < \alpha(x) < 1, \]
\[ y(0) = \theta, \]  
(5.11)  
(5.12)

where \( b_1, c_2, \) and \( g \) are continuous functions on \([0,1]\) and \( \theta \) is a constant. Let
\[ F(x, y) = c_1(x)y^2(x). \]

Let
\[ H(y, \lambda) = D^{\alpha(x)}y(x) + b_1(x)y(x) - g(x) + \lambda F(x, y) = 0 \]
(5.13)

where \( \lambda \in [0,1] \). If \( \lambda = 0 \), we get
\[ D^{\alpha(x)}y(x) + b_1(x)y(x) - g(x) = 0 \]
which can be solved by using RKHSM as we described in the previous section. If \( \lambda = 1 \), we turns out to be problem (5.11). Following the Homotopy Perturbation method [26], we expand the solution in terms of the Homotopy parameter \( \lambda \) as
\[ y = y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \ldots. \]
(5.14)

Substitute Equation (5.14) into Equation (5.13) and equating the coefficients of the identical powers of \( \lambda \) to get
\( \lambda^0 : \quad D^{\alpha(x)}y_0(x) + b_1(x)y_0(x) = g(x), \quad u_0(0) = 0, \)
\( \lambda^1 : \quad D^{\alpha(x)}y_1(x) + b_1(x)y_1(x) = -F(x, \sum_{i=0}^{\infty} \lambda^i y_i) \big|_{\lambda=0}, \quad y_1(0) = 0, \)
\( \lambda^2 : \quad D^{\alpha(x)}y_2(x) + b_1(x)y_2(x) = -\frac{dF(x, \sum_{i=0}^{\infty} \lambda^i y_i)}{d\lambda} \big|_{\lambda=0}, \quad y_2(0) = 0, \)
\( \lambda^3 : \quad D^{\alpha(x)}y_3(x) + b_1(x)y_3(x) = -\frac{d^2F(x, \sum_{i=0}^{\infty} \lambda^i y_i)}{d\lambda^2} \big|_{\lambda=0}, \quad y_3(0) = 0, \)
\( \vdots \)
\( \lambda^k : \quad D^{\alpha(x)}y_k(x) + b_1(x)y_k(x) = -\frac{d^{k-1}F(x, \sum_{i=0}^{\infty} \lambda^i y_i)}{d\lambda^{k-1}} \big|_{\lambda=0}, \quad y_k(0) = 0. \)

Using RKHSM which is described in the previous section and we obtain

\[
y_k(x) = \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f_k(x_j) \psi_i(x) + \gamma_k \psi_{n+1}(x), \quad k = 0, 1, 2, \ldots
\]

where

\[
f_0(x) = g(x), \quad f_1(x) = -F(x, \sum_{i=0}^{\infty} \lambda^i y_i) \big|_{\lambda=0} \quad \vdots \quad f_k(x) = -\frac{d^{k-1}F(x, \sum_{i=0}^{\infty} \lambda^i y_i)}{d\lambda^{k-1}} \big|_{\lambda=0}, \quad k = 2, 3, \ldots
\]

One can see that

\[
y(x) = \sum_{k=0}^{\infty} y_k(x) = \sum_{k=0}^{\infty} \left( \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f_k(x_j) \psi_i(x) + \gamma_k \psi_{n+1}(x) \right). \quad (5.15)
\]

We approximate the solution of Problem (5.11)-(5.12) by

\[
y_{N,M}(x) = \sum_{k=0}^{M} \left( \sum_{i=1}^{n} \sum_{j=1}^{i} \alpha_{ij} f_k(x_j) \psi_i(x) + \gamma_k \psi_{n+1}(x) \right). \quad (5.16)
\]

### 6 Results and Discussion

We apply the method outlined in the previous sections to solve numerically the following three examples. Let \( M = 12 \) for all examples considered in this paper.

**Example 6.1.** Let

\[
D^{\alpha(x)}u(x) + e^x u(x) + 2u^2(x) = g(x), \quad x \in (0, 1)
\]

subject to

\[
u(0) = 1
\]

where

\[
\alpha(x) = \frac{3 + \cos x}{5}, \quad f(x) = \frac{2}{\Gamma(3 - \alpha(x))} x^{2-\alpha(x)} + e^x (x^2 + 1) + 2(x^2 + 1)^2.
\]

The exact solution is \( u(x) = x^2 + 1 \). Figure 1 shows the graphs of the approximate and the exact solutions. The absolute error is shown in Table 1.
Example 6.2. Let

\[ D^{\alpha(x)}u(x) + e^{2x}u(x) + e^{x}u^2(x) = g(x), \quad x \in (0, 1) \]

subject to

\[ u(0) = 1 \]

where

\[ \alpha(x) = \frac{3 + \sin x}{5}, \quad f(x) = \frac{24}{\Gamma(5 - \alpha(x))}x^{4-\alpha(x)} + e^{2x}(x^4 + 1) + e^x(x^4 + 1)^2. \]

The exact solution is \( u(x) = x^4 + 1 \). Figure 2 shows the graphs of the approximate and the exact solutions. The absolute error is shown in Table 2.
Example 6.3. Let

\[ D^{\alpha(x)} u(x) + (x^2 + 1)u(x) + e^x u^2(x) = g(x), \quad x \in (0, 1) \]  

subject to

\[ u(0) = 0 \]  

where

\[ \alpha(x) = \frac{e^x}{3}, \quad f(x) = \frac{6}{\Gamma(4 - \alpha(x))} x^3 - x^5 + x^3 + x^6 e^x. \]

The exact solution is \( u(x) = x^3 \). Figure 3 shows the graphs of the approximate and the exact solutions. The absolute error is shown in Table 3.

| \( x \) | \( |u(x) - u_{app}(x)|\) |
|--------|--------------------------|
| 0.0    | 0.0                      |
| 0.1    | 2.1 \times 10^{-10}     |
| 0.2    | 2.2 \times 10^{-10}     |
| 0.3    | 3.1 \times 10^{-10}     |
| 0.4    | 3.3 \times 10^{-10}     |
| 0.5    | 3.4 \times 10^{-10}     |
| 0.6    | 3.4 \times 10^{-10}     |
| 0.7    | 3.6 \times 10^{-10}     |
| 0.8    | 3.7 \times 10^{-10}     |
| 0.9    | 3.7 \times 10^{-10}     |
| 1.0    | 3.8 \times 10^{-10}     |
Conclusions

We study the fractional variable-order Riccati equation. The modified reproducing kernel Hilbert space method (RKHSM) is employed to compute an approximation to the proposed problem. The construction of the reproducing kernel based on the orthonormal shifted Legendre polynomials is presented. The validity of the RKHSM is ascertained by presenting three of our examples. Theoretical and numerical results are presented. Error estimation to the proposed method is proven. The numerical examples show that the proposed method is very efficient and can be used for similar problems.

Conflict of Interests

The authors declares that there is no conflict of interests regarding the publication of the paper.

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