Approximate solution for Heat equation via the Pade approximation

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Abstract
In this paper the classical explicit approximation to \( \frac{\partial^2 U}{\partial t^2} = \frac{\partial U}{\partial t} \) is given by approximating exponential function by Pade approximate. The condition of stability is obtained by proving a theorem.

Keywords: partial differential equation, Finite difference, Pade approximation, Heat equation.

1 Introduction
In mathematics, a partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. (A special case are ordinary differential equations (ODEs), which deal with functions of a single variable and their derivatives.) PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model. The theory of partial differential equations has been one of the most important fields of study in mathematics. PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalized similarly in terms of PDEs.

In this paper we want to use finite difference and (2, 1) Pade approximation for heat equation. In mathematics a Pade approximant is the ”best” approximation of a function by a rational function of given order — under this technique, the approximant’s power series agrees with the power series of the function it is approximating. The technique was developed around 1890 by Henri Pade, but goes back to Georg Frobenius who introduced the idea and investigated the features of rational approximations of power series.

The Pade approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge. For these reasons Pade approximants are used extensively in computer calculations.

The paper organized as follows:
In section 2 we will introduce the finite difference equations for heat equation via Pade approximation for exponential function. In section 3 the \( L_0 \) stability will be discussed and a condition for stability of the method will be estimate and conclusion is drown in section 4.

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2 Preliminaries

In this paper \( \frac{\partial U}{\partial x} \) and \( \frac{\partial U}{\partial t} \) mean the first order derivative respect to \( x \) and \( t \) respectively. Similarly \( \frac{\partial^2 U}{\partial x^2} \) and \( \frac{\partial^2 U}{\partial t^2} \) show the second order derivative.

**Definition 2.1.** The second order partial differential equations in the dependent variable \( u \) and the independent variables \( x, t \) is defined as follows:

\[
a \frac{\partial^2 U}{\partial x^2} + b \frac{\partial^2 U}{\partial x \partial t} + c \frac{\partial^2 U}{\partial t^2} + d \frac{\partial U}{\partial x} + e \frac{\partial U}{\partial t} + fU + g = 0
\]

where the coefficients are functions of \( x \) and \( t \) and do not vanish simultaneously. We shall assume that the function \( U \) and the coefficients are twice continuously differentiable in some domain \( R \).

**Definition 2.2.** The equation

\[
a \frac{\partial^2 U}{\partial x^2} + b \frac{\partial^2 U}{\partial x \partial t} + c \frac{\partial^2 U}{\partial t^2} + d \frac{\partial U}{\partial x} + e \frac{\partial U}{\partial t} + fU + g = 0
\]

is said to be elliptic when \( b^2 - 4ac < 0 \), parabolic when \( b^2 - 4ac = 0 \), and hyperbolic when \( b^2 - 4ac > 0 \). For example the best known of two dimensional equations are Poisson’s equation

\[
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial t^2} + g = 0 \quad \text{elliptic equation}
\]

and heat equation

\[
\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \quad \text{parabolic equation}
\]

**Definition 2.3.** The rational function

\[
R_{S,T}(\theta) = \frac{1 + p_1 \theta + \ldots + p_T \theta^T}{1 + q_1 \theta + \ldots + q_S \theta^S} = \frac{P_T(\theta)}{Q_S(\theta)}
\]

is called the \((S, T)\) Padé approximant of order \((S + T)\) to \( e^\theta \). For example the \((2, 1)\) Padé approximant is as follow

\[
R_{2,1}(\theta) = \frac{1 + \frac{1}{2} \theta}{1 - \frac{1}{2} \theta + \frac{1}{6} \theta^2}
\]

and its leading error term is \(-\frac{1}{72} \theta^4\).

**Definition 2.4.** When \( |R_{S,T}(k \lambda_s)| < 1 \) for \( s = 1, \ldots, N - 1 \) the equations are said to be \( A_0 \) – stable.

**Definition 2.5.** As a consequence, a set of difference equation is said to be \( L_0 \) – stable if \( |R_{S,T}(k \lambda_s)| < 1 \) for \( s = 1, \ldots, N - 1 \) and \( R_{S,T}(k \lambda_s) \to 0 \) as \( k \lambda_s \to -\infty \), where \( \lambda_s \) is real, negative, and non-zero.

3 Finite Difference Equations Via Padé Approximation

Consider the heat equation:

\[
\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \quad 0 < x < X, \quad t > 0
\]
where $U$ satisfies the initial condition:

$$U(x, 0) = g(x) \quad 0 \leq x \leq X$$

and has known boundary values at $x = 0$ and $X$, $t > 0$. If we suppose:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2} \left( U(x-h,t) - 2U(x,t) + U(x+h,t) \right) + O(h^2)$$

then the equation (3.1) can be written as:

$$\frac{dU}{dt} = \frac{1}{h^2} \left( U(x-h,t) - 2U(x,t) + U(x+h,t) \right) + O(h^2) \quad (3.2)$$

Subdivide the interval $0 \leq x \leq X$ into $N$ equal subintervals by the grid lines $x_i = ih$, $i = 0, \ldots, N$, where $Nh = X$ and write down equation (3.2) at every mesh point $x_i = ih$, $i = 1, \ldots, N-1$, along time level $t$. It then follows that the values $V_i(t)$ approximating $U_i(t)$ will be the exact solution values of the system of $(N-1)$ ordinary differential equations

$$\frac{dv_i(t)}{dt} = \frac{1}{h^2} \left( V_0 - 2V_i + V_2 \right)$$

$$\frac{dv_2(t)}{dt} = \frac{1}{h^2} \left( V_1 - 2V_2 + V_3 \right)$$

$$\vdots$$

$$\frac{dv_{N-1}(t)}{dt} = \frac{1}{h^2} \left( V_{N-2} - 2V_{N-1} + V_N \right)$$

Where $V_0$ and $V_N$ are known boundary-values. These can be written in matrix form as

$$\frac{dV(t)}{dt} = AV(t) + b$$

Where $V(t) = [V_1 \ V_2 \ \cdots \ V_{N-1}]^T$, $b$ is a column vector of zeros and known boundary-values and matrix $A$ of order $(N-1)$ is given by

$$A = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -2 \end{bmatrix}$$

The solution of the ordinary differential equation is

$$\frac{dV}{dt} = AV(t) + b$$

Where $A$ and $b$ are independent of $t$ and $V(t)$ satisfies the initial condition $V(0) = g$, so we have

$$\frac{dV}{dt} - AV = b \quad (3.3)$$
Consider the homogeneous equation
\[ \frac{dV}{dt} - AV = 0 \Rightarrow V(t) = c \exp(-At) \]

Hence by differentiation of the (3.4) we have:
\[ 0 = \frac{dV}{dt} \exp(-At) - AV \exp(-At) = \exp(-At) \left( \frac{dV}{dt} - AV \right) \]

By equation (3.3) it follows that
\[ \frac{d}{dt} (V \exp(-At)) = b \exp(-At) \]
\[ \Rightarrow V \exp(-At) = -\frac{1}{A} \exp(-At) + c \]

so V is developed as follow :
\[ V = -bA^{-1} + c \exp(At) \]

On the other hand by initial value of (1):
\[ V(0) = g \Rightarrow c = g + bA^{-1} \]
\[ \Rightarrow V(t) = -bA^{-1} + (g + bA^{-1}) \exp(At) \]

Hence value of V in other points is obtained by following iterated equation:
\[ V(t+k) = -bA^{-1} + (V(t) + bA^{-1}) \exp(Ak) \]

If \( b = 0 \) then \( V(t+k) \) can be written as follow:
\[ V(t+k) = \exp(Ak)V(t) \] (3.4)

Now we approximate exponential function by (2, 1) Padé approximation:
\[ R_{2,1}(\theta) = \frac{1 + \frac{1}{3} \theta}{1 - \frac{1}{3} \theta + \frac{1}{6} \theta^2} \]

Result is
\[ \exp(kA) = \frac{I + \frac{1}{3} kA}{I - \frac{2}{3} kA + \frac{1}{6} k^2 A^2} \] (3.5)

Hence by replacing equation (3.5) in (3.4) an approximate iterative equation is obtained by:
\[ V(t+k) = \frac{I + \frac{1}{3} kA}{I - \frac{2}{3} kA + \frac{1}{6} k^2 A^2} V(t) \]
\[ \Rightarrow (I - \frac{2}{3} kA + \frac{1}{6} k^2 A^2) V(t+k) = (I + \frac{1}{3} kA) V(t) \] (3.6)

Where the matrix structure is as the follow:
\[ I + \frac{1}{3} kA = \begin{bmatrix}
1 - \frac{2}{3} r & \frac{1}{3} r & \\
\frac{1}{3} r & 1 - \frac{2}{3} r & \frac{1}{3} r \\
& & \ddots & \ddots & \ddots \\
& & & \frac{1}{3} r & 1 - \frac{2}{3} r
\end{bmatrix} \] (3.7)
and
\[ I - \frac{2}{3}kA + \frac{1}{6}K^2A^2 = \]
\[
\begin{bmatrix}
  r^2 + \frac{4}{3}r + 1 & -\frac{2}{3}(r^2 + r) & \frac{1}{3}r^2 \\
  -\frac{2}{3}(r^2 + r) & r^2 + \frac{4}{3}r + 1 & -\frac{2}{3}(r^2 + r) \\
  \frac{1}{3}r^2 & -\frac{2}{3}(r^2 + r) & r^2 + \frac{4}{3}r + 1 \\
  & & \\
  & & \\
  & & \\
  \frac{1}{3}r^2 & -\frac{2}{3}(r^2 + r) & r^2 + \frac{4}{3}r + 1
\end{bmatrix}
\]

So the equation (3.6) can be written as follow
\[
\left(I - \frac{2}{3}kA + \frac{1}{6}K^2A^2 \right) \begin{bmatrix} u_{1,j+1} \\ \vdots \\ u_{N-1,j+1} \end{bmatrix} = \left(I + \frac{1}{3}kA \right) \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{N-1,j} \end{bmatrix}
\]

By replacing (3.7) and (3.8) in (3.9) we get the following equation :
\[
\frac{1}{6}r^2u_{i-2,j+1} - \frac{2}{3}(r^2 + r)u_{i-1,j+1} + (r^2 + \frac{4}{3}r + 1)u_{i,j+1} - \frac{2}{3}(r^2 + r)u_{i+1,j+1} + \frac{1}{6}r^2u_{i+2,j+1} \\
= \frac{1}{3}ru_{i-1,j} + (1 - \frac{2}{3}r)u_{i,j} + \frac{1}{3}ru_{i+1,j}
\]

Or
\[
\frac{1}{6}r^2 (u_{i-2,j+1} + u_{i+2,j+1}) - \frac{2}{3}(r^2 + r) (u_{i-1,j+1} + u_{i+1,j+1}) + (r^2 + \frac{4}{3}r + 1)u_{i,j+1} \\
= \frac{1}{3}ru_{i-1,j} + (1 - \frac{2}{3}r)u_{i,j} + \frac{1}{3}ru_{i+1,j}
\]

4 A Necessary Condition for Stability

Now to more ensure that the problem has a solution we must be sure that the method is stable. It means the small perturbations in the initial data, will only lead to small changes in the solutions. To prove \(L_0\)- stability two following theorem should be established.

Theorem 4.1. The difference equation by (2,1) Pade approximate is \(L_0\)- stable.

Proof. For \(L_0\)- stability we have
\[
R_{2,1}(-z) = \frac{I - \frac{1}{2}z}{I + \frac{r}{2}z + \frac{1}{2}z^2} \Rightarrow \left\{ \begin{array}{c} |R_{2,1}(-z)| < 1 \\
\lim_{z \to -\infty} R_{2,1}(-z) = 0 \end{array} \right.
\]

It is clear the method is \(L_0\)- stable.

Theorem 4.2. Topic \(M\) then for
\[
\left\{ \begin{array}{c} r < \frac{1 - \sqrt{TM}}{4M} \\
or \\
\frac{1 + \sqrt{TM}}{4M} \end{array} \right.
\]

the method is stable.

Proof. Suppose that \(\lambda_s, s = 1, \cdots, N - 1\) are the eigenvalue of the matrix \(A\), so
\[
\lambda_s = -\frac{4}{h^2} \sin^2 \frac{s\pi}{2N}
\]
and $\mu_s, s = 1, \cdots, N - 1$ are the eigenvalue of the matrix $\exp(kA) = \frac{1 + \frac{1}{2}kA}{1 - \frac{1}{2}kA + \frac{1}{6}k^2A^2}$. Therefore

$$\mu_s = \frac{1 + \frac{1}{2}k\lambda_s}{1 - \frac{1}{2}k\lambda_s + \frac{1}{6}k^2\lambda_s^2}$$

If we put So

$$\mu_s = \frac{1 - 4r\sin^2\frac{\pi}{2N}}{1 + \frac{8}{3}r\sin^2\frac{\pi}{2N} + \frac{8}{3}r^2\sin^4\frac{\pi}{2N}}$$

For stability we must have

$$|\mu_s| < 1 \Rightarrow -1 < \mu_s < 1$$

Or

$$-1 < \frac{1 - 4r\sin^2\frac{\pi}{2N}}{1 + \frac{8}{3}r\sin^2\frac{\pi}{2N} + \frac{8}{3}r^2\sin^4\frac{\pi}{2N}} < 1$$

Hence if we put $\sin^2\frac{\pi}{2N} = M$, be achieved

$$\left\{ \begin{array}{l}
    r < \frac{1 - \sqrt{TM}}{4M} \\
    or \\
    r > \frac{1 + \sqrt{TM}}{4M}
\end{array} \right.$$  

\[\square\]

5 Conclusion

We presented different method for solving \[ \frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \]. This method based on using $(2, 1)$ Padé approximation to exponential function. We obtained necessary conditions for stability of this method and we have shown that the method is $L_0$- stable.

References


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