Assessing the Effectiveness of a Single Curriculum for a Group of Students with Different Mathematics Literacy

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Abstract

In recent years, applicants to Azad universities are different than those in last years. This difference arose from the unequal ability in learning mathematics by students. Traditionally university students were K-12 graduates and because of severe competition to enter the university, they had equivalent math literacy. In contrast recent applicants to universities have been graduated from different majors in high school in different past years. Since passing more years after graduating from high school causes degradation in math literacy, this kind of applicants has more divergent math literacy. When having university students with more divergent math literacy, it may be required to have different syllabus for different students at the same course. In this paper we model the learning phases of a math course as a network in which the arcs are sub-topics and the logical precedence between arcs exhibits the prerequisite relation between sub-topics. The ability to accomplish the course is equal to pass all sub-topics. This problem can be modeled as finding the shortest path in a network with probabilistic arc lengths. Traditionally, finding the shortest path in a probabilistic network is done with PERT. Since PERT is known to be over optimistic, we have developed a new method to find a more realistic estimate for completion time in the network on sub-topics in a course.

Keywords: Assessment, mathematics literacy, probabilistic network, shortest path problem.

1 Introduction

Applicants to Azad universities comprise of two groups. First are young k-12 graduates and the second are middle aged people. The most important aspect of the second group is that they have been graduated from high school in deferent last years. Passing more years from the high school ages beside been employed in different careers and professions, causes the second group to be at more different abilities and interest in mathematics. Consequently this issue causes different math literacy within the second group and between first and second group of applicants to Azad University. Previously more applicants to Azad University were among the first group but in recent years the weight of the second group is increasing.

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When teacher lectures the course topics in such classes, obviously different layers of learners form, some of them grab the course material at a glance while a minority requires more examples and explanation. Having such differences in learning time required for each topic, leads us to an assumption in which “it may be required to have different syllabus for different students at the same course”.

Learning process of a math course can be modeled as a network in which its arcs are sub-topics and the logical precedence between arcs exhibits the prerequisite relation between learning the sub-topics. Accomplishing the course is equivalent to pass all the arcs (sub-topics) of such network. The problem of finding the time required to accomplish the learning project for a math course can be modeled as finding the shortest path in a network with probabilistic arc lengths. This problem is discussed under the title of planning and scheduling projects. Traditionally, finding the shortest path in a probabilistic network is done with PERT. Since PERT is known to be over optimistic, we have developed a new method to find a more realistic estimate for completion time in the network on sub-topics in a course.

In the following sections of this paper first we review the techniques for finding the shortest path in probabilistic networks and then discuss their shortcomings. In section 4 we review some bounds on the mean and variance of a merge event. In section 5 we present our calculated bounds on such statistic and evaluate its efficiency in action.

2 Learning a course as accomplishing a network of activities

Zhen-ting et. al. (2010) [1] argue that critical path method (CPM) and program evaluation and review technique (PERT) constitute the main tools utilized for planning and scheduling. Although we know that most activity durations are stochastic in nature, CPM which is regarded as a traditional method thrives to schedule a project and take deterministic durations into account. Thus, random variables and some distribution function can present activity duration in an effective manner. The underlying assumption of PERT maintains that each activity consists in a random variable whose behavior is based on some known probability distributions.

The results generated by PERT for project completion time are almost known to underestimate the real value observed in action. One of the most important causes to this problem usually is reported as merge event bias occurred during the PERT calculations. Traditional Monte Carlo simulation is used to overcome this fault. In this paper we use the recent findings in statistics to model the mean value and variance of the completion time of a merge event and refine these findings to non-identically distributed activity durations. Simulations made to compare the results generated by our developed algorithm versus PERT shows that we can provide more effective estimated amounts of mean value and variance of merge events and consequently project completion time.

The DuPont Corporation proposed the critical path method (CPM) in the late 1950s. This almost coincided the time when the program evaluation and review technique (today known as PERT) was developed by General Dynamics and the US Navy (Laslo & Gurevich, 2011) [2]. CPM and PERT are two frequently-used methods utilized in all sorts of projects. In CPM, it is assumed that project activity durations remain fixed. So, this method embarks on first the calculation of the project paths then the identification of the paths which possess the longest completion time. The path determining the project completion time is referred to as the critical path abbreviated as CP. There may be several parallel critical paths and parallel non-CPs in a project through the network.

As mentioned above, the activity durations are considered to be fixed in the CPM method but the statistical PERT was primarily designed for planning and controlling projects that involve sort of uncertainty (Zohar and Gurevich, 2011). Such uncertainty is chiefly related to the time and cost which each activity calls for. PERT draws on logic diagrams so that it can carry out an analysis of activity durations. This also enables it to focus on the event. With the assumption of fixed activity durations, PERT becomes able to give an estimation of the probability of meeting specified completion dates. The duration is something random in
nature, which hinders the definition of CPs. Thus, in order to determine the probability of a contractual due time, PERT draws on a quantified risk assessment.

CPs can be easily solved by CPM but the problem is that since both the durations of activities and paths constitute random variables CPM is not able to display a suitable performance in stochastic networks. The enumeration of all the probable paths connecting the start and end node available in the project network is regarded as one of the most common approaches (Chu et. al., 2014a) [3]. Attempt to search for CP in stochastic network is something justified having its own reasons (Chu et. al., 2014b) [4]:

1) In order to solve CP, all completion paths' durations available in the network have to be quantified and listed. Perhaps it is computationally inefficient especially for projects with numerous activities.
2) The critical path (CP) relies on the results obtained from the sorting of all the completion paths available in the project. This sorting calls for a considerable amount of calculations among the Probability Distribution Functions (PDF) related to path durations.
3) Regarding the maximum or minimum operations in the network, some sort of bias is found.
4) The shared sub-paths existing in the network creates path dependence that engenders bias between the results of this method and real world results.
5) In a stochastic network, all the completion can be probable CPs. As a result, it is necessary to have a criterion in order to be able to make a comparison of the probability values of all the completion paths, and select one as the critical path (CP).

3 Common approaches to deal with uncertainty

As it was noted before, analytical and simulation are two well-known approaches to deal with stochastic activity durations. Here we review PERT as an analytical approach and MCS as a simulation based approach.

3.1. PERT

As far as activity duration and the project network are concerned, PERT was the first attempt to calculate the uncertainty existing in them. Evidence indicates the shortcomings of PERT but many continue to suggest this method as the solution to uncertainty in activity duration networks.

The team working on PERT endeavored to model uncertainty. Their suggestion was to give three estimations for all of the activity durations, which were labeled as the optimistic, the most-likely, and the pessimistic estimate. According to Lootsma (1989) [6] the team drew on an appropriate beta distribution for the purpose of modeling activity duration as a random variable. Then, they put forward a simple method which tried to quantify the expectation and the variance of the event times in an approximate manner. A systematic error exists in the calculation procedure of this version of PERT. In this version, in cases where several activities join together the earliest event time constitutes the maximum number of generally random variables. This causes the expected event times. And therefore, the expected minimum project duration to be estimated lower than they are. This drawback motivated some researchers to develop an improved version of the method by a more sophisticated maximization and also a simulation. The maximization employed was not approximate. Later these researchers acknowledged that the task of finding a distribution or the moments of the minimum project duration was too difficult, theoretically too complicated even in case of independent activity durations, and computationally too expensive (Lootsma, 1989) [6]. So, they had to adopt a worst-case analysis procedure in which they draw on the maximization over the class of joint distributions not incompatible with the marginal activity durations in order to obtain an upper bound expected violation of a predetermined project delivery time.
Some remarkable limitations of PERT include (Guo, 2001 and mosaic projects) [5]:

i. No Statistical Validation
Existence of simplifying assumptions in the process of approximating the expected value that does not exactly match with the analytical results from statistics.

ii. The existence of Exclusively One Distribution
With regard to the type of work involved in an activity, there will be a variety of the potential outcomes for that activity. PERT is confined to the 'modified Beta' distribution whereas the modern tools utilize various distributions in order to more correctly depict the variability that exists in each activity. Some distributions used by modern tools include Triangular, Normal (Gaussian), Beta, and Flat. This causes a decrease in calculation accuracy.

iii. Calculation of SD as an Approximation
The way in which PERT calculates SD generates significant errors. More complicated errors are required to conduct an accurate assessment of the actual SD for a dataset.

iv. Merge Event Bias by PERT
The PERT possesses a single-pass calculation which is based on the critical path (CP) notion. It ignores the fact that the completion time of some critical paths may be longer under some conditions. In the sketch outlined above, the project outcome will be delayed provided that the critical path is obtaining an outcome near to its 'most likely' date, but the sub-critical path is proceeding towards a pessimistic outcome. The calculations done by PERT are unable to recognize this state.

Merge bias was recognized by those who designed in originally in the early 1960's. It was also considered as acceptable and referred to as the 'PERT Merge Bias'. That PERT recognizes exclusively the PERT critical path thus ignoring the parallel slack paths. Such paths are able to add to the risk potential at any merge point. This causes PERT to give unreal lower estimates of the schedule risk at any merge point. There is a simulation termed as Monte Carlo Simulation which is usually used to determine the bias extent. Merge bias grows if the activity within the network consists of numerous activities and perhaps more merge events. This effect is highlighted when there is a decrease in float on sub-critical paths, and/or there is an increase in the number of sub-critical paths that are near to the CP. The likelihood that the overall project is completed on time decreases. This problem is concealed by the PERT merge bias effect with the consequence that optimistic predictions are always produced by PERT. More realistic predictions of the probable range of completion dates can be generated by techniques like Monte Carlo simulation that calculate the entire network.

3.2. Monte Carlo Simulations (MCS)
The Rand Corporation used Monte Carlo for the scheduling purpose first in 1963. It drew on the data collected for the analysis called normal PERT’ analysis that time. (mosaic projects).

Using a simulation procedure, researchers applied Monte Carlo simulation to analyze networks in 1960s. Each activity's probable distribution function is obtained or assumed prior to the simulation of a construction network (Guo, 2001) [5]. The completion probability of each activity is given stochastic values in the process of each replication in the simulation. The calculation of the completion probability of the activities makes it possible to compute their durations using both the CP standard deviation and the expected duration. If we sum up the duration of all activities in the path, we will be able to compute each path's duration. The duration of the longest path is equal to the network duration. In probability-based approaches such as PERT, first the activity duration has to be represented by a random variable. Then, the random variable has to be characterized by a probability distribution function (PDF).
This makes it possible to compute the random variable statistics such as the mean value and variance. It also makes it possible to compute the project completion time and its probability distribution function (Long and Ohsato, 2008) [8]. Because some activities in the above projects are unique, and historical data about activity duration are absent, it is possible that a project manager is not able to characterize the random variables through probability distributions in an accurate manner but only through accurate mean value and variance. An advantage of MCS over the PERT is that it makes it possible to compute the criticality index. The criticality index is defined as the likelihood that an activity will be in the critical path state. We may have a proportion of runs in which an activity encounters the critical path. The criticality index is computed as the proportion of such runs. In contrary, PERT assumes it binary (either 100% or 0%).

Although MCS eliminates the disadvantages of PERT but still has some shortcomings itself. Monte Carlo simulation (MCS) encounters its own problems, including problems with the PDF of a single activity, and the correlation that exists between variables (Guo, 2001) [5]. The main strength point of MCS lies in the fact that the results that it generates produced an unbiased estimation of the completion distribution of the project. Subjective judgments can be made since there may be different distributions for activities, and we lack the required data in this regard. MCS makes it possible to compute the criticality index defined as the likelihood that an activity will be in critical path state.

Drawbacks of MCS are as follows:

a) It is difficult to calculate the correlation between activities, and
b) It is a time-consuming and expensive tool with respect to computer time. The final disadvantage of MCS which still remains unsolved is that it does not give the mean value and variance of the events in the project network directly; and the user is subjected to do an additional calculation to find these values (Guo, 2001) [5].

4 Mean value and Variance of the Completion time for sample maximum

According to Pontrandolfo (2000) [9] most of the procedures designed for analysis purpose like as the PERT traditional method resulted in an estimation of the earliest completion time that is both optimistic and biased. This situation arises from the notion of “merge event bias problem” mentioned previously. This is a stochastic problem in nature, which refers to a random variable. This random variable is again the maximum value of a set of random variables. So, no one can say that this is a variable statistically considered as independent.

For instance, we turn to consider two activities A and B. Both of them come to a conclusion with the same node related to an event denoted as P. Suppose that TA and TB - the durations of activities A and B- are a random variable, and that the date when event P occurs is also a random variable TP. From Logic point of view, TP can be defined by the expression as max{TA, TB}. It follows that E[max{TA, TB}] equals the expected date E[TP] that the event P happens but according to the PERT traditional method, the assumption was as follows: E[TP]=max{E[TA ], E[TB]}. Such an assumption led to the underestimation of the exact value, So, the merge event bias resulting from this assumption is ascribed to the uncertainty existing in activity duration estimations along with the logic of network nodes. The bias is emphasized by the complexity that prevails in the network. In fact, it raises with the number of parallel paths that lead to the end event of the network (Pontrandolhofo, 2000).

In statistics, TP is known as a statistics called sample maximum. If all TA, TB, TC,… are Identically Independently Distributed (IID) random variables, there are various upper/lower bounds recommended for mean value and variance of TP by several researchers. For example Moriguti (1951) [10] recommends the following upper bound on the variance of the sample max:

\[ V(X_n) < n \int_{1/2}^{1} x (F)^2 dF = \frac{1}{2} n \sigma^2 \]  

(4.1)
In which \( X_n \) is the sample maximum and \( n \) is the number of sample members. \( \sigma^2 \) corresponds to variance of the population.

Balakrishnan et. al. (2003) [11] proposed the following bounds on the mean value and variance of the \( i \)th order statistics denoted as \( X_{i:n} \).

\[
\mu - \sigma \sqrt{\frac{n-i}{i}} \leq E[X_{i:n}] \leq \mu + \sigma \sqrt{\frac{i-1}{n-i+1}}, 1 \leq i \leq n. \tag{4.2}
\]

\[
0 \leq V(X_{i:n}) \leq \max \left\{ \frac{n}{i} \cdot \frac{n}{n+1-i} \right\} V(X_i), 1 \leq i \leq n. \tag{4.3}
\]

In which \( X_i \) is an arbitrary dependent identically distributed random variable with a finite variance \( \text{Var}(X_i) \) and \( X_{i:n} \) denotes the \( i \)th order statistics of the sample with homogeneous variance and mean value. This upper bound on \( \text{Var}(X_{i:n}) \) is restated by Rychlik (2008) [13]. The above bound when \( i = 1 \) or \( n \) is edited for a finite population with size \( N \) (Ciginas & Dalius, 2014 [14] and Balakrishnan et. al., 2003 [11]):

\[
V(X_{i:n}) \leq n \frac{N-n}{N-1} V(X_i), i = 1 \text{ or } n. \tag{4.4}
\]

Bertsimas et. al. (2004) [12] recommends the following bound on mean value of highest order statistics \( X_{n:n} \). This bound is also applicable for sample drawn of a population with identical distribution having an individual variance \( \sigma_i^2 \) and mean value \( \mu_i \).

\[
E[X_{n:n}] \leq \max \{ \mu_i \} + \sqrt{\frac{n-1}{n} \sum_{i=1}^{n} \sigma_i^2}, \tag{4.5}
\]

Papadatos (1995) [15] reviews that:

For \( n = 2k - 1 \), \( \text{Var}(X_{k:n}) \leq \sigma^2 \tag{4.6} \)

For \( n = 2k \), \( \text{Var}[(X_{k:n} + X_{k+1:n})/2] \leq \sigma^2 \tag{4.7} \)

For \( k \neq (n + 1)/2 \), there exist a continuous distribution function \( F(x) \) such that \( \text{Var}(X_{k:n}) > \sigma^2 \). \tag{4.8} \]

Moriguti (1951) [10] provides a relation between mean value and variance of sample max and efficient lower and upper bounds on \( \text{Var}(X_n) \):

\[
\frac{\sqrt{V(X_n)}}{E[X_n]} \approx \frac{1}{2^{n-1}}, \tag{4.9}
\]

\[
\lambda_n \sigma^2 \leq V(X_n) \leq \frac{n \sigma^2}{2}, \lambda_n = \frac{\pi}{2^n} \left[ 1 + O \left( \frac{1}{n} \right) \right]. \tag{4.10}
\]

5 New Upper Bounds on Mean and Variance of Sample Max

5.1. Our Developed Bounds

All the above reviewed papers provide bounds on variance of sample max drawn from a single population; except Bertsimas et. al. (2004) [12] which provides bounds on mean value of sample max from different populations, the other reviewed papers assume a single population.

Based on numerous experiments we develop the following bounds on mean value and variance of sample max. The sample consists of \( x_i \)'s. Every \( x_i \) follows an arbitrary distribution with \( \mu_i \) and \( \sigma_i \).
\[ E[X_{\text{max}}] = \max \left\{ \mu_i + \sqrt{n-1} \sigma_i \right\}, \quad E[X_{\text{min}}] = \min \left\{ \mu_i - \sqrt{n-1} \sigma_i \right\}, \quad (5.11) \]

\[ \sigma_{X_{\text{max}}} = \max \left\{ \sigma_{X_i} \right\} \sqrt{\frac{n+1}{2}}, \quad \sigma_{X_{\text{min}}} = \sigma_{X_{\text{max}}}, \quad (5.12) \]

We can define even more tight bounds on standard deviation of sample max and sample min as follows:

\[ \theta = \arg \max_{i=1,...,n} \left\{ \mu_i + \sqrt{n-1} \sigma_i \right\}, \quad \tau = \arg \min_{i=1,...,n} \left\{ \mu_i - \sqrt{n-1} \sigma_i \right\}, \quad (5.13) \]

\[ \sigma_{X_{\text{max}}} = \sqrt{\frac{n+1}{2}} \sigma_{\theta}, \quad \sigma_{X_{\text{min}}} = \sqrt{\frac{n+1}{2}} \sigma_{\tau}. \quad (5.14) \]

5.2. Better estimates for Mean value and variance of the activities with three estimates

Standard beta distribution with parameters \( \alpha \) and \( \beta \) is known as to be:

\[ f(x) = \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise}. \end{cases} \quad (5.15) \]

It is necessary that both \( \alpha \) and \( \beta \) be positive real numbers. For the mean value for such random variable \( X \) we have:

\[ \mu = \left( \frac{\alpha}{\alpha + \beta} \right), \quad (5.16) \]

\[ \sigma^2 = \left( \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} \right). \quad (5.17) \]

Classic PERT assumes a modified Beta distribution on activity durations. Such that for each activity duration \( x_i \) we have:

\[ a \leq x_i \leq b, \quad \text{Mode}(x_i) = m. \quad (5.18) \]

Therefore the above distribution function is revised as:

\[ f(x) = \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{(b-a)^{\alpha+\beta-1}} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise}. \end{cases} \quad (5.19) \]

It is still necessary that both \( \alpha \) and \( \beta \) be positive real numbers. We can compute the mean, variance and distribution mode of modified Beta distribution using its probability distribution function:

\[ \mu = a + (b-a) \left( \frac{\alpha}{\alpha + \beta} \right), \quad (5.20) \]

\[ \sigma^2 = (b-a)^2 \left( \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} \right), \quad (5.21) \]

\[ m = a + (b-a) \left( \frac{\alpha - 1}{\alpha + \beta - 2} \right). \quad (5.22) \]

Classic PERT estimates the mean and variance of such random variables by using the three estimates data gathered as follows:

\[ \mu = \left( a + 4m + b \right) \frac{1}{6}, \quad (5.23) \]
\[ \sigma^2 = \left( \frac{b-a}{6} \right)^2. \]  
(5.24)

In which \( a, b \) and \( m \) are earliest, latest and most probable value (known as mode of the distribution) for activity durations correspondingly. These values are gathered from data records. In order to have more precise bounds on mean and variance of merge event, first we have to calculate mean and variance based on the three available estimates on activity duration (\( a, m \) and \( b \)). This is done by using equations (5.20-5.22):

\[
\begin{align*}
  a &= m, m < b \Rightarrow \alpha = 1, \beta > 1 \Rightarrow \text{we take} \; \beta = 2, \\
  a < m, m = b \Rightarrow \beta = 1, \alpha > 1 \Rightarrow \text{we take} \; \beta = 2, \\
  a < m < b \Rightarrow \beta > 1, \alpha > 1 \Rightarrow \begin{cases} 
    \text{if} \; m = \frac{a+b}{2} \Rightarrow \alpha = \beta \rightarrow \text{take} \; \alpha = \beta = 4 \\
    \text{if} \; m > \frac{a+b}{2} \Rightarrow \alpha > \beta \rightarrow \text{take} \; \beta = \frac{b-a}{b-m} \\
    \text{if} \; m < \frac{a+b}{2} \Rightarrow \alpha < \beta \rightarrow \text{take} \; \alpha = \frac{b-a}{m-a} 
  \end{cases} 
\end{align*}
\]
(5.25-5.29)

After selecting appropriate parameters \( \alpha \) and \( \beta \) with regard to given \( a, b, \) and \( m \), now we can compute the mean and variance of the corresponding beta distribution.

\[
\begin{align*}
  &\mu = \frac{2a + 2m + 2b}{6}, \quad \sigma^2 = 2 \left( \frac{b-a}{6} \right)^2, \\
  &\mu = \frac{2a + 2m + 2b}{6}, \quad \sigma^2 = 2 \left( \frac{b-a}{6} \right)^2, \\
  &\mu = a + \left( \frac{b-a}{2} \right), \quad \sigma^2 = \left( \frac{b-a}{6} \right)^2, \\
  &\mu = a + (b - a) \left( \frac{b-a}{3b-2m-a} \right), \quad \sigma^2 = \frac{2(b-a)^3(b-m)^2}{(3b-2m-a)^2(4b-3m-a)}, \\
  &\mu = a + (b - a) \left( \frac{2m-2a}{b+2m-3a} \right), \quad \sigma^2 = \frac{2(b-a)^3(m-a)^2}{(2m-3a+b)^2(b+3m-4a)} 
\end{align*}
\]
(5.30-5.34)

As it can be seen, there are several formulas for calculating mean and variance of activity duration (5.30-5.34) for different values for \( a, b, \) and \( m \). It is in contrast with the simpler and less accurate method used in classic PERT.
6 Computational Results: The Case of Assessing the Completion Time for Learning a Mathematics Course

In order to compare the results generated by our method with those generated by PERT we solve an individual problem. The focused problem is the time required for a student of Industrial Management major to learn a basic math course. This course consists of eight sub-topics, each one illustrated by an arrow in figure 1. The time required for each student to learn each sub-topic is placed on its corresponding arrow. Activities (4-7) and (4-5) are dummy arcs to represent the precedence relation between activity (3-4) for (5-6) and (7-8) respectively. Table 1 summarizes the calculation of \( \mu_i \) and \( \sigma_i^2 \) for each activity duration by two methods: our developed one and classic PERT. The results of forward calculation by our developed method and PERT are depicted in Figure 2 and Figure 3 respectively. \( \mu_i \) and \( \sigma_i^2 \) of Earliest occurrence time for each event \( E_i \) is depicted above its circle. As it can be inferred from these figures, the regarding values for merge event at circle called 5 and 7 are significantly different in each method; especially the values generated by PERT are less than those generated by our method. This complies with before said characteristics of PERT in which underestimates the mean value and variance of project completion time.

![Figure 1: The Activity On Arrow (AOA) network of the focused problem and three estimates for each activity duration.](image)

Table 1: Mean and variance of the activity durations in the focused problem.

<table>
<thead>
<tr>
<th>Activity Name</th>
<th>a</th>
<th>m</th>
<th>b</th>
<th>Our Method</th>
<th>PERT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \mu_i )</td>
<td>( \sigma_i^2 )</td>
<td>( \mu_i )</td>
<td>( \sigma_i^2 )</td>
<td></td>
</tr>
<tr>
<td>1-2</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>1.5</td>
<td>0.028</td>
</tr>
<tr>
<td>2-3</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>3.6</td>
<td>0.886</td>
</tr>
<tr>
<td>3-4</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>3.92</td>
<td>1.392</td>
</tr>
<tr>
<td>2-5</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>0.444</td>
</tr>
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<td>5-6</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>3.27</td>
<td>1.328</td>
</tr>
<tr>
<td>6-7</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>3.6</td>
<td>0.886</td>
</tr>
<tr>
<td>7-8</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>0.111</td>
</tr>
<tr>
<td>8-9</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>3.29</td>
<td>0.49</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Based on the above results we can infer that having variety of students sitting in a classroom and being taught with a single curriculum, results in more different learning times than a classroom with more homogeneous students. The difference between learning time of students is more highlighted if it is calculated with a more precise method such as that we have presented in this paper.

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