Interpolation of bounded sequences by \(\alpha\)-dense curves

G. García

Universidad Nacional de Educación a Distancia (UNED), CL. Candalix S/N, 03202 Elche (Alicante) Spain.

Abstract
In 1905 Lebesgue showed that there is a sequence of continuous functions, put \(f_n : [0, 1] \rightarrow [0, 1]\), which interpolates any sequence in \([0, 1]\), that is, given \((a_n)_{n \geq 1} \subset [0, 1]\) there is \(t \in [0, 1]\) such that \(f_n(t) = a_n\) for each positive integer \(n\). This result was improved (in the sense of Theorem 1.1) in 1998 by Y. Benyamini. In this paper, we generalize the Benyamini’s result in Theorem 4.1. The key tool for this goal are the so called \(\alpha\)-dense curves. We apply our results to approach the solution of a certain infinite-dimensional linear program with a countable number of constraints.

Keywords: Sequences interpolation, \(\alpha\)-dense curves, Peano curves, Global Optimization.

1 Introduction

In this paper we note the closed unit interval \([0, 1]\) by \(I\), the positive integers and integers numbers sets, as usual, by \(\mathbb{N}\) and \(\mathbb{Z}\) respectively.

In 1905 (see [6, p. 210]) Lebesgue showed that there is a sequence of continuous functions \(f_n : I \rightarrow I\) such that for any given sequence \((a_n)_{n \geq 1} \subset I\) there is at least one \(t \in I\) such that \(f_n(t) = a_n\) for all \(n \in \mathbb{N}\). This fact is intimately related with the existence of the so called space-filling curves in the Hilbert cube \(H = [0, 1]^\omega\), \(\omega\) being the first countable ordinal; see [14, p. 111-112]. Recalling, again [14], that given a positive integer \(d > 1\) or \(d = \omega\) a space-filling curve in \(I^d\) is a continuous map from \(I\) onto \(I^d\). The existence of such curves is guaranteed by the Hahn-Mazurkiewicz theorem (see [14, 16]), which states that every compact, connected and locally connected set of a metric space is the continuous image of \(I\).

The above Lebesgue result on interpolation of bounded sequences is not the only one of this type. We focus here on a result due to Y. Benyamini published in 1998 [1, Th. 5], namely:

**Theorem 1.1.** There is a continuous function \(f : \mathbb{R} \rightarrow [-1, 1]\) such that for each doubly infinite sequence \((y_n)_{n \in \mathbb{Z}} \subset [-1, 1]\) there is \(t \in I\) such that

\[
\forall n \in \mathbb{Z}.
\]

The proof of Benyamini’s theorem is based on the universal surjectivity of the Cantor set \(\mathcal{C}\) (Hausdorff’s theorem, see for instance [14, Th. 6.6, p. 100]): as the compact set \([-1, 1]^\mathbb{Z}\) is the continuous image of \(\mathcal{C}\), there is a continuous mapping \(\phi : \mathcal{C} \rightarrow [-1, 1]^\mathbb{Z}\), put \(\phi(t) = (\psi_n(t))_{n \in \mathbb{Z}}\), with \(\phi(\mathcal{C}) = [-1, 1]^\mathbb{Z}\). We identify \(\mathcal{C}\) as a closed subset of \([0,1/2]\). Then, \(\mathcal{C} + n\) and \(\mathcal{C} + m\) are disjoint for integers \(n \neq m\) and \(\mathcal{C}_0 := \bigcup\{\mathcal{C} + n : n \in \mathbb{Z}\}\) is a closed subset of

*Corresponding author. Email address: gonzalogarciamacias@gmail.com
So, the function \( f : \mathbb{C}_0 \rightarrow [-1,1] \) given by \( f(t+n) := \phi_n(t) \) for all \( t \in \mathbb{C} \) and \( n \in \mathbb{Z} \) can be extended (by linear interpolation) to the whole interval \( I \). Such extension of the function \( f \) is the desired function in Theorem 1.1.

On the other hand, we will prove a result which generalizes, in the sense of Theorem 4.1, the Benyamini’s theorem. Unlike the results of Lebesgue or Benyamini, we will not use the Hahn-Mazurkiewicz theorem neither Hausdorff theorem to construct a function \( g_\alpha : \mathbb{R}_1 \rightarrow I \), with \( \mathbb{R}_1 := [1, +\infty) \), such that for each sequence \( (x_n)_{n \geq 1} \subset I \), there is \( t \in I \) with

\[
|g_\alpha(t + n) - x_n| \leq \alpha, \quad \forall n \in \mathbb{N}, \tag{1.2}
\]

where \( 0 < \alpha < 1 \) is given. The fact of taking sequences in the interval \( I \) instead doubly infinite sequences in \([-1,1]\) is not relevant, as we will see, it is just for simplicity. In addition, the function \( g_\alpha \) could be extended symmetrically to the whole \( \mathbb{R} \) but this fact is not relevant for our purposes.

The construction of the function \( g_\alpha : \mathbb{R}_1 \rightarrow I \) satisfying (1.2) is based on the so called \( \alpha \)-dense curves, which we will discuss in detail in Section 2. Noticing that the \( \alpha \)-dense curves are (in the sense of Hausdorff metric) a generalization of the space-filling curves, the existence of a function \( g_\alpha : \mathbb{R}_1 \rightarrow I \) obeying (1.2) is natural from the existence of a function \( f : \mathbb{R} \rightarrow I \) obeying (1.1). However, as we will see in the following sections, we want to emphasize that the construction of the function \( g_\alpha \) is more explicit than the function \( f \) of Theorem 1.1 and the sequence of functions \( f_n \) of the Lebesgue interpolation result.

To conclude, as application of our results, we show in Proposition 5.1 a method to approach the solution (when exists) of certain infinite-dimensional linear program with a countable number of constraints.

### 2 Denseable sets and \( \alpha \)-dense curves

In this section \( (E,d) \) will be a metric space. In 1997 the concept of \( \alpha \)-dense curve was introduced by Mora and Cherruault [10]:

**Definition 2.1.** Let \( \alpha \geq 0 \) and \( B \subset (E,d) \) be a non-empty and bounded set. A continuous mapping \( \gamma_\alpha : I \rightarrow (E,d) \) is said to be an \( \alpha \)-dense curve in \( B \) if

1. \( \gamma_\alpha(I) \subset B \).
2. For any \( x \in B \), there is \( t \in I \) such that \( d(x, \gamma_\alpha(t)) \leq \alpha \).

If for any \( \alpha > 0 \) there is an \( \alpha \)-dense curve in \( B \), then the set \( B \) is said to be denseable.

Note that if \( B \) is closed, the positive parameter \( \alpha \) in Definition 2.1 coincides with the Hausdorff distance (see, for instance, [14, p. 151]) from \( \gamma_\alpha(I) \) to \( B \). So, in this sense, the \( \alpha \)-dense curves are a generalization of the space-filling curves because for the particular case \( \alpha = 0 \) and \( B := I^d \), the curve \( \gamma_0 \) is precisely a space-filling curve in \( I^d \). Also, note that given \( B \subset E \) non-void and bounded, there is always an \( \alpha \)-dense curve in \( B \) for any \( \alpha \geq \text{Diam}(B) \), diameter of \( B \). Indeed, fixed \( x_0 \in B \), the mapping \( \gamma(t) := x_0 \), for each \( t \in I \) is clearly an \( \alpha \)-dense curve in \( B \), for any \( \alpha \geq \text{Diam}(B) \).

**Example 2.1.** ([2, Prop. 9.5.4, p. 144]). The cosines curve. Given a positive integer \( d > 1 \), the closed unit cube \( I^d \) is a denseable set. In fact, for each positive integer \( m \), the map

\[
t \in I \mapsto \gamma_m(t) := \left( t, \frac{1}{2}(1 - \cos(m\pi t)), \ldots, \frac{1}{2}(1 - \cos(m^{d-1}\pi t)) \right),
\]

is a \( \sqrt{\frac{d}{m}} \)-dense curve in \( I^d \).

Other examples of \( \alpha \)-dense curves can be found in [2, 13]. As expected, not every bounded subset of \( \mathbb{R}^d \) (even connected and compact) is a denseable set:

**Example 2.2.** Let

\[
B := \left\{ (x, \sin(x^{-1})) : x \in [-1,0) \cup (0,1) \right\} \cup \left\{ (0,y) : y \in [-1,1] \right\} \subset \mathbb{R}^2.
\]

Then, given any continuous mapping \( \gamma : I \rightarrow \mathbb{R}^2 \) with \( \gamma(I) \subset B \), by connectedness, \( \gamma(I) \) must be contained in some connected component of \( B \). So, taking any \( 0 < \alpha < 1 \) it is clear that there is not any \( \alpha \)-dense curve in \( B \) and therefore \( B \) is not denseable.
However, the class of densifiable sets is large:

**Proposition 2.1.** Let \( K \subset (E,d) \) a precompact and path-wise connected set. Then, \( K \) is densifiable.

**Proof.** Let \( \alpha > 0 \). By the precompactness of \( K \) there is \( \{x_1, \ldots, x_n\} \subset K \) such that

\[
K \subset \bigcap_{i=1}^{n} \bar{B}(x_i, \alpha),
\]

(2.3)

where \( \bar{B}(x_i, \alpha) \) denotes the closed ball centered at \( x_i \) and radius \( \alpha \), for \( i = 1, \ldots, n \). Now, taking \( \gamma_\alpha : I \rightarrow (E,d) \) a path in \( K \) joining the points \( x_1, \ldots, x_n \) it is clear by (2.3) that \( \gamma_\alpha \) is an \( \alpha \)-dense curve in \( K \) and this completes the proof. \( \square \)

The main application of the \( \alpha \)-dense curves is the resolution of global optimization problems: given a continuous mapping \( f : B \rightarrow \mathbb{R} \), with \( B \subset E \) non-void and bounded, an \( \alpha \)-dense curve in \( B \), put \( \gamma_\alpha \), allows us to obtain an approximation method to turn the global optimization problem (for details on this issue, see [2, 8, 9, 11, 12])

\[
\inf \left\{ f(x) : x \in B \right\},
\]

into the unidimensional

\[
\min \left\{ f(\gamma_\alpha(t)) : t \in I \right\},
\]

provided that the above infimum be finite and is possible to define such a curve for arbitrarily small \( \alpha > 0 \), that is, provided that the set \( B \) be densifiable. More specifically, we have the following result (see [2, Th. 5.4.4, p. 85])

**Theorem 2.1.** With the above notation we have

\[
\inf \left\{ f(x) : x \in B \right\} - \min \left\{ f(\gamma_\alpha(t)) : t \in I \right\} \leq \omega(f, \alpha),
\]

where \( \omega(f, \alpha) \) is the modulus of continuity of \( f \) of order \( \alpha \), recall, \( \omega(f, \alpha) := \sup \{|f(x) - f(y) : x, y \in B, d(x,y) \leq \alpha\} \).

The above method has proved to be efficient and highly accurate, a fact that is evidenced in the following example.

**Example 2.3.** Let \( f(x,y) := \exp(\sin(2\pi xy) + \cos(2\pi xy)) \) for each \( x,y \in I \) and, for each \( m \in \mathbb{N} \), let \( \gamma_m \) the cosines curve of Example 2.1. With the help of the software Maple\textsuperscript{®}, we know that the minimum of \( f \) is achieved on \((x^*, y^*) \approx (0.790569415042074009, 0.790569415042074009)\), and \( f(x^*, y^*) \approx 0.243116734434214221 \).

On the other hand, in Tab. 1 we show some approximations of \( f(x^*, y^*) \) following the above method. We have noted by \( t_m \in I \) the value for which the function \( f(\gamma_m(t)) \) attains its minimum.

<table>
<thead>
<tr>
<th>Table 1: Some approximations of the minimum of ( f ) by the cosines curves of Example 2.1</th>
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<tbody>
<tr>
<td>( m )</td>
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However, in Theorem 4.1, we will use the \( \alpha \)-dense curves to construct a continuous function \( g_\alpha : \mathbb{R}_1 \rightarrow I \) satisfying the inequality (1.2). Moreover, as application of such result, in Section 5 and following the above research line, we propose a method to approach the solution (if there is any) of certain infinite-dimensional linear program with a countable number of constraints.
3 Densifiability of the Hilbert cube

Recall, that the Hilbert cube $\mathbb{H} := I^\mathbb{N}$ can be endowed the metric
\[
\delta(x,y) := \sum_{n \geq 1} 2^{-n} |x_n - y_n| \quad \text{for each } x := (x_n)_{n \geq 1}, y := (y_n)_{n \geq 1} \in \mathbb{H},
\]
which turns $(\mathbb{H}, \delta)$ into a complete metric space (see, for instance, [16]).

In [7, Lem. 2] (see also [14, Chap. 7]) explicit constructions of space-filling curves in $\mathbb{H}$ are showed, while in [13] a method to construct $\alpha$-dense curves in $\mathbb{H}$ is showed. We know by Proposition 2.1 that $\mathbb{H}$ is densifiable, but in this section we propose (below, Proposition 3.1) an explicit construction of an $\alpha$-dense curve in $\mathbb{H}$, for an arbitrary $0 < \alpha < 1$.

The simplicity of the following technical result is clear.

**Lemma 3.1.** Let $\varphi : I \mapsto I$ be a continuous and surjective function of period $0 < \alpha < 1$. For any interval $A \subset I$ with length $|A| > \alpha$, one has $A \cap \varphi^{-1}(B) \neq \emptyset$ for all non-void closed set $B \subset I$.

Now, a key result to our main goal:

**Proposition 3.1.** Let $\varphi : I \mapsto I$ be a continuous and surjective function of period $0 < \alpha < 1$. Then, the mapping $\gamma_\alpha : I \mapsto (\mathbb{H}, \delta)$ given by
\[
t \in I \mapsto \gamma_\alpha(t) := (t, \varphi(t), \varphi^2(t), \ldots, \varphi^n(t), \ldots),
\]
where the exponent $k$ denotes the composition of $\varphi$ with itself $k$ times, is an $\alpha$-dense curve in $(\mathbb{H}, \delta)$.

**Proof.** Let $y := (y_n)_{n \geq 1} \in \mathbb{H}$ and $n \in \mathbb{N}$. Taking $\varepsilon > \alpha$ define for each $i = 1, \ldots, n$ the set $B_i := [y_i - \varepsilon, y_i + \varepsilon] \cap I$. Since the length of $B_1$ is greater than $\alpha$, by the above lemma, the set
\[
B_1 \cap \varphi^{-1}(B_2) \cap \varphi^{-2}(B_3) \cap \ldots \cap \varphi^{-n+1}(B_n),
\]
where $\varphi^{-k}$, for $1 \leq k \leq n - 1$, denotes the composition of $\varphi^{-1}$ with itself $k$ times, will be non-empty if
\[
\varphi^{-1}(B_2) \cap \varphi^{-2}(B_3) \cap \ldots \cap \varphi^{-n+1}(B_n),
\]
is a non-empty closed set in $I$. But, (3.5) can be written as
\[
\varphi^{-1}(B_2) \cap \varphi^{-2}(B_3) \cap \ldots \cap \varphi^{-n+2}(B_n),
\]
and as the length of $B_2$ is greater that $\alpha$, again by the above lemma, the set defined in (3.6) will be non-empty if
\[
\varphi^{-1}(B_3) \cap \ldots \cap \varphi^{-n+2}(B_n),
\]
is a non-empty closed set in $I$. Therefore, repeating this process we are led, finally, to consider the set
\[
B_{n-1} \cap \varphi^{-1}(B_n),
\]
which is non-empty by Lemma 3.1. Then, given a positive integer $n$ we have
\[
\bigcap_{i=1}^n \varphi^{-i+1}(B_i) \neq \emptyset,
\]
and from (3.8) follows that the family of closed set $\mathcal{B} := \{\varphi^{-n+1}(B_n) : n \in \mathbb{N}\}$ has the finite intersection property. Because $I$ is compact and $\mathcal{B} \subset I$ we find (see, for instance, [16, Th. 17.4, p. 118]) that
\[
\bigcap_{n \geq 1} \varphi^{-n+1}(B_n) \neq \emptyset.
\]

So, noticing (3.9), there is $t \in I$ such that $\varphi^{n-1}(t) \in B_n$ for each $n \in \mathbb{N}$, meaning as usual that $\varphi^0(t) := t$, and this fact implies that
\[
\delta(y, \gamma_\alpha(t)) = \sum_{n \geq 1} 2^{-n} |y_n - \varphi^{n-1}(t)| < \varepsilon \sum_{n \geq 1} 2^{-n} = \varepsilon.
\]
Therefore, $\gamma_\alpha$ is an $\varepsilon$-dense curve in $\mathbb{H}$. Because $\varepsilon > \alpha$ is arbitrary, in view of (3.10) we conclude that $\gamma_\alpha$ is an $\alpha$-dense curve in $\mathbb{H}$, as needed, and the proof is now complete.

\[\square\]
Some comments are necessary. Note that the method showed in the above result is not the only way to construct an \( \alpha \)-dense curve in \( \mathbb{H} \), as we will see in the following lines.

Fixed \( \alpha > 0 \) pick \( n \in \mathbb{N} \) such that

\[
\sum_{k \geq n+1} 2^{-k} \leq \frac{\alpha}{2} \quad \text{(3.11)}
\]

Let \( \gamma_{\alpha} : I \rightarrow \mathbb{R}^{n} \) be an \( \frac{\alpha}{2} \)-dense curve in \( I^{n} \), for instance, the cosines curve of Example 2.1. Put, \( \bar{\gamma}_{\alpha}(t) := (\gamma_{\alpha}^{(1)}(t), \ldots, \gamma_{\alpha}^{(n)}(t)) \) for each \( t \in I \). So, given any \( y := (y_{k})_{k \geq 1} \in \mathbb{H} \) we can take \( t \in I \) such that

\[
\| (y_{1}, \ldots, y_{n}) - \bar{\gamma}_{\alpha}(t) \|_{n} \leq \frac{\alpha}{2}, \quad \text{(3.12)}
\]

\( \| \cdot \|_{n} \) being the Euclidean norm in \( \mathbb{R}^{n} \). Now, we define \( \gamma_{\alpha} : I \rightarrow (\mathbb{H}, \delta) \) as

\[
\gamma_{\alpha}(t) := (\gamma_{\alpha}^{(1)}(t), \ldots, \gamma_{\alpha}^{(n)}(t), \psi_{n+1}(t), \psi_{n+2}(t), \ldots) \quad \forall t \in I,
\]

where \( \psi_{k} : I \rightarrow I \) are arbitrary continuous functions, for each \( k \geq n + 1 \). So, by (3.11) and (3.12) we have

\[
\delta(y, \gamma_{\alpha}(t)) = \sum_{k=1}^{n} 2^{-k}|y_{k} - \gamma_{\alpha}^{(k)}(t)| + \sum_{k \geq n+1} 2^{-k}|y_{k} - \psi_{k}(t)| \leq
\]

\[
\leq \| (y_{1}, \ldots, y_{n}) - \bar{\gamma}_{\alpha}(t) \|_{n} \sum_{k=1}^{n} 2^{-k} + \sum_{k \geq n+1} 2^{-k} < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha,
\]

and therefore \( \gamma_{\alpha} \) is an \( \alpha \)-dense curve in \( \mathbb{H} \).

However, note that the curve constructed in Proposition 3.1 is such that

\[
|y_{k} - \varphi^{k-1}(t)| \leq \alpha \quad \forall k \in N, \quad \text{(3.13)}
\]

and compare (3.13) with (3.12). This fact will be crucial to our goal. Other method to construct \( \alpha \)-dense curves in \( (\mathbb{H}, \delta) \) can be found in [13].

On the other hand, if we take \( \mathbb{H}_{0} := J^{p_{0}} \), where \( J \) is a bounded and closed interval, the above proposition remains true for \( \mathbb{H}_{0} \) with slight changes. We can take \( \varphi : J \rightarrow J \) continuous, surjective and \( \alpha \)-periodic with \( 0 < \alpha < \text{length of } J \), and \( \gamma_{\alpha} : J \rightarrow (\mathbb{H}_{0}, \delta) \) as in the proof of Proposition 3.1.

To conclude this section, an example:

**Example 3.1.** The \( \mathbb{R}_{p} \)-cosines curve. Let \( \varphi : I \rightarrow I \) the map defined as

\[
\varphi(t) := \frac{1}{2} \left( 1 - \cos(m \pi t) \right) \quad \forall t \in I.
\]

It is clear that \( \varphi \) is continuous, surjective and \( \frac{2}{m} \)-periodic. So, by Proposition 3.1, for each integer \( m > 2 \) the curve \( \gamma_{m} : I \rightarrow (\mathbb{H}, \delta) \) given by

\[
\gamma_{m}(t) := (t, \varphi(t), \varphi^{2}(t), \ldots, \varphi^{n}(t), \ldots) \quad \forall t \in I,
\]

is a \( \frac{2}{m} \)-dense curve in \( (\mathbb{H}, \delta) \).

4 The main result

Now, we can state and show the main result of this paper:

**Theorem 4.1.** Given any \( \alpha \in (0, 1) \), there is a continuous function \( g_{\alpha} : \mathbb{R}_{1} \rightarrow I \) such that for each sequence \( (x_{n})_{n \geq 1} \in \mathbb{H} \) there is \( t \in I \) such that

\[
|g_{\alpha}(t + n) - x_{n}| \leq \alpha \quad \forall n \in N.
\]
More specifically, we pose the problem of finding a solution of the problem (P) (if there is any) by the solution of a suitable single variable optimization problem.

Proposition 5.1. Given a dense curve \( \gamma_a : I \rightarrow (\mathbb{H}, \delta) \) in \( \mathbb{H} \) as in Proposition 3.1, put

\[
\gamma_a(t) := (\gamma_a^{(1)}(t), \gamma_a^{(2)}(t), \ldots, \gamma_a^{(n)}(t), \ldots) \quad \forall t \in I.
\]

Now, define the map \( g_a : \mathbb{R} \rightarrow I \) as

\[
g_a(t + n) := \gamma_a^{(n)}(t) \quad \text{for all} \ t \in I \text{ and } n \in \mathbb{N}.
\]

It is clear that \( g_a \) is well defined. Also, \( g_a \) is continuous, because the topology on \( \mathbb{H} \) is defined in such way that for each fixed \( n \) the \( n \)-th coordinate function of \( \gamma_a \) is continuous.

The function \( g_a \) is the desired function: given any \( (x_n)_{n \geq 1} \in \mathbb{H} \), as \( \gamma_a \) is an \( \alpha \)-dense curve in \( (\mathbb{H}, \delta) \), there is \( t \in I \) such that

\[
|x_n - \gamma_a^{(n)}(t)| \leq \alpha \quad \forall n \in \mathbb{N},
\]

and so, by (4.14) we conclude that

\[
|g_a(t + n) - x_n| = |\gamma_a^{(n)}(t) - x_n| \leq \alpha \quad \forall n \in \mathbb{N},
\]

and the results follows. \( \square \)

5 Application to countably infinite linear programs

Recall (see, for instance, [3] and references therein) that a countably infinite linear program, in short CILP, is an infinite-dimensional linear program with a countable number of constraints. These problems are often used to model and solve many optimization problems, to quote some of them: cost flow problems on infinite networks (see [15] and references therein), infinite horizon stochastic programs (see [5] and references therein) or countable-state Markov decision processes (see [4] and references therein).

In what follows, we note by \( \mathbb{R}^\mathbb{N} \) the space of all real-valued sequences, and its elements by bold letters \( x, y, \ldots \) which components, as usual, noted with the same letter but with subindex and free bold. Also, \( A := (a_{ij})_{i,j \in \mathbb{N}} \) will be a doubly-infinite matrix and as usual in the existing literature, we assume that fixed \( i \in \mathbb{N} \) only a finite numbers of terms in the sequence \( (a_{ij})_{j \geq 1} \) are non-zero, and the same if we fix \( j \in \mathbb{N} \) for the sequence \( (a_{ij})_{i \geq 1} \). Given \( b \in \mathbb{R}^\mathbb{N} \) put

\[
\mathcal{X} := \{ x \in \mathbb{H} : \sum_{j \geq 1} a_{ij}x_j = b_i, \forall i \in \mathbb{N} \};
\]

and pose the optimization problem

\[
(P) \quad \inf \left\{ \sum_{j \geq 1} 2^{-j}x_j : x \in \mathcal{X} \right\},
\]

interpreting, as usual, that inf\( \{ \emptyset \} = -\infty \). Our goal is, as in the case of finite dimension exposed in Section 2, approach the solution of the problem \( (P) \) (if there is any) by the solution of a suitable single variable optimization problem. More specifically, we pose the problem

\[
(P)_\alpha \quad \min \left\{ f_a(t) : t \in I \right\},
\]

where \( f_a : I \rightarrow (\mathbb{H}, \delta) \), is a suitable continuous mapping, and approach the infimum given by the problem \( (P) \) by the solution of \( (P)_\alpha \). Note that we have replaced the term "inf" by "min" in problem \( (P)_\alpha \) because as \( f_a \) is continuous, by the compactness of \( I \) follows the existence of the above minimum by a well known theorem due to Weierstrass.

Now, the following result holds:

Proposition 5.1. Assume that \( \mathcal{X} \neq \emptyset \) and \( a_{ij} \geq 0 \) for each \( i, j \in \mathbb{N} \). Then, fixed \( 0 < \alpha < 1 \), there is a curve \( \gamma_a : I \rightarrow (\mathbb{H}, \delta) \), put \( \gamma_a(t) := (\gamma_a^{(n)}(t))_{n \geq 1} \) for each \( t \in I \), such that

\[
0 \leq \xi^* - \eta^*_a \leq \alpha,
\]

where \( \xi^* \) and \( \eta^*_a \) are the infimum given by the problem \( (P)_\alpha \) and \( (P) \), respectively.
where the numbers $\xi^*$ and $\eta^*_\alpha$ are given by

$$\xi^* := \inf \left\{ \sum_{j \geq 1} 2^{-j} x_j : x \in \mathcal{X} \right\} \quad \text{and} \quad \eta^*_\alpha := \min \left\{ \sum_{j \geq 1} 2^{-j} \gamma_\alpha(t) : t \in I \right\}.$$ 

Proof. Let $F : (\mathcal{X}, \delta) \rightarrow \mathbb{R}$ given by $F(x) := \sum_{j \geq 1} 2^{-j} x_j$ for each $x \in \mathcal{X}$. Then, $F$ is continuous and for each $x \in \mathcal{X}$

$$0 \leq F(x) = \sum_{j \geq 1} 2^{-j} x_j \leq \sum_{j \geq 1} 2^{-j} = 1,$$

and therefore, the number $\xi^*$ is well defined from the boundedness of $F$. Now, we show that $\mathcal{X}$ is convex. Given $x, y \in \mathcal{X}$ and $0 < \lambda < 1$, for each $i \in \mathbb{N}$ we have

$$\sum_{j \geq 1} a_{ij}(\lambda x_j + (1 - \lambda) y_j) = \lambda \sum_{j \geq 1} a_{ij}x_j + (1 - \lambda) \sum_{j \geq 1} a_{ij}y_j = \lambda b_i + (1 - \lambda) b_i = b_i,$$

and so, the set $\mathcal{X}$ is convex. Noticing Proposition 2.1, we find that $\mathcal{X}$ is a densifiable set, and therefore, there is an $\alpha$-dense curve in $\mathcal{X} : \gamma_\alpha : I \rightarrow (\mathbb{H}, \delta)$, put, $\gamma_\alpha(t) := (\gamma_\alpha^n(t))_{n \geq 1}$ for each $t \in I$.

On the other hand, as for each $x, y \in \mathcal{X}$ the inequality

$$|F(x) - F(y)| \leq \sum_{j \geq 1} 2^{-j}|x_j - y_j| = \delta(x, y),$$

holds, we infer that $\omega(F, \varepsilon) \leq \varepsilon$ for each $\varepsilon > 0$, recalling that $\omega(F, \varepsilon)$ is the modulus of continuity of $F$ of order $\varepsilon$. Then, by Theorem 2.1, we find that

$$0 \leq \xi^* - \eta^*_\alpha = \inf \left\{ F(x) : x \in \mathcal{X} \right\} - \min \left\{ F(\gamma_\alpha(t)) : t \in I \right\} \leq \omega(F, \alpha) \leq \alpha,$$

and the proof is now complete. 

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