Solving nonlinear Fredholm Volterra integro-differential equation by the \((G'/G)\) -expansion method

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Abstract
In this paper, we apply \((G/G)\) -expansion methods for solving the interested nonlinear Fredholm-Volterra integro-differential equation.

Keywords: \((G/G)\) -expansion methods, Nonlinear Fredholm Volterra integro-differential equation, Partial Differential Equation, Derivative, Second linear ordinary differential equation.

1 Introduction
Nonlinear equations are used for the study of many cases. Various methods proposed to solve nonlinear equations, including the Backlund transformation [8], Inverse method [3] Wronskian determinant technique [5], Numerical methods \([4,2]\), Hirota bilinear method [6]. There are other methods like Jacobi elliptic function method [7], homotopy perturbation method [9]. In addition, One of the methods used to solve nonlinear equations is the \((G/G)\) -expansion methods. This model applied for the first time by Wang et al [10]. Structure of the paper is the following: In section 2, some concepts, definitions and description of \((G/G)\) -expansion method are brought. In section 3, application of mentioned method to solve nonlinear Fredholm Volterra integro-differential equation is illustrated. In section 4, conclusion is drawn.

2 Preliminaries and description of the \((G'/G)\) - expansion method

Wang in [10] applied the \((G/G)\) -expansion method for solving the certain nonlinear partial differential equations \((PDEs)\)\(^1\). Thus considered the nonlinear \((PDEs)\) finding \(u(x,t)\), in the from

\[ p\left(u, u_t, u_x, u_{tt}, u_{xx}, ...\right) = 0, \]  \((2.1)\)

where \(P\) is a polynomial includes nonlinear terms and the highest order derivatives. Combining the independent variables \(x\) and \(t\) with one variable \(\xi = kx + \omega t\), reduces Eq. \((1)\) to the ordinary differential equation( ODE)

\[ p(U, kU', \omega U'', k^2 U'', k \omega U'', \omega^2 U'', ...\right) = 0, \]  \((2.2)\)
where $k, \omega$ are constants and the prime denotes the derivative with respective to $\xi$. Suppose solution $U(\xi)$ Eq. (2.2) can be expressed as a finite series as follows:

$$U(\xi) = \sum_{i=0}^{m} \alpha_i (G'/G)'.$$

(2.3)

The function $G(\xi)$ satisfies the second linear ordinary differential equation (LODE) as follows:

$$G''(\xi) + \lambda G' + \mu G = 0,$$

(2.4)

here the prime denotes the derivative in terms of $\xi$, where $\alpha_i$ for $i = 0, 1, \ldots, m$, $\lambda, \mu$ are real constants to be determined later. Then, we solve Eq. (2.4),

for $\lambda^2 - 4\mu > 0$ we have:

$$G(\xi) = e^{\frac{\lambda}{2\xi}} (c_1 e^{2(\lambda^2 - 4\mu)^{1/2}/2\xi} + c_2 e^{-2(\lambda^2 - 4\mu)^{1/2}/2\xi}),$$

(2.5)

Now we compute $(G'/G)$ as follows:

$$G'/G = \frac{(\lambda^2 - 4\mu)^{1/2}}{2} \left( \frac{c_1 \sinh(\frac{(\lambda^2 - 4\mu)^{1/2}}{2\xi}) + c_2 \cosh(\frac{(\lambda^2 - 4\mu)^{1/2}}{2\xi})}{c_1 \cosh(\frac{(\lambda^2 - 4\mu)^{1/2}}{2\xi}) + c_2 \sinh(\frac{(\lambda^2 - 4\mu)^{1/2}}{2\xi})} \right) - \frac{\lambda}{2},$$

(2.6)

While $\lambda^2 - 4\mu < 0$ we have

$$G(\xi) = e^{\frac{-\lambda}{2\xi}} (c_1 \cos(\frac{(4\mu - \lambda^2)^{1/2}}{2\xi}) + c_2 \sin(\frac{(4\mu - \lambda^2)^{1/2}}{2\xi}),$$

(2.7)

so,

$$G'/G = \frac{(4\mu - \lambda^2)^{1/2}}{2} \left( \frac{-c_1 \sin(\frac{(4\mu - \lambda^2)^{1/2}}{2\xi}) + c_2 \cos(\frac{(4\mu - \lambda^2)^{1/2}}{2\xi})}{c_1 \cos(\frac{(4\mu - \lambda^2)^{1/2}}{2\xi}) + c_2 \sin(\frac{(4\mu - \lambda^2)^{1/2}}{2\xi})} \right) - \frac{\lambda}{2},$$

(2.8)

To determine $U$ explicitly, we take the following four steps:[1]

Step 1. Determine the integer $m$ by substituting Eq. (2.4) along with Eq. (2.3) into Eq. (2.2) and balancing the highest order nonlinear term(s) and the highest order partial derivative.

Step 2. Substituting Eq. (2.4) with the value of $m$ determined in Step 1, along with Eq. (2.3) into Eq. (2.2) and collect all terms with the same order of $(G'/G)$ together, the left-hand side of Eq (2.2) is converted to another polynomial in $(G'/G)$. Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for $k$, $\omega, \alpha_0$ and $\alpha_i$, for $i = 1, 2, \ldots, m$.

Step 3. Solve the system of algebraic equations obtained in Step 2, for $k, \omega, \alpha_0$ and $\alpha_i$, ....$\alpha_m$, for $i = 1, 2, \ldots, m$, by use of Mathematica.

Step 4. Use the results obtained in above Steps to derive a series of fundamental solutions $u(\xi)$ of Eq. (2.2) depending on $(G'/G)$, since the solutions of Eq. (2.4) have been well known for us, then we can obtain the exact solutions of the given Eq. (2.1).

Now we are going to find the solution of nonlinear Fredholm Volterra integro-differential equation.

3 Numerical examples

Consider the integro-differential equation

$$u^*(x) + xu'(x) = e^x (2 + x^2 + 3x) - 0.5892858x + \int_0^x u(t)^2 \, dt + \int_0^x xt (1 + u(t))^2 \, dt.$$  

(3.9)

Then, by taking differential twice we have the following nonlinear ODE

$$u^{(4)}(x) + xu^{(3)}(x) + 2u^{(2)}(x) - u'(x)u(x) - e^x (x^2 + 7x + 10) = 0.$$  

(3.10)
According to \((G'/G)\) – expansion method and assume that the solution of Eq. (3.10) be in the following form:

\[
u(x) = v(\xi), \quad \xi = kx,
\]

(3.11)

where \(k\) is constant,

On the other hand we known that the Taylor expansion of \(x = \ln(v + 1)\) is,

\[
\ln(v + 1) = v - \frac{v^2}{2} + \frac{v^3}{3} - ... \quad (3.12)
\]

Now, Substituting Eq. (3.12) instead of \(x\), and Eq. (3.11) in the Eq.(3.10) we have:

\[
k^4v^{(4)} + k^3v^{(3)} - 1/2k^2v^3(3) - 2k^2v^{(2)} - kv - 1/4v^5 + 3/4v^4 + 7/2v^3 - 9/2v^2 - 17v - 10 = 0. \quad (3.13)
\]

Step 1. Firstly balancing between the highest order nonlinear terms(s) and the highest order partial derivative.

\[m + 4 = 5m, \quad \text{hence } m = 1.\]

Step 2. We determine the structure of \(v\). Then we find necessary powers and derivatives of \(v\) determine

\[
v = \sum_{i=1}^{m} \alpha_i (G'/G)^i \alpha_0, \quad (3.14)
\]

\[
v = \alpha_1 (G'/G) + \alpha_0, \n\]

\[
v^2 = \alpha_1^2 (G'/G)^2 + 2\alpha_1 \alpha_2 (G'/G) + \alpha_0^2,
\]

\[
v^3 = \alpha_1^3 (G'/G)^3 + 3\alpha_1 \alpha_2^2 (G'/G)^2 + 3\alpha_1^2 \alpha_3 (G'/G) + \alpha_0^3,
\]

\[
v^4 = \alpha_1^4 (G'/G)^4 + 4\alpha_1 \alpha_2^3 (G'/G)^3 + 6\alpha_1^2 \alpha_3^2 (G'/G)^2 + 4\alpha_1^3 \alpha_4 (G'/G) + \alpha_0^4,
\]

\[
v^5 = \alpha_1^5 (G'/G)^5 + 5\alpha_1 \alpha_2^4 (G'/G)^4 + 10\alpha_1^2 \alpha_3^3 (G'/G)^3 + 10\alpha_1^3 \alpha_4^2 (G'/G)^2 + \alpha_0^5.
\]

Continued Step 2 and collecting all terms same order of \((G'/G)\) together, we have:

\[
(G'/G)^3: 3\alpha_1^2 k^3 = 0,
\]

\[
(G'/G)^4: 24\alpha_1 k^4 - 6\alpha_2 k^3 + 6\alpha_2 k^3 + 6\alpha_3 k^3 - 1/4\alpha_4 = 0,
\]

\[
(G'/G)^5: 60\alpha_1 k^5 - 12\alpha_2 k^4 + 6\alpha_2 k^4 + (4\alpha_3 k^3, \mu + 7/2\alpha_3 k^3, \lambda + 12\alpha_3 k^3, \mu + 3\alpha_4 k^3 + 3\alpha_5 k^3
\]

\[
-5/4\alpha_4 k^4 + 3/4\alpha_5 = 0,
\]

\[
(G'/G)^6: 40\alpha_1 k^6 + 50\alpha_2 k^4 + 7\alpha_2 k^4 - 8\alpha_2 k^4 - 7\alpha_2 k^3 - 12\alpha_3 k^3, \lambda + 4\alpha_4 k^3, \mu + 1/2\alpha_4 k^3, \lambda
\]

\[
+8\alpha_6 k^5, \mu + 7\alpha_6 k^5, \lambda + 6\alpha_6 k^5, \lambda + 4\alpha_6 k^5, \mu + 10/4\alpha_6 k^5, \lambda + 3\alpha_7 k^5 + 7/2\alpha_7 = 0,
\]
\[(G'/G)^{3} : 60\alpha_{1}k^{4} \mu^{2} + 15\alpha_{1}k^{4} \lambda^{2} - 8\alpha_{1}k^{3} \lambda \mu - \alpha_{1}k^{2} \lambda^{3} - 8\alpha_{0}\lambda^{3}k^{2} \lambda^{2} + \alpha_{1}k^{3} \mu^{2} \\
+ 12\alpha_{0}k^{2} \lambda^{2} \mu + 8\alpha_{0}\lambda^{2}k^{3} \lambda \mu + 4\alpha_{0}\mu k^{3} \lambda^{2} + 7/2\lambda^{2}k^{3} \lambda^{2} + 6\alpha_{0}k^{2} \lambda + \alpha_{1}k \lambda \\
+ \alpha_{0}\lambda k - 10/4\alpha_{0}k^{2} + 18/4\alpha_{0}k^{2} + 21/2\alpha_{0}k^{2} + 9/2\alpha_{0}^{2} = 0, \]
\[(G'/G)^{3} : 16\alpha_{1}k^{4} \mu^{2} + 22\alpha_{1}k^{4} \lambda^{2} \mu + \alpha_{1}k^{4} \lambda^{4} - 8\alpha_{0}\lambda^{3}k^{3} \lambda \mu - 2\alpha_{1}k^{3} \lambda^{3} - 2\alpha_{1}k^{3} \lambda^{2} \mu \\
+ 2\alpha_{0}\lambda^{2}k^{3} \mu^{2} + \alpha_{0}\lambda^{2}k^{3} \lambda^{2} \mu + 4\alpha_{0}\lambda^{2}k^{3} \lambda^{2} \mu + 11/2\alpha_{0}\lambda^{2}k^{3} \lambda^{2} \mu + 4\alpha_{0}k^{2} \mu + 2\alpha_{0}k^{2} \lambda^{2} \\
+ \alpha_{1}k \lambda + \alpha_{0}\lambda k \lambda - 5/4\alpha_{1}k^{2} + 3\alpha_{0}\lambda k \lambda + 21/2\alpha_{0}k^{2} - 9\alpha_{0}\lambda k + 17\alpha_{1} = 0, \]
\[(G'/G)^{3} : 8\alpha_{1}k^{4} \lambda \mu^{2} + \alpha_{1}k^{4} \lambda^{2} \mu + \alpha_{0}\lambda^{2}k^{3} \mu^{2} - 2\alpha_{0}\lambda^{2}k^{3} \mu^{2} - \alpha_{0}\lambda^{2}k^{3} \lambda^{2} \mu + 1/2\alpha_{0}\lambda^{2}k^{3} \lambda^{2} \mu \\
+ 2\alpha_{1}k^{2} \lambda \mu + \alpha_{0}\lambda k \mu - 1/4\alpha_{1}k^{2} + 3/4\alpha_{0}k^{2} + 7/2\alpha_{0}k^{2} - 2\alpha_{0}^{3} - 17\alpha_{0} = 0. \]

Step 3. Solving the set of algebraic equations using mathematica we get the following results:
\[\alpha_{0} = -1, \alpha_{1} = 0\]
\[\alpha_{0} = 1 \pm 11^{1/2}, \alpha_{1} = 0\]
\[\alpha_{0} = 1 \pm 5^{1/2}, \alpha_{1} = 0\]

4 Conclusion

In this paper, we discussed about a method for solving nonlinear Fredholm Volterra integro-differential equation. This method explains an exact solution. For future works we are going to use it for solving nonlinear partial Fredholm Volterra integro-differential equation.

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