A note on Ultra-discrete equations

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Abstract

Recently ultra-discrete equations attract a lot of attention. One reason could be due to their simpler settings, formulation and handling. This note discuss one of the algebraic settings of these equations and the derivation of these equations from discrete equations. Further, we comment on some special solutions for certain ultra-discrete equations. We conclude the note with discussion of some of the known integrability tests for these equations.

Keywords: Max-Plus semi-field; Ultra-discrete equations, Tropical geometry.

2000 Mathematics Subject Classification: 39A10, 34N05, 65Q10, 14T05

1 Introduction

Recently, many mathematicians were interested in ultra-discrete equations. These equations gave them another prospective to view their results and systems. A diverse wide range of different equations and systems in many branches of applied mathematics could be recast as ultra-discrete equations. The setting of these equations in many instances is much simpler than the original setting. This could be a major motivation for mathematicians to be interested in them. The algebraic setting of these equations varies between max-plus algebra, min-plus algebra and minimax algebra. Some of these systems appeared in the middle of the last century e.g. the max-plus algebra (appeared first in Kleene’s paper on nerve sets and automata as it was stated in [40]), the min-plus algebra and the minimax algebra [11]. There are lots of fields (e.g. computer science, computer languages, finite automata, optimization problems on graph, stochastic systems ....etc) that used successfully the settings of these algebras to recast their problems in a simpler manner. In [11] there are illustrations for a number of applications in different fields recast their problems using these new algebras and solved them.

In this note, we start by illustrating one of the algebraic setting related to ultra-discrete equations i.e. the max-plus semi-field in section 2. We follow this by a discussion of the derivation of ultra-discrete equations and comments on special solutions of certain ultra-discrete equations in section 3. Since integrability is a desirable criterion for any system of equations, we conclude this note by illustrating and commenting on some known integrability detectors for ultra-discrete equations in section 4. One of the tests is based on Singularity Confinement [12]. It is given by N. Joshi and S. Lafortune [30, 34]. It was shown [44] that some non-integrable ultra-discrete equations have confined singularities. This led mathematicians to attempt to improve the Singularity Confinement method by complementing it with another test. Research suggested that this test is based on growth measure criterion. Therefore, some detectors have been proposed such as a test based on Singularity Confinement and Lyapunov exponent [23]. Finally, we end this

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section with some remarks about a new suggested integrability detector. There have been some encouraging numerical results suggesting that Tropical Nevanlinna theory might be used to comment on the integrability of ultra-discrete equations [40]. We end this note by giving some summarized remarks in section 5. In this note our primary goal is to shed some light on ultra-discrete equations, their derivation from discrete equations and their integrability tests. We present this material from an outsider of the field point of view. Although, the interested reader might find the material here is very basic and presented in a primitive way. The references in this note could be considered as a good database to start your research in the area involving ultra-discrete equations.

2 Preliminaries and Notations

2.1 The Max-Plus semi-field

The semi-ring under consideration is the max-plus semi-ring \((\mathbb{R}_{\text{max}}, \oplus, \odot)\). Here \(\mathbb{R}_{\text{max}}\) is the set of real numbers augmented with \(-\infty\). The operations of addition and multiplication are defined as follow:

\[
\begin{align*}
a \oplus b &= \max\{a, b\}, \\
a \odot b &= a + b,
\end{align*}
\]

(2.1)

where \(a \) and \(b \in \mathbb{R}\). By definition \(a \oplus -\infty = a, \forall a \in \mathbb{R}_{\text{max}}\). Hence, the additive identity is \(-\infty\). There is no additive inverse for any element \(a \in \mathbb{R}\). There is some suggested solutions for the problem of lack of inverses discussed in [29] and in the references of [35]. The max-plus semi-ring arose in many contexts in applied mathematics. For instance, it appears in control theory, optimization and mathematical physics [19, 22, 27]. This type of semi-ring is referred to as Idempotent semi-ring in the sense that \(x \oplus x \oplus \ldots \oplus x = x\), [26]. The term Tropical semi-ring is used to refer to such semi-ring. This term was first used by French mathematicians to honour their Brazilian colleague Imre Simon according to the authors of [41]. This semi-ring was first appeared in middle of the last century in Kleene’s paper on nerve sets and automata according to [40]. Another semi-ring that is Idempotent and Tropical the min-plus semi-ring where \(\mathbb{R}_{\text{min}} = \mathbb{R} \cup \{\infty\}\) and \(a \oplus b = \min\{a, b\}, a \odot b = a + b\).

In this section we will give a formal definition of the max-plus semi-field over \(\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}\). It is a semi-ring \((\mathbb{R}_{\text{max}}, \oplus, \odot)\) where addition and multiplication operations are defined as in (2.1) with all elements \(a \in \mathbb{R}_{\text{max}}\) except for \(-\infty\) have multiplicative inverses. By definition

\[
a \odot -\infty = -\infty, \quad \forall a \in \mathbb{R}_{\text{max}}.
\]

(2.2)

The multiplicative identity is 0 and the multiplicative inverse is \(-a\) for any \(a \in \mathbb{R}_{\text{max}}\) except for \(-\infty\). In this semi-field all the usual commutative, associative and distributive axioms hold [35].

\[
\begin{align*}
\begin{align*}
a \oplus b &= b \oplus a, \\
a \oplus (b \odot c) &= (a \oplus b) \odot c, \\
a \odot b &= b \odot a, \\
a \odot (b \oplus c) &= (a \odot b) + (a \odot c), \\
a \odot (b \oplus c) &= (a \odot b) \oplus (a \odot c),
\end{align*}
\end{align*}
\]

(2.3)

where \(a, b \) and \(c \in \mathbb{R}_{\text{max}}\). There is also one property that holds for the operation \(\oplus\) and is called the idempotent law (2.4) which opens the door for a very rich algebra called the idempotent algebra, more details could be found in [25] and [28].

\[
a \oplus a = a,
\]

(2.4)

where \(a \in \mathbb{R}_{\text{max}}\). Sometimes the max-plus semi-field is called idempotent semi-field because of property (2.4). Another term is used for max-plus semi-field is tropical semi-field. According to [28] this term was first introduced in computer science to represent the discrete version of max-plus algebra \(\mathbb{R}_{\text{max}}\) or min-plus algebra \(\mathbb{R}_{\text{min}}\) and their subalgebras. Properties in (2.2 -2.4) could be used as a starting point for the theory of linear algebra over the system
$$\mathbb{R}_{\text{max}}$$.

We will denote the $r$-fold (product of an element $a$ with itself $r$ times) by a power,

$$a^{(r)} = a \otimes a \otimes \cdots \otimes a.$$  \hspace{1cm} (2.5)

To distinguish the regular exponent that is produced from the ordinary multiplication operation and the exponent that is a result of the operation $\otimes$ we used a brackets around the exponent in (2.5). From (2.1) it is clear that $a^{(r)} = ra$, $\forall a \in \mathbb{R}_{\text{max}}$ and $r$ is a positive integer. Let us define the zero and negative exponents by

$$a^{(0)} = 0, \quad a^{(-r)} = -ra \quad (r > 0).$$  \hspace{1cm} (2.6)

Another axiom could be added here

$$(a \oplus b)^{(r)} = a^{(r)} \oplus b^{(r)} \quad (r \geq 0),$$  \hspace{1cm} (2.7)

where the proof is given in [9]. Now we are ready to introduce a new notation and concept for a quotient in our max-plus algebra. We will denote the quotient by $\oslash$ to distinguish it from the usual quotient in the classic algebra. Now if we have two expressions in the max-plus algebra $U$ and $V$, then we define the quotient by [11],

$$U \oslash V = U \oslash V^{(-1)} = U - V.$$  \hspace{1cm} (2.8)

Let us take an example to illustrate this definition.

$$(4 \oplus 7 \otimes x^{(-1)}) \oplus (3 \otimes x \oplus x) = \max\{4, 7 - x\} - \max\{3 + x, x\},$$  \hspace{1cm} (2.9)

where we have used the definitions in (2.8) and (2.6). In addition, we could define the operator $\min$ in the max-plus semi-field as follows,

$$\min\{a, b\} = a + b - \max\{a, b\} = (a \oplus b) \ominus (a \odot b).$$  \hspace{1cm} (2.10)

The max-plus semi-field can be introduced from a different prospective similar to quantum theory. The parameter $h$ below will play a role similar to Planck’s constant in quantum theory. Consider the semi-ring $\mathbb{R}_{+, +, \times}$ where $\mathbb{R}_{+}$ is the set of all nonnegative real numbers and $+$ and $\times$ are the classical addition and multiplication operations. Define a map $\Phi_h : \mathbb{R}_{+} \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$\Phi_h(x) = h \ln x \quad (h > 0).$$  \hspace{1cm} (2.11)

Now let the classical addition and multiplication be mapped from $\mathbb{R}_{+}$ to $\mathbb{R} \cup \{-\infty\}$ by $\Phi_h$. Hence, for $a = \Phi_h(x) = h \ln(x)$ and $b = \Phi_h(y) = h \ln(y)$, where $x = e^{a/h}$, $y = e^{b/h}$, we have

$$\Phi_h(x + y) = h \ln(x + y) = h \ln(\exp(a/h) + \exp(b/h)) = a \oplus_h b,$$

$$\Phi_h(xy) = h \ln(xy) = h \ln(x) + h \ln(y) = \Phi_h(x) + \Phi_h(y) = a + b = a \odot b.$$  \hspace{1cm} (2.12)

Note that

$$a \oplus b = \lim_{h \rightarrow 0} a \oplus_h b = \lim_{h \rightarrow 0} h \ln(\exp(a/h) + \exp(b/h))) = \max\{a, b\}. \hspace{1cm} (2.12)$$

Also, note that from (2.11) the additive identity $0$ mapped to $-\infty$ and the multiplicative identity $1$ mapped to $0$. Therefore, $\mathbb{R} \cup \{-\infty\}$ forms a max-plus semi-field with respect to the operations $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$ with identity $-\infty$ for the operation $\oplus$ and $0$ as the identity for the operation $\odot$. All the axioms and the properties discussed above are valid here. In analogy with quantum theory $\mathbb{R}_{+, +, \times}$ could be viewed as a quantum object and $\mathbb{R}_{\text{max}}$ as a result of its dequantization [25]. The process in identity (2.12) is known as dequantization or ultra-discretization. The reference [36] considers this semi-ring with an associated linear problems. Despite the restrictive nature of this semi-ring the author of [36] shows that it is possible to define a connection matrix. The connection matrix in the Birkhoff sense and his school for systems of linear difference equations over this semi-ring. Further, from this
prospective the author provides evidence for integrability of ultra-discrete counterpart of the second Painlevé equation $P_{II}$.

Recently, strong links between integrable cellular automata and tropical geometry were found. The first ultra-discrete equation was related to box and ball systems according to [40]. The ultra-discrete equations are obtained through the process of ultra-discretization. Obviously, they are expressed in terms of the max-plus semi-field expressions. In the next sections we discuss and explore the ultra-discrete equations and their integrability.

3 Ultra-discrete equations

Ultra-discrete equations are the equations where both independent and dependent variables take discrete values. Hence, they are the equations obtained by discretization of the dependent variable in discrete equations. If both variables take integer values in the ultra-discrete equation, then we call it Cellular Automaton (CA). In [31], the authors defined CA as a rule-based computing machine. It was first initiated by von Neumann in the early 1950's [1]. Later in 1980's many systematic studies and research were conducted. Mainly, it was led by Wolfram [2, 3, 4, 5, 6, 7, 8]. He investigated the rich dynamics and complexity of CA's which yield his well known classification of 1D cellular automata. Details of this is given in [31] and its references. A relation between systems of CA's and systems of differential equations and partial differential equations is summarized and illustrated in [31]. A well known example of Cellular Automata is the Box-Ball system. Generally, it arose in two fields. Namely, quantum and classical integrable systems through crystalization and ultra-discretization procedures respectively [46]. It is known that this system is an integrable CA on one dimensional lattice. Its integrability was verified by showing that it is linked to many integrable properties and aspects as shown in [46]. Furthermore, a formula to obtain $N$-soliton solutions was constructed in [15].

It had been shown in [15] that this system is related to the well known nonlinear wave equation Korteweg-de Vries (KdV). The flow chart below describes the process of ultra-discretization as it was used in [15].

Many mathematicians obtained ultra-discrete equations by applying the limit in (3.13) to discrete equations,

$$\lim_{\epsilon \to 0^+} \epsilon \log(e^{X_i/\epsilon} + e^{X_j/\epsilon} + \cdots) = \max\{X_1, X_2, \cdots\}. \quad (3.13)$$
They used the change of variable \( x = e^{X/n} \) in the discrete equation, then applied (3.13) to obtain the corresponding ultra-discrete equation. Observe that this substitution requires that \( x \) to be positive. The Box-Ball system was obtained by applying the ultra-discrete limit (3.13) to the corresponding discrete equation related to the KdV equation. Now let us illustrate the derivation of ultra-discrete equations by the following example.

**Example 3.1.** We find an ultra-discrete equation related to a discrete version of the First Painlevé equation:

\[ d-P_1: x_{n+1}x_{n-1} = \alpha \lambda^a + \frac{1}{x_n}, \]

where \( a, \lambda \) are constants. Let us start by writing the equation as:

\[ x_n x_{n+1} x_{n-1} = \alpha \lambda^a x_n + 1. \]

Assume that \( x_j = e^{X/n} \) for \( j \in \{n-1,n,n+1\} \) and for \( a, \lambda > 0 \) we let \( a = e^{\beta/\varepsilon}, \lambda = e^{1/\varepsilon} \). Substitute these expressions in (3.14), we get

\[ \exp \left( \frac{X_n + X_{n+1} + X_{n-1}}{\varepsilon} \right) = \exp \left( \frac{X_n + \beta + a}{\varepsilon} \right) + e^0. \]

Taking the natural logarithm followed by the limit as \( \varepsilon \to 0^+ \) for both sides and applying (3.13) yield

\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + \beta + n, 0), \]

which is an ultra-discrete equation related to \((d-P_1)\).

Some of the natural questions that the reader might have are about solutions of ultra-discrete equations. How to obtain them? Are they expressed in terms of transcendental functions? What about the growth criteria of these solutions? Some literature had been devoted to answer some of these questions. The author of [43] presented an ultra-discrete analogue of \( q - P(A_n) \) equation. He derived exact solutions with two parameters for this equation. In addition, the authors of [17] investigated existence of special solutions to ultra-discrete analogues of Painlevé equations. In particular, they showed that these equations possessed an explicit invariant that is solved in terms of ultra-discrete analogues of elliptic functions. While the authors in [45] were interested in investigating the relation between ultra-discrete equations and Nevanlinna theory through the terminology of Tropical Nevanlinna theory. Hence, the authors explored methods of solving ultra-discrete equations in terms of tropical meromorphic functions. Also, they presented general solutions of certain classes of ultra-discrete equations, and classes of special rational and hypergeometric solutions of ultra-discrete Painlevé equations. In addition, they described and illustrated how to use the Tropical Nevanlinna theory tools in studying the growth of tropical meromorphic solutions of ultra-discrete equations.

### 4 Integrability and ultra-discrete equations

Integrability is a desirable criterion for any system of equations. Many research were conducted towards the quest of integrability in ultra-discrete equations. Proposing detectors to detect integrability of ultra-discrete equations and to derive them generated an extensive area of research recently.

The first question that the reader might ask is what we mean by integrable ultra-discrete equation? Literature [18, 23, 36, 39] showed that integrable ultra-discrete equations possess properties compatible with integrable differential equations. They considered this to be evidence and indication for integrability in ultra-discrete equations. Some of these properties are Lax-Pair, auto-Bäcklund and Miura transformations, existence of degeneration Cascades, special solutions of various kinds and Contiguity relations for solutions. On the other hand, slow growth of some characteristics of equations were related to integrability. Arnold [10] and Veselov [13] were the pioneers in observing this relation. Therefore, in this note we follow this description whenever we mention integrable ultra-discrete equations.

In differential equations, Painlevé property (i.e. all movable singularities of all solutions are poles) is a necessary condition for integrability. There have been many proposed tests in discrete equations analogues for the Painlevé property in differential equations. A number of mathematicians interpreted Veselov’s statement “... Integrability has an essential correlation with the weak growth of certain characteristics” in their quest for integrability detectors [24]. Consequently, many integrability detectors for discrete equations were proposed some of which are Algebraic entropy [20], Diophantine integrability [32] and Nevanlinna theory approach [33, 38]. Viallet and his colleagues interpreted Arnold’s idea of complexity by proposing a detector of integrability based on algebraic entropy [20, 21].
integrability detectors was taking another direction by studying the singularity of the discrete equations. This study led to another proposed integrability detector i.e. *Singularity Confinement* [12]. Singularity Confinement is linked to the disappearance of the singularity of a discrete equation after a few iterations provided that the initial data is preserved. Interestingly, in the discrete world results derived in [37] showed that links between integrable and non-integrable systems can be found.

Proposing detectors of integrability for ultra-discrete equations is an area of research that attracted a lot of attention lately. Some mathematicians attempted to extend the integrability detectors of the discrete equations to propose their ultra-discrete counter parts. Others, assumed that deriving ultra-discrete equations from their integrable discrete counterparts is sufficient to guarantee integrability of ultra-discrete equations.

In [30, 34] the authors extended the Singularity Confinement to ultra-discrete equations. They proposed a procedure based on the concept of Singularity Confinement as a new detector of integrability for ultra-discrete equations. They used their criterion to derive ultra-discrete analogues of Painlevé equations and other ultra-discrete equations as counter parts of integrable differential equations. The next example illustrate their procedure [30].

**Example 4.1.**

\[ \frac{x_{n+1}}{x_n} = \frac{k + 1}{x_n}, \]  

where \( k \) is constant. Note that this equation is a member of the class of integrable second order discrete Quispel-Roberts-Thompson equations (i.e. QRT mapping), the autonomous version. Equation (4.15) admits the conserved quantity \( I \) given in

\[ I = \frac{x_n + x_{n-1}}{x_n} + \frac{k}{x_{n-1}} + \frac{1}{x_n x_{n-1}}. \]

The singularity is when \( x_n \to 0 \). Also, \( x_{n-1} \) is nonzero arbitrary constant. This singularity is considered movable because both the pre-image \( x_{n-1} \) and the initial point of iteration \( n, x_n \) are arbitrary.

To study the behaviour around \( x_n = 0 \), we set \( x_n = -\frac{1}{k} + \epsilon, \) for small value \( |\epsilon| > 0 \) (see [34]). Here, \( x_{n+1} \to 0 \) as \( \epsilon \to 0 \), the other iterates are:

\[
\begin{align*}
x_{n+1} &= -\frac{k}{x_0} \epsilon + O(\epsilon^2), \\
x_{n+2} &= \frac{x_{n-1}}{\epsilon k} + O(1), \\
x_{n+3} &= -\frac{x_{n-1}}{\epsilon k} + O(1), \\
x_{n+4} &= \frac{k^2}{x_{n-1}} \epsilon + O(\epsilon^2), \\
x_{n+5} &= -\frac{1}{k} + O(\epsilon), \\
x_{n+6} &= x_{n-1} + O(\epsilon).
\end{align*}
\]

The next iterates are well defined as \( \epsilon \to 0 \). Therefore, the singularity in (4.15) is confined. Also, we could obtain discrete versions of Painlevé equations by deautonomizing i.e. by assuming \( k \) is a function of \( n \) denoted by \( \phi_n \),

\[ \frac{\phi_n}{\phi_{n-1}} = \phi_0 + \frac{1}{x_n}. \]

Now requesting the singularity \( x_0 = 0 \) to be confined as in the autonomous case give rise to the following condition,

\[ \frac{\phi_0 \phi_{n+5}}{\phi_{n+3} \phi_{n+2}} = 1. \]

It has been remarked by the authors in [34] that the singularity confinement property applies only to movable singularities. The nonconfinement of fixed singularities is not inconsistent with integrability according to the authors.

It has been shown that ultra-discretization method for (4.15) yields

\[ X_{n+1} + X_n + X_{n-1} = \max\{X_n + K, 0\}. \]
Note that (4.20) is defined for any real initial conditions. To derive integrable ultra-discrete equations using the Singularity Confinement analysis method, we follow the steps used in [34]. In (4.20), we consider initial data at an arbitrary $n$ where the right hand side of (4.20) fails to be differentiable. Since ultra-discrete equations are piece-wise continuous linear equations their derivatives at each point is integer except at end points of each interval. This happens whenever $X_n = -K$ in (4.20). Set $X_n = -K + \varepsilon$, $|\varepsilon| > 0$ where $X_{n-1} > 2|K|$ but otherwise arbitrary. Iterate the equation in (4.20) and we get the following result:

<table>
<thead>
<tr>
<th>Iterate</th>
<th>$\varepsilon &gt; 0$</th>
<th>$\varepsilon &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_n$</td>
<td>$-K + \varepsilon$</td>
<td>$-K + \varepsilon$</td>
</tr>
<tr>
<td>$X_{n+1}$</td>
<td>$K - X_{n-1}$</td>
<td>$K - X_{n-1} - \varepsilon$</td>
</tr>
<tr>
<td>$X_{n+2}$</td>
<td>$X_{n-1} - \varepsilon$</td>
<td>$X_{n-1}$</td>
</tr>
<tr>
<td>$X_{n+3}$</td>
<td>$X_{n-1}$</td>
<td>$X_{n-1} + \varepsilon$</td>
</tr>
<tr>
<td>$X_{n+4}$</td>
<td>$K - X_{n-1} + \varepsilon$</td>
<td>$K - X_{n-1}$</td>
</tr>
<tr>
<td>$X_{n+5}$</td>
<td>$-K - \varepsilon$</td>
<td>$-K - \varepsilon$</td>
</tr>
<tr>
<td>$X_{n+6}$</td>
<td>$X_{n-1}$</td>
<td>$X_{n-1}$</td>
</tr>
</tbody>
</table>

It is clear from the above table that $X_{n+1}, X_{n+2}, X_{n+3}$ and $X_{n+4}$ are non-differentiable at $X_n = -K$. Since $X_{n+1}$ for $\varepsilon > 0$ and $\varepsilon < 0$ are not equal for $i \in \{1, 2, 3, 4\}$, observe that in $X_{n+5}$ and $X_{n+6}$ the iterates coincide. Hence, the differentiability is recovered in $X_{n+5}$ and $X_{n+6}$. This shows the well posedness of the ultra-discrete equation even for nearly singular values of initial values of (4.20) according to the authors. Consequently, the singularity was confined with the initial data survived.

Further, they claim that this characterizes integrability. But the authors showed as well that

$$X_{n+1} + 3X_n + X_{n-1} = \max\{X_n + K, 0\}$$

do not possess the Singularity Confinement property where the singularity is not fixed.

In discrete equations, it had been shown that non-integrable discrete equation passed the Singularity Confinement test [20]. This led to proposing another integrability test based on the degree growth of some initial data under the action of the equation and this test is called Algebraic entropy. This test was first initiated by Viallet et al. [20, 21]. In [44], examples of integrable ultra-discrete equation with unconfined singularities and non-integrable ultra-discrete equation with confined singularities were derived. Hence, the author of [44] suggested that growth criteria of ultra-discrete equations solutions to be investigated as well to give an indication of equations integrability. This inspired the quest of integrability detectors for ultra-discrete equations. Therefore, augmenting Singularity Confinement with another growth measure criterion motivated a number of mathematicians.

Another detector of integrability for ultra-discrete equations was proposed. It is based on Singularity Confinement and Lyapunov exponent. This detector was proposed by the authors of [23]. To illustrate their method the authors considered the following equation. Note that it is a general form of equation (4.15).

$$x_{n+1}^\sigma x_{n-1} = \alpha^{\lambda/\varepsilon} x_n + 1. \quad (4.21)$$

For $\sigma = 0, 1, 2$ with $\lambda = 1$ equation (4.21) is related to QRT. Further, for generic $\lambda$ with $\sigma = 0, 1, 2$ this equation is known as discrete analogue to Painlevé I. Hence, it is considered as integrable equation. To derive an ultra-discrete version of (4.21) we apply the following transformation:

$$x_n = e^{x_n/\varepsilon}, \quad \alpha = e^{\lambda/\varepsilon}, \quad \lambda = e^{\lambda/\varepsilon}. \quad (4.22)$$

Applying the ultra-discretization procedure we get

$$X_{n+1} + \sigma X_n + X_{n-1} = \max(0, X_n + nL + A). \quad (4.23)$$

Here, equation (4.23) is considered integrable for $\sigma = 0, 1, 2$ and $L = 0$ (autonomous case). This is due to two reasons. The first is that each equation has a conserved quantity. The second is that each equation admits a general solution in terms of ultra-discrete analogue of elliptic functions [17]. The authors of [23] found the singularity patterns of (4.23)
solution \( X_n \) for \( n = 2 \) when \( A = L = 0 \) i.e. there are no singular parameters \( \alpha \) and \( \lambda \). They showed that the singularity patterns of the solution \( x_n \) of equation (4.21) is transformed consistently through ultra-discretization procedure to the singularity patterns of equation (4.23) solution \( X_n \).

On the contrary, in the case of singular parameters \( \alpha \) and \( \lambda \) in (4.21) the singularity patterns of the solution cannot be transformed consistently from (4.21) to (4.23). Note that the authors assume that the parameter singularities occur if in (4.22) \( A \) and \( L \) are not zero. This is related to singularities in \( \sigma \) and \( \lambda \) and consequently in \( x_n \) in (4.21). Therefore, the method needs to be improved in the non-autonomous case by investigating another criterion to test for integrability.

Hence, the authors were interested in studying the growth rate of perturbation in the singularities patterns of equation (4.23) when \( \sigma = 2 \) and \( \sigma = 3 \).

They assumed that \( X_0 \) is a solution for equation (4.23) when \( \sigma = 2 \) and \( X'_n \) is one perturbed by \( \rho \). The amplitude of perturbation \( \delta_{tn} = \frac{X_n - X'_n}{\rho} \). They showed that for the following case where \( \sigma = 2, A = 3, L = 2, X_0 = 0, X'_0 = \phi \) and \( X_1 = X'_1 = 1 \) for equation (4.23), \( \delta_{tn} \) grows linearly for \( n < 0 \). While, \( \delta_{tn} \) does not grow for \( n > 0 \). Consequently, they concluded that the solution \( X_0 \) is not chaotic and global behaviour of the solution can be estimated.

A measure of growth and complexity of a system which is used as an integrability check is Lyapunov exponent. It is a mean growth rate of perturbation which is defined by \( \lambda_{\sigma}(\lambda_{-\sigma}) \) is the exponent [14]. When \( \lambda_{\sigma} > 0 \), the perturbation grows exponentially, hence, the system becomes chaotic. The authors, showed that for equation (4.23) with \( \sigma = 3, X_0 = 0, X'_0 = \rho \) and \( X_1 = X'_1 = 1 \), the quantity \( \lambda_{10n} \) is asymptotically about 0.66 for large \( n \). Therefore, they concluded that equation (4.23) with \( \sigma = 3 \) is chaotic.

Illustrated examples in [44] show that exponential growth of the values of iterates indicates non-integrability and linear growth indicates linearisability. Although the growth properties may be used as an aided tool in the integrability quest in ultra-discrete equation but it is not as powerful tool as in the discrete case. Therefore, searching for integrability detector for ultra-discrete equations using the Tropical algebraic geometry settings could be more successful and effective. Therefore, in [45] the authors examined and explored the growth of tropical meromorphic solutions of ultra-discrete equations.

In [40], the authors intend to define a measure of complexity of solutions of ultra-discrete equations. They are motivated by the idea of finding a criterion that will single out integrable ultra-discrete equations from others. Therefore, they are using ideas and concepts from Tropical Nevanlinna theory to obtain a natural criterion that is ultra-discrete analogue of Painlevé property in differential equations. They constructed a strong base of theory serves them in their quest of the integrability detector for ultra-discrete equations. One of the lemmas that they proved in [40] is the following.

**Lemma 4.1.** Let \( K \) be a positive constant and let \( y \) be a max-plus meromorphic solution of

\[
y(x + 1) + 3y(x) + y(x - 1) = \max(y(x) + K, 0)
\]

such that \( y(0) > 0 \) and \( y(1) < K \). Then \( y \) has infinite order.

Here the max-plus meromorphic functions are continuous piecewise linear functions of a real variable whose one-sided derivative are integers at every point. Preliminary numerical results suggesting that the existence of infinitely many finite order max-plus meromorphic solutions in the sense of Tropical Nevanlinna theory is a good candidate to be an ultra-discrete analogue of the Painlevé property [40, 42].

5 Summary

In this note we were interested in shedding some light on ultra-discrete equations and the algebraic setting of these equations. In addition, we aim to collect the most relevant material related to these equations in one place. This will make this note a good starting place for an interested reader in these equations. Therefore, we started this note by a description of one of the fundamental algebraic setting of these equations i.e. max-plus semi-field. Then we discussed the derivation of these equations from discrete equations and commented on a number of special solutions for certain ultra-discrete equations. As in the discrete equations, ultra-discrete equations have rich structure and the theory that is related to them is not extensively explored yet. Lately, it attracted a lot of attention and many research
were conducted. Hence, we concluded this note by a topic which is considered to be an active area of research i.e. integrability in the ultra-discrete equations. We summarize some of the well known proposed integrability detectors. A test based on Singularity Confinement [30, 34] was illustrated. Although, this test was successful in deriving integrable ultra-discrete equations but examples of integrable ultra-discrete equation with unconfined singularities and non-integrable ultra-discrete equation with confined singularities were derived [44]. The authors of [23], want to improve the Singularity Confinement test by augmenting it with a measure of growth rate such as Lyapunov exponent. Furthermore, the authors of [40] were interested in using the setting of Tropical Nevanlinna theory to derive an integrability detector for ultra-discrete equations. Preliminaries numerical results suggested that the existence of infinitely many finite order max-plus meromorphic solutions in the sense of Tropical Nevanlinna theory is a good candidate to be an ultra-discrete analogue of the Painlevé property.

Conflict of Interests
The author declare that she has no any conflict of interests.

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