Comparison between Spectral perturbation and Spectral relaxation approach for unsteady heat and mass transfer by MHD mixed convection flow over an impulsively stretched vertical surface with chemical reaction effect

T. M. Agbaje¹*, S. S. Motsa¹

(1) School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal Private Bag X01 Scottsville 3209, Pietermaritzburg, South Africa

Abstract

In this study, the spectral perturbation method (SPM) is utilized to solve the momentum, heat and mass transfer equations describing the unsteady MHD mixed convection flow over an impulsively stretched vertical surface in the presence of chemical reaction effect. The governing partial differential equations are reduced into a set of coupled non-similar equations and then solved numerically using the SPM. The SPM combines the standard perturbation method idea with the Chebyshev pseudo-spectral collocation method. In order to demonstrate the accuracy and efficiency of the proposed method, the spectral perturbation (SPM) numerical results are compared with numerical results generated using the spectral relaxation method (SRM) and a good agreement between the two methods is observed up to a minimum of eight decimal digits. Several simulation are conducted to ascertain the accuracy of the SPM and the SRM. The computational speed of the SPM is demonstrated by comparing the SPM computational time with the SRM computational time. A residual error analysis is also conducted for the SPM and the SRM in order to further assess the accuracy of the SPM. The study shows that the spectral perturbation method (SPM) is more efficient in terms of computational speed when compared with the SRM. The study also shows that the SPM can be used as an efficient and reliable tool for solving strongly nonlinear boundary value partial differential equation problems that are defined under the Williams and Rhyne [3] transformation. In addition, the study shows that accurate results can be obtained using the perturbation method and thus, the conclusions earlier drawn by researchers regarding the accuracy of perturbation methods is corrected.

Keywords: Chebyshev spectral perturbation method, Chebyshev spectral relaxation method

1 Introduction

In the past few decades, the study of mixed convection flow with or without the heat and mass transfer under the effect of magnetic field and chemical reaction has attracted the interest of considerable number of researchers because of its numerous applications in several branches of science and engineering and in the transport processes. These
applications include the power and cooling industry for drying, cooling of nuclear reactors, chemical vapor deposition on surfaces and Magneto hydrodynamic (MHD) power generators [1, 2]. In the design of chemical process equipment, distribution of temperature and moisture over agriculture fields, combined heat and mass transfer plays an essential role [1, 2]. In addition, Magneto-hydrodynamics (MHD) flow also plays a crucial role in the polymer, engineering metallurgical, petroleum and agriculture industries [1, 2].

The problem of unsteady heat and mass transfer by MHD mixed convection flow over an impulsively stretched vertical surface with chemical reaction effect has been studied numerically in this dissertation. The problem considered extended the analysis of [4] by introducing chemical reaction effects. [4] previously used the implicit finite difference method known as Keller - box to analyze the melting effect on unsteady heat and mass transfer by MHD mixed convection flow over an impulsively stretched vertical surface in a quiescent fluid in the absence of chemical reaction. Recently, researchers have shown an increased interest in the study of the combined effects of heat and mass transfer with chemical reactions due to its numerous importance to scientists and engineers. The effect of chemical reaction with heat transfer cannot be neglected because of its universal occurrence in many branches of science and engineering. Possible application of this type of flow can be found in many engineering processes such as oxidation of solid materials, production of polymers, food processing, synthesis of ceramic materials and tubular reactors [1].

The equations describing unsteady heat and mass transfer by MHD mixed convection flow over an impulsively stretched vertical surface with chemical reaction effect are characterized by systems of nonlinear partial differential equations (PDEs) defined on a semi-infinite domain in both time and space. The PDE based unsteady flow equations are more difficult to solve when compared to that of steady flows which are often modeled by nonlinear ordinary differential equations (ODEs). To simplify the analysis of the class of nonlinear PDEs describing the unsteady boundary layer flow, Williams and Rhyne [3] presented a convenient transformation of converting the infinite time scale $\tau = [0, \infty)$ of unsteady problems to a finite domain $\xi = [0, 1]$ of integration. In recent times, there has been an increasing number of research that utilize the transformation of Williams and Rhyne [3] on the solution of unsteady flows using perturbation expansions for $\xi < 1$. These investigations includes the study of Seshadri et al.[6] who used the perturbation series approach for the solution of unsteady mixed convection flow along a heated vertical plate. The perturbation series approach was also employed by Nazar et al. [7, 8, 9] to obtain first-order perturbation approximation of the solution of unsteady boundary layer flow due to a stretching surface in a rotating fluid, unsteady boundary layer flow in the region of the stagnation point on a stretching sheet, and unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium respectively.

The limitation of the perturbation series approach used in [6, 7, 8, 9] is that the obtained perturbation series differential equations, despite being linear, cannot be solved exactly beyond the equation for first term or second term solution. This observation was highlighted by Liao [10] who noted that the difficulty in obtaining higher order solutions was due to the appearance of a combination of error function terms and exponential function term in the first solutions. In the research conducted by [6, 7, 8, 9] only the first order solutions were presented. It was pointed out by Liao [10, 11] that the perturbation results were only valid for a minimal frame of time.

An analytical approach was reported by Liao [10] as an alternative approach to overcome some of the limitations of the perturbation method. The approach is based on the homotopy analysis method (HAM) and gives results which are uniformly valid for all time. Ever since then, many researchers have utilized the method in solving unsteady flow problems. These includes Liao [11], Xu et al. [12, 13], Cheng et al. [14], Xu and Liao [15], Ali and Mehmood [16], Mehmood et al. [17, 18], Sajid et al. [19, 20], Kumari and Nath [21, 22], Hayat et al. [23], Nadeem et al. [24]. According to Liao [10], one major advantage of the homotopy analysis method (HAM) over the standard perturbation method for the solution of unsteady boundary layer flow problems is that the (HAM) provides flexibility in the choice of initial approximation and linear operator which can be chosen carefully so that the higher order approximation can be integrated analytically. This HAM advantage contradicts the conclusion earlier drawn by researchers about the perturbation methods that analytical solution cannot be obtained beyond first order approximation for higher order perturbation equations in unsteady flow problems. Surveys such as that conducted out by [11] have noted that in the application of the HAM technique, the nonlinear PDEs modeling the unsteady flow problems are reduced to an infinite number of linear ordinary differential equations which are governed by an auxiliary linear operator which can be used to control the convergence of the solution.

The aim of this present study is firstly, to introduce a new approach called the spectral perturbation method (SPM) for solving nonlinear PDEs. In this current work, we apply for the first time, the spectral perturbation method (SPM) to solve nonlinear PDEs describing the unsteady heat and mass transfer by MHD mixed convection flow over an impul-
sively stretched vertical surface with chemical reaction effect. The SPM, blends the standard perturbation approach with the Chebyshev pseudo-spectral method to generate numerical solution of higher order perturbation equations describing the flow which are not possible to solve analytically. With the SPM, solutions to partial differential equations can be obtained by applying discretization only in the space direction. Applying discretization only in the space direction and integrating using the Chebyshev spectral collocation method makes the SPM computationally efficient. The Chebyshev pseudo-spectral method is employed because of its high level accuracy. Also, using the spectral methods, only few grid points are required to yield accurate results. In addition, using the spectral method to integrate the perturbation equations, very accurate solutions that are valid for all time domain are obtained. Another aim of the study is to extend the analysis of El-Kabeir and Rashad [4] in this investigation by adding the chemical reaction effects. Results generated using the spectral perturbation method (SPM) were compared and validated using the spectral relaxation method (SRM) and the two methods were found to be in excellent agreement. The spectral relaxation method (SRM) is an innovative iterative method that has shown to be convenient in obtaining solution to nonlinear equations. The SRM is based on simple decoupling and rearrangement of the governing nonlinear equations in a Gauss-Seidel manner. The resulting sequence of linear differential equations are then discretized and solved using the Chebyshev pseudo-spectral method. The SRM has been used successfully by considerable number of investigators for the solution of fluid mechanics based ODEs and PDEs problems. (See for instance, Motsa et al. [25, 26, 27, 28], Kameswaran et al. [29, 30], Motsa and Makukula [31], Motsa [32], Shateyi [33], Shateyi and Marewo [34] Shateyi and Makinde [35], Shateyi and Prakash [36 , Makukula et al. [37]). The results obtained shows that the proposed spectral perturbation method (SPM) can be used efficiently to solve partial differential equations by applying the discretization only in the space direction.

2 Mathematical Formulation

Following [4, 5, 22], we investigate the unsteady, laminar heat and mass transfer by MHD mixed convection boundary layer flow of an electrically conducting fluid over a heated vertical linearly stretched sheet with a supporting external laminar flow in the presence of a chemical reaction. A uniform magnetic field is applied in the transverse direction y normal to the plate. It is assumed that the wall is impulsively stretched with a velocity \( u \), which is proportional to the distance \( x \) along the sheet surface. The sheet surface is maintained at a variable temperature \( T = T_w = T_{\infty} + bx \) and a variable concentration \( C = C_w = C_{\infty} + dx \). The stream is kept at a constant temperature \( T_{\infty} \) and a constant concentration \( C_{\infty} \) far from the sheet surface. Initially \( t < 0 \), the temperature \( T_{\infty} \) and concentration \( C_{\infty} \) of the ambient fluid saturated porous medium are quiescent. At \( t = 0 \), the fluid is impulsively started in motion with the velocity \( U \) and both the temperature and concentration at the sheet are suddenly increased from \( T_{\infty} \) to \( T_w \) \((T_w > T_{\infty})\) and \( C_{\infty} \) to \( C_w \) \((C_w > C_{\infty})\) and subsequently maintained at that temperature and concentration. Furthermore, the magnetic Reynolds number is assumed to be small so that the induced magnetic field is neglected. In addition, the hall effect and electric field are assumed negligible. The small magnetic Reynolds number assumption uncouples the Naiver-Stokes equation from Maxwell’s equations [4, 5]. All physical properties are assumed constant except the density in the buoyancy force term. By invoking all of the boundary layer and Boussinesq approximations, the simplified basic two-dimensional boundary layer equations governing the flow of an unsteady heat and mass transfer by MHD flow over an impulsively stretched vertical surface with chemical reaction effect are derived from [4, 5] and written below as follows:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \nu \frac{\partial^2 u}{\partial y^2} + g \beta_T (T - T_{\infty}) + g \beta_C (C - C_{\infty}) - \frac{\sigma B_0^2}{\rho} u + \alpha \frac{\partial^2 u}{\partial y^2}, \tag{2.1}
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{\rho_c} \frac{\partial^2 T}{\partial y^2}, \tag{2.2}
\]

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_m \frac{\partial^2 C}{\partial y^2} - K (C - C_{\infty}). \tag{2.4}
\]
where \( t, u \) and \( v \) are the time, fluid tangential velocity and normal velocity components along the \( x \) and \( y \) axes respectively, \( T \) is the temperature, \( C \) is the concentration, \( g \) is the acceleration due to gravity, \( \rho \) is the fluid density, \( \nu \) is the kinematic viscosity, \( \alpha \) is the thermal diffusivity, \( D_m \) is the chemical molecular diffusivity, \( \sigma \) is the fluid electrical conductivity, \( B_0 \) is the magnetic induction, \( K \) is the dimensional chemical reaction parameter, \( \beta_T \) is the thermal expansion coefficient and \( \beta_c \) is the concentration coefficient.

The appropriate initial and boundary conditions for this problem can be written as [23]:

\[
t < 0: \quad u = 0, \quad v = 0, \quad T = T_w, \quad C = C_w, \quad \forall x, y,
\]

\[
t \geq 0: \quad u = ax, \quad v = 0, \quad T = T_w = T_m + bx, \quad C = C_w = C_m + dx, \quad \text{at} \quad y = 0,
\]

\[
t \geq 0: \quad u \rightarrow 0, \quad T \rightarrow T_m, \quad C \rightarrow C_m \quad \text{as} \quad y \rightarrow \infty,
\]

(2.5)

where \( a, b \) and \( d \) are constants.

The stream function \( \psi \) is introduced and defined as [4, 5]:

\[
u = -\frac{\partial \psi}{\partial x},
\]

(2.6)

Further, it is convenient to choose time scale \( \xi \) so that the region of the time integration can be finite. Such transformations have been introduced by Williams and Rhyne [3] in a related study. The transformations are expressed as:

\[
\xi = 1 - \exp(-\tau), \quad \tau = at,
\]

(2.7)

with \( b \) a positive constant and \( t \) is the time variable. The Williams and Rhyne’s [3] transformations (2.7) are used to convert from the infinite (original) time scale \( 0 \leq \tau < \infty \) to the finite scale \( 0 \leq \xi \leq 1 \) so that the interval of integration is collapsed from an infinite domain to a finite domain.

The similarity variables given in [5] was used and written below as:

\[
\eta = \sqrt{\frac{a}{\nu \xi}}, \quad \psi = \sqrt{\frac{a \nu \xi}{\eta}} f(\xi, \eta),
\]

\[
\theta(\xi, \eta) = \frac{T - T_m}{T_w - T_m}, \quad \phi(\xi, \eta) = \frac{C - C_m}{C_w - C_m}.
\]

(2.8)

Equation (2.1) is identically satisfied and the governing equations (2.2) - (2.4) along with the boundary conditions (2.5) can be presented in the form:

\[
\frac{f'''}{2} + \frac{\eta}{2} (1 - \xi) f'' + \xi \left[ f f'' - (f')^2 - Haf' + \lambda (\theta + N \phi) \right] = \xi (1 - \xi) \frac{\partial f'}{\partial \xi},
\]

(2.9)

\[
\frac{1}{Pr} \phi'' + \frac{1}{2} \eta (1 - \xi) \theta' + \xi \left[ f \phi' - f' \phi - \gamma \phi \right] = \xi (1 - \xi) \frac{\partial \phi}{\partial \xi},
\]

(2.10)

subject to

\[
f(\xi, 0) = 0, \quad f'(\xi, 0) = 1, \quad f'(\xi, \infty) = 0,
\]

(2.12)

\[
\theta(\xi, 0) = 1, \quad \theta(\xi, \infty) = 0,
\]

(2.13)

\[
\phi(\xi, 0) = 1, \quad \phi(\xi, \infty) = 0.
\]

(2.14)

In the above equations, prime denotes the derivative with respect to \( \eta \) and the parameters are defined as:

\[
Ha = \frac{\sigma B_0^2}{\rho a^3}, \quad N = \frac{\beta_c (C_m - C_w)}{Pr (T_w - T_m)}, \quad \lambda = \frac{(g \beta (T_m - T_w) x / \nu^2)}{(U_c / \nu)^2} = \frac{Gr_s}{Re_s^2},
\]

\[
Pr = \frac{\nu}{\alpha}, \quad Sc = \frac{\nu}{D_s}, \quad \gamma = \frac{K}{a},
\]

(2.15)
where $Ha$ is the magnetic number (square of Hartman number), $N$ is concentration to the thermal buoyancy ratio, $\lambda$ is mixed convection parameter, $\gamma$ is dimensionless chemical reaction parameter, $Pr$ is Prandtl number, and $Sc$ is Schmidt number. It can be noted that $\lambda > 0$ equals to the aiding flow case and $\lambda < 0$ equals to the opposing flow case. Also, $Gr_x$, $Re_x$ are the local Grashof number and the local Reynold number, respectively.

The expressions for the skin friction coefficients $C_{fx}$, local Nusselt number $Nu_x$ and local Sherwood number $Sh_x$ are given in [4, 5] as:

$$C_{fx} = \frac{-\mu (\partial u / \partial y)_{y=0}}{\rho U_0^2} = -Re_x^{-1/2} \xi^{-1/2} f''(\xi, 0),$$  
(2.16)

$$Nu_x = \frac{-x(\partial T / \partial y)_{y=0}}{T_w - T_{\infty}} = -Re_x^{-1/2} \xi^{-1/2} \theta'(\xi, 0),$$  
(2.17)

$$Sh_x = \frac{-x(\partial C / \partial y)_{y=0}}{C_w - C_{\infty}} = -Re_x^{-1/2} \xi^{-1/2} \phi'(\xi, 0).$$  
(2.18)

For the initial unsteady state flow, when $\xi = 0$, corresponding to $\tau = 0$, equations (2.9 - 2.11) becomes:

$$f''' + \frac{\eta}{2} f'' = 0,$$  
(2.19)

$$\frac{1}{Pr} \theta'' + \frac{\eta}{2} \theta' = 0,$$  
(2.20)

$$\frac{1}{Sc} \phi'' + \frac{\eta}{2} \phi' = 0,$$  
(2.21)

subject to the boundary conditions,

$$f(0) = 0, \ f'(0) = 1, \ f'(\infty) = 0, \ \theta(0) = 1, \ \theta'(\infty) = 0, \ \phi(0) = 1, \ \phi'(\infty) = 0.$$  
(2.22)

Equations (2.19 - 2.21) alongside with the boundary conditions (2.22) admit closed form analytical solutions given in [23] as:

$$f(0, \eta) = \eta \ \text{erfc} \left( \frac{\eta}{2} \sqrt{Pr} \right) + \frac{2}{\sqrt{\pi}} \left[ 1 - \exp \left( -\frac{\eta^2}{4} \right) \right],$$  
(2.23)

$$\theta(0, \eta) = \text{erfc} \left( \eta \sqrt{Pr} \right),$$  
(2.24)

$$\phi(0, \eta) = \text{erfc} \left( \eta \sqrt{Sc} \right).$$  
(2.25)

where the complementary error function $\text{erfc}$, is defined as:

$$\text{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} \exp(-r^2)dr.$$  
(2.26)

### 3 Method of solution

In this section the spectral perturbation method (SPM) is used to solve the partial differential equations (2.9 - 2.11) subject to the boundary conditions (2.12 - 2.14). Perturbation methods in general construct a solution for a problem by generating asymptotic expansions of the perturbation parameter [38]. In perturbation methods, higher order perturbation approximation are difficult to get which may result in less accurate results if only one or two series solutions are used. For instance, the perturbation approach utilized by Seshadri et al. [6] and Nazar et al. [7] yields only the first order approximate solutions. Below, the applicability of the proposed spectral perturbation method on equations (2.9 - 2.11) is demonstrated.
3.1 Spectral Perturbation Method (SPM)
In the spectral perturbation method (SPM), we generate series equations using the standard perturbation approach
and then solve the series equations integrated in the space direction $\eta$ numerically using the Chebyshev spectral
collocation method. With the spectral methods, we can solve higher order perturbation equations easily. Following
\[ f(\xi, \eta) = \sum_{k=0}^{+\infty} \xi^k f_k(\eta), \] (3.27)
\[ \theta(\xi, \eta) = \sum_{k=0}^{+\infty} \xi^k \theta_k(\eta), \] (3.28)
\[ \phi(\xi, \eta) = \sum_{k=0}^{+\infty} \xi^k \phi_k(\eta), \] (3.29)
Substituting (3.27 - 3.29) in equations (2.9 - 2.11) and boundary conditions (2.12 - 2.14) and balancing terms of equal
power of $\xi$, we obtain,
\[ f'''' + \frac{\eta}{2} f''' = 0, \quad f(0) = 0, \quad f'(0) = 1, \quad f''''(\infty) = 0, \] (3.30)
\[ \theta'''' + Pr \frac{\eta}{2} \theta''' = 0, \quad \theta(0) = 1, \quad \theta''''(\infty) = 0, \] (3.31)
\[ \phi'''' + Sc \frac{\eta}{2} \phi''' = 0, \quad \phi(0) = 1, \quad \phi''''(\infty) = 0 \] (3.32)
\[ f'''' + \frac{\eta}{2} f''' - kf'' = \frac{\eta}{2} f'' - (k-1)f'_{k-1} + Ha f'_{k-1} - \lambda \theta_{k-1} - \lambda N \phi_{k-1} \]
\[ + \sum_{i=0}^{k-1} \left[ f'_{k-i} - f_{k-1} - i f''_{i} \right] \] (3.33)
\[ f_{k}(0) = 0, \quad f'_{k}(0) = 0, \quad f''''(\infty) = 0, \quad k \geq 1. \] (3.34)
\[ \theta_{k}'''' + \frac{\eta}{2} Pr \theta_{k}''' - Prk \theta_{k} = \frac{\eta}{2} Pr \theta_{k-1}' - Pr(k-1) \theta_{k-1} \]
\[ - \sum_{i=0}^{k-1} Pr \left[ f'_{k-1-i} \theta'_{i} - f'_{k-1-i} \theta_{i} \right] \] (3.35)
\[ \theta_{k}(0) = 0, \quad \theta''''(\infty) = 0, \quad k \geq 1. \] (3.36)
\[ \phi_{k}'''' + \frac{\eta}{2} Sc \phi_{k}''' - Sc \phi_{k} = \frac{\eta}{2} Sc \phi'_{k-1} - Sc(k-1) \phi_{k-1} + \gamma Sc \phi_{k-1} \]
\[ - \sum_{i=0}^{k-1} Sc \left[ f'_{k-1-i} \phi'_{i} - f'_{k-1-i} \phi_{i} \right] \] (3.37)
\[ \phi_{k}(0) = 0, \quad \phi''''(\infty) = 0, \quad k \geq 1. \] (3.38)
The Chebyshev spectral collocation method is applied to integrate (3.33 - 3.38). The spectral method is based on
the Chebyshev polynomials defined on the domain $[-1, 1]$ by
\[ T_{i}(x) = \cos \left[ i \cos^{-1}(x) \right]. \] (3.39)
Before implementing the spectral method, we first transform the physical domain on which the governing equation
is defined to the region $[-1, 1]$ where the spectral method can then be applied. This can be done with the aid of
the domain truncation procedure, the problem is solved in the interval $[0, L]$ in place of $[0, \infty)$, where $L$ is the scaling
parameter taken to be large. This leads to the transformation
\[ x = \frac{2\eta}{L} - 1, \quad -1 \leq x \leq 1 \] (3.40)
The fundamental aim following the spectral collocation method is the introduction of a differential matrix $D$. The differential matrix $D$ used to approximate the derivatives of the unknown variables $f_k(\eta)$, $\theta_k(\eta)$, $\phi_k(\eta)$ at the collocation points can be defined as:

$$\frac{df_k}{d\eta}(\eta_j) = \sum_{l=0}^{N_s} D_{jl} f_k(x_l) = D F_k, \quad j = 0, 1, \ldots, N_s,$$

$$\frac{d\theta_k}{d\eta}(\eta_j) = \sum_{l=0}^{N_s} D_{jl} \theta_k(x_l) = D \Theta_k, \quad j = 0, 1, \ldots, N_s,$$

$$\frac{d\phi_k}{d\eta}(\eta_j) = \sum_{l=0}^{N_s} D_{jl} \phi_k(x_l) = D \Phi_k, \quad j = 0, 1, \ldots, N_s,$$

where $(N_s + 1)$ is the number of collocation points, $D = 2D/L$, and

$$F_k = [f_k(x_0), f_k(x_1), \ldots, f_k(x_{N_s})]^T,$$

$$\Theta_k = [\theta_k(x_0), \theta_k(x_1), \ldots, \theta_k(x_{N_s})]^T,$$

$$\Phi_k = [\phi_k(x_0), \phi_k(x_1), \ldots, \phi_k(x_{N_s})]^T,$$

is the vector function at the collocation points. We obtain the higher order derivatives as powers of $D$, that is;

$$f_k^{(p)} = D^p F_k,$$

$$\theta_k^{(p)} = D^p \Theta_k,$$

$$\phi_k^{(p)} = D^p \Phi_k,$$

where $p$ is the order of the derivatives. The matrix $D$ is of size $(N_s + 1) \times (N_s + 1)$ and its entries are defined in [41, 42] as:

$$D_{jl} = \frac{c_j (-1)^{j+l}}{c_l \tau_j - \tau_l}, \quad j \neq l; \quad j, l = 0, 1, 2, N_s,$$

$$D_{ll} = -\frac{\tau_l}{2(1 - \tau_l^2)}, \quad 1 \leq j = l \leq N_s - 1,$$

$$D_{00} = \frac{2N_s^2 + 1}{6} = -D_{N_sN_s},$$

with

$$c_l = \begin{cases} 2, & l = 0, N_s \\ -1, & -1 \leq l \leq N_s - 1. \end{cases}$$

Substituting (3.42 - 3.50) in (3.33 - 3.37) gives

$$A_{1,k-1} F_k = B_{1,k-1},$$

$$A_{2,k-1} \Theta_k = B_{2,k-1},$$

$$A_{3,k-1} \Phi_k = B_{3,k-1},$$
subject to the following boundary conditions

\[
\sum_{j=0}^{N_i} D_{0j} f_k(x_j) = 0, \quad \sum_{j=0}^{N_i} D_{N_{ij}} f_k(x_j) = 0, \quad f_k(x_{N_i}) = 0, \tag{3.56}
\]

\[
\theta_k(x_0) = 0, \quad \theta_k(x_{N_i}) = 0, \tag{3.57}
\]

\[
\phi_k(x_0) = 0, \quad \phi_k(x_{N_i}) = 0, \tag{3.58}
\]

where \(A_{1,k-1}, A_{2,k-1}, A_{3,k-1}, B_{1,k-1}, B_{2,k-1}, \) and \(B_{3,k-1}\) are defined as:

\[
A_{1,k-1} = D^3 + \text{diag}\left(\frac{n}{2}\right)D^2 - kD, \tag{3.59}
\]

\[
A_{2,k-1} = D^2 + \text{diag}\left(\frac{n}{2}Pr\right)D - kPrI, \tag{3.60}
\]

\[
A_{3,k-1} = D^2 + \text{diag}\left(\frac{n}{2}Sc\right)D - kScI, \tag{3.61}
\]

\[
B_{1,k-1} = \frac{n}{2}(D^2F_{k-1} + (k - 1)DF_{k-1} + Ha(DF_{k-1}) - \lambda\Theta_{k-1} - \lambda N\Phi_{k-1} + \text{Sum}F, \tag{3.62}
\]

\[
B_{2,k-1} = \frac{n}{2}Pr(D\Theta_{k-1}) - Pr(k - 1)\Theta_{k-1} - \text{Sum}\Theta, \tag{3.63}
\]

\[
B_{3,k-1} = \frac{n}{2}Sc(D\Phi_{k-1}) - Sc(k - 1)\Phi_{k-1} + \gamma Sc\Phi_{k-1} - \text{Sum}\Phi, \tag{3.64}
\]

where \(\eta = [\eta_0, \eta_1, \cdots, \eta_{N_i}]\). SumF, Sum\(\Theta\) and Sum\(\Phi\) are defined as:

\[
\text{SumF} = \sum_{i=0}^{k-1} \left[\left(DF_{k-1-i}\right)DF_{i} - F_{k-1-i}\left(D^2F_{i}\right)\right],
\]

\[
\text{Sum}\Theta = Pr \sum_{i=0}^{k-1} \left[\left(DF_{k-1-i}\right)\Theta_i - F_{k-1-i}\left(D\Theta_i\right)\right],
\]

\[
\text{Sum}\Phi = Sc \sum_{i=0}^{k-1} \left[\left(DF_{k-1-i}\right)\Phi_i - F_{k-1-i}\left(D\Phi_i\right)\right],
\]

with \(I\) representing an \((N_i + 1) \times (N_i + 1)\) identity matrix and diag() is a diagonal matrix obtained from the vector \((x_0, x_1, \ldots, x_{N_i})\). The boundary conditions (3.56) are imposed on the first \(N_i\)th row (second from the last row) and \((N_i + 1)\)st row (last row) rows and first and last columns of (3.53) to obtain

\[
\begin{bmatrix}
D_{0,0} & D_{0,1} & \cdots & \cdots & D_{0,N_{i-1}} & D_{0,N_i} \\
D_{N_i,0} & D_{N_i,1} & \cdots & \cdots & D_{N_{i-1},N_i} & D_{N_i,N_i} \\
0 & 0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 f_k(x_0) \\
 f_k(x_1) \\
 \vdots \\
 f_k(x_{N_i-1}) \\
 f_k(x_{N_i})
\end{bmatrix}
= \begin{bmatrix}
0 \\
B_{1,k-1}(x_1) \\
\vdots \\
B_{1,k-1}(x_{N_i-1}) \\
0
\end{bmatrix}, \tag{3.65}
\]

while the boundary conditions (3.57) and (3.58) are imposed the first and last rows and columns of equations (3.54) and (3.55) respectively to obtain

\[
\begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 \theta_k(x_0) \\
 \theta_k(x_1) \\
 \vdots \\
 \theta_k(x_{N_i-1}) \\
 \theta_k(x_{N_i})
\end{bmatrix}
= \begin{bmatrix}
0 \\
B_{2,k-1}(x_1) \\
\vdots \\
B_{2,k-1}(x_{N_i-1}) \\
0
\end{bmatrix}, \tag{3.66}
\]
and
\[
\begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 1
\end{bmatrix} \mathbf{A}_{3,k-1}
\]
\[
\begin{bmatrix}
\Phi_k(x_0) \\
\Phi_k(x_1) \\
\vdots \\
\Phi_k(x_{N_k})
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\] (3.67)

Hence, starting from a known \(F_0, \Theta_0, \Phi_0\), the solutions \(F_k, \Theta_k, \Phi_k, k \geq 1\) can be obtained from equations (3.65 - 3.67) as;
\[
F_k = \mathbf{A}_{1,k-1}^{-1} \mathbf{B}_{1,k-1},
\]
\[
\Theta_k = \mathbf{A}_{2,k-1}^{-1} \mathbf{B}_{2,k-1},
\]
\[
\Phi_k = \mathbf{A}_{3,k-1}^{-1} \mathbf{B}_{3,k-1}.
\] (3.68 - 3.70)

3.2 Spectral Relaxation Technique (SRM)

In this section, the development of the spectral relaxation method (SRM) to obtain solution of the governing partial differential equations (2.9 - 2.11) is being discussed. To start the SRM algorithm, it is convenient to reduce the order of equation (2.9) from three to two. To this end, we first set \(f' = u\), so that equation (2.9) becomes
\[
u'' + (1 - \xi) \left( \frac{\eta}{2} u' - \xi \frac{\partial u}{\partial \xi} \right) + \xi \left[ f u' - u^2 - Hau + \lambda \theta + \lambda Nf \phi \right] = 0.
\] (3.71)
The SRM [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37] uses the Gauss-Seidel concept to decouple the governing nonlinear systems of equations (2.9 - 2.11). From the decoupled equations we develop an iteration scheme by evaluating linear terms in the current iteration level denoted by \((r+1)\) and nonlinear terms in the previous iteration level denoted by \((r)\). Implementing the SRM on the resulting system of nonlinear partial differential equations yields the following linear partial differential equations;
\[
u''_{r+1} + a_{1,r} u'_{r+1} + a_{2,r} u_{r+1} + a_{3,r} = \xi (1 - \xi) \frac{\partial u_{r+1}}{\partial \xi},
\]
\[
f'_{r+1} = u_{r+1},
\]
\[
\theta''_{r+1} + b_{1,r} \theta'_{r+1} + b_{2,r} \theta_{r+1} = \xi (1 - \xi) \frac{\partial \theta_{r+1}}{\partial \xi},
\]
\[
\phi''_{r+1} + c_{1,r} \phi'_{r+1} + c_{2,r} \phi_{r+1} = \xi (1 - \xi) \frac{\partial \phi_{r+1}}{\partial \xi},
\]
\[
u_{r+1}(0,\xi) = \theta_{r+1}(0,\xi) = \phi_{r+1}(0,\xi) = 1, f_{r+1}(0,\xi) = 0,
\]
\[
u_{r+1}(\infty,\xi) = \theta_{r+1}(\infty,\xi) = \phi_{r+1}(\infty,\xi) = 0,
\] (3.72 - 3.76)
where the coefficient parameters \(a_{1,r}, a_{2,r}, a_{3,r}, b_{1,r}, b_{2,r}, c_{1,r}, c_{2,r}\) are defined as;
\[
a_{1,r} = \left( \frac{1}{2} \eta (1 - \xi) + \xi f_r \right), \quad a_{2,r} = -\xi H a, \quad a_{3,r} = -\xi (u_r^2 + \lambda (\theta_r + Nf \phi)),
\]
\[
b_{1,r} = Pr \left( \frac{1}{2} \eta (1 - \xi) + \xi f_r \right), \quad b_{2,r} = -\xi Pr (u_r),
\]
\[
c_{1,r} = Sc \left( \frac{1}{2} \eta (1 - \xi) + \xi f_r \right), \quad c_{2,r} = -\xi Sc (u_r + \gamma).
\]
The initial approximation for solving equations (3.72 - 3.76) are obtained as the solutions at \(\xi = 0\). Hence, \(f_0(\xi, \eta), \theta_0(\xi, \eta), \phi_0(\xi, \eta)\) are given in equations (2.23 - 2.25) and \(u_0(\xi, \eta)\) is given as
\[
u_0(\xi, \eta) = \text{erfc} \left( \frac{\eta}{2} \right).
\] (3.77)
Equations (3.72 - 3.76) can be solved iteratively for the unknown functions starting from the initial approximations given in (2.22 - 2.25) and (3.77). The iteration schemes (3.72), (3.74) and (3.75) are solved iteratively for $u_{r+1}(\xi, \eta)$, $\theta_{r+1}(\xi, \eta)$ and $\phi_{r+1}(\xi, \eta)$ when $r = 0, 1, 2, \cdots$. The solution for $u_{r+1}$ is utilized in (3.73) which is, in turn, solved for $f_{r+1}$. To solve equations (3.72 - 3.75), the equations are discretized using the Chebyshev spectral collocation method in the $\eta$ - direction and the implicit finite difference method in the $\xi$ - direction. The underlying idea behind the Chebyshev spectral collocation method has been explained above. The finite difference scheme is used with centering about a mid-point between $\xi^{n+1}$ and $\xi^n$. We define the mid-point as $\xi^{n+\frac{1}{2}} = (\xi^{n+1} + \xi^n)/2$. Thus, implementing the centering about $\xi^{n+\frac{1}{2}}$ to any function, say $u(\xi, \eta)$ and its associated derivative we obtain,

$$
u\left(\xi^{n+\frac{1}{2}}, \eta\right) = u^{n+\frac{1}{2}} = \frac{u^{n+1} + u^n}{2}, \quad \left(\frac{\partial u}{\partial \xi}\right)^{n+\frac{1}{2}} = \frac{u^{n+1} - u^n}{\Delta \xi}.$$  

(3.78)

The spectral method is first applied on equations (3.72 - 3.75), before applying the finite differences to obtain

$$[D^2 + a_{1,r}D + a_{2,r}] U_{r+1} + a_{3,r} = \xi(1 - \xi) \frac{dU_{r+1}}{d\xi},$$  

(3.79)

$$u_{r+1}(x_0, \xi) = 0, \quad u_{r+1}(x_N, \xi) = 1,$$

$$D F_{r+1} = U_{r+1}, \quad f_{r+1}(x_N, \xi) = 0,$$

$$[D^2 + b_{1,r}D + b_{2,r}] \Theta_{r+1} = \xi(1 - \xi) \frac{d\Theta_{r+1}}{d\xi},$$  

(3.81)

$$\theta_{r+1}(x_0, \xi) = 0, \quad \theta_{r+1}(x_N, \xi) = 1$$

$$[D^2 + c_{1,r}D + c_{2,r}] \Phi_{r+1} = \xi(1 - \xi) \frac{d\Phi_{r+1}}{d\xi},$$  

(3.82)

where

$$U_{r+1} = \begin{bmatrix} u_{r+1}(x_0, \xi) \\ u_{r+1}(x_1, \xi) \\ \vdots \\ u_{r+1}(x_{N-1}, \xi) \\ u_{r+1}(x_N, \xi) \end{bmatrix}, \quad F_{r+1} = \begin{bmatrix} f_{r+1}(x_0, \xi) \\ f_{r+1}(x_1, \xi) \\ \vdots \\ f_{r+1}(x_{N-1}, \xi) \\ f_{r+1}(x_N, \xi) \end{bmatrix}, \quad a_{3,r} = \begin{bmatrix} a_{3,r}(x_0, \xi) \\ a_{3,r}(x_1, \xi) \\ \vdots \\ a_{3,r}(x_{N-1}, \xi) \\ a_{3,r}(x_N, \xi) \end{bmatrix},$$  

(3.83)

$$a_{1,r} = \begin{bmatrix} a_{1,r}(x_0, \xi) \\ a_{1,r}(x_1, \xi) \\ \vdots \\ a_{1,r}(x_{N-1}, \xi) \\ a_{1,r}(x_N, \xi) \end{bmatrix}$$  

(3.84)

$$a_{2,r} = \begin{bmatrix} a_{2,r}(x_0, \xi) \\ a_{2,r}(x_1, \xi) \\ \vdots \\ a_{2,r}(x_{N-1}, \xi) \\ a_{2,r}(x_N, \xi) \end{bmatrix}$$  

(3.85)

$$\Theta_{r+1} = \begin{bmatrix} \theta_{r+1}(x_0, \xi) \\ \theta_{r+1}(x_1, \xi) \\ \vdots \\ \theta_{r+1}(x_{N-1}, \xi) \\ \theta_{r+1}(x_N, \xi) \end{bmatrix}, \quad b_{1,r} = \begin{bmatrix} b_{1,r}(x_0, \xi) \\ b_{1,r}(x_1, \xi) \\ \vdots \\ b_{1,r}(x_{N-1}, \xi) \\ b_{1,r}(x_N, \xi) \end{bmatrix}.$$  

(3.86)
Next, the finite difference scheme is applied on (3.79 - 3.82), in the $\xi$-direction with centering about the mid-point $x^{n+\frac{1}{2}}$ to obtain the following systems of decoupled equations

$$A_1 U_{n+1} = B_1 U^n + K_1,$$

$$A_2 \Theta_{n+1} = B_2 \Theta^n + K_2,$$

$$A_3 \Phi_{n+1} = B_3 \Phi^n + K_3,$$

$$DF_{n+1} = U_{n+1}^\prime,$$

subject to the following initial and boundary conditions

$$u_{r+1}(x_N, \xi^n) = \theta_{r+1}(x_N, \xi^n) = \phi_{r+1}(x_N, \xi^n) = 1,$$

$$u_{r+1}(x_0, \xi^n) = \theta_{r+1}(x_0, \xi^n) = \phi_{r+1}(x_0, \xi^n) = 0,$$

$$f_{r+1}(x_n, \xi^n) = 0, \quad n = 0, 1, 2, \ldots,$$

$$f_{r+1}(\eta_j, 0) = \eta_{\text{erfc}} \left( \frac{\eta_j}{2} \right) + \frac{2}{\sqrt{\pi}} \left[ 1 - \exp \left( - \frac{\eta_j^2}{4} \right) \right],$$

$$u_{r+1}(\eta_j, 0) = \text{erfc} \left( \frac{\eta_j}{2} \right),$$

$$\theta_{r+1}(\eta_j, 0) = \text{erfc} \left( \frac{\sqrt{Pr} \eta_j}{2} \right),$$

$$\phi_{r+1}(\eta_j, 0) = \text{erfc} \left( \frac{\sqrt{Sc} \eta_j}{2} \right), \quad j = 0, 1, 2, \ldots, N_x.$$
The matrices above are defined as follows

\[
A_1 = \frac{1}{2} \left( D^2 + a_{1,r}^{n+\frac{1}{2}} D + a_{2,r} \right) - \frac{\xi^{n+\frac{1}{2}} \left( 1 - \xi^{n+\frac{1}{2}} \right)}{\Delta \xi} I,
\]

(3.101)

\[
A_2 = \frac{1}{2} \left( D^2 + b_{1,r}^{n+\frac{1}{2}} D + b_{2,r} \right) - \frac{p_r \xi^{n+\frac{1}{2}} \left( 1 - \xi^{n+\frac{1}{2}} \right)}{\Delta \xi} I,
\]

(3.102)

\[
A_3 = \frac{1}{2} \left( D^2 + c_{1,r}^{n+\frac{1}{2}} D + c_{2,r} \right) - \frac{S \xi^{n+\frac{1}{2}} \left( 1 - \xi^{n+\frac{1}{2}} \right)}{\Delta \xi} I,
\]

(3.103)

\[
B_1 = -\frac{1}{2} \left( D^2 + a_{1,r}^{n+\frac{1}{2}} D + a_{2,r} \right) - \frac{\xi^{n+\frac{1}{2}} \left( 1 - \xi^{n+\frac{1}{2}} \right)}{\Delta \xi} I,
\]

(3.104)

\[
B_2 = -\frac{1}{2} \left( D^2 + b_{1,r}^{n+\frac{1}{2}} D + b_{2,r} \right) - \frac{p_r \xi^{n+\frac{1}{2}} \left( 1 - \xi^{n+\frac{1}{2}} \right)}{\Delta \xi} I,
\]

(3.105)

\[
B_3 = -\frac{1}{2} \left( D^2 + c_{1,r}^{n+\frac{1}{2}} D + c_{2,r} \right) - \frac{S \xi^{n+\frac{1}{2}} \left( 1 - \xi^{n+\frac{1}{2}} \right)}{\Delta \xi} I,
\]

(3.106)

\[
K_1 = -a_{3,r}^{n+\frac{1}{2}},
\]

(3.107)

\[
K_2 = O,
\]

(3.108)

\[
K_3 = O.
\]

(3.109)

where \( I \) is an \((N_x + 1) \times (N_x + 1)\), \( U, F, \Theta, \) and \( \Phi \) are the vectors of the functions \( u, f, \theta, \) and \( \phi \) when evaluated at the grid points and \( O \) is a vector of zeros of size \((N_x + 1) \times 1\). The boundary conditions (3.94 - 3.96) are imposed on the first and last rows of (3.90 - 3.93) as follows;
This work were generated using the computed SPM results was validated against numerical results obtained using the SRM. The results presented in equations (3.90 - 3.93) can be solved iteratively to give approximate solutions for \( u_{r+1}^{n+1}(x_j, \xi^n) \), \( \theta_{r+1}^{n+1}(x_j, \xi^n) \), and \( \phi_{r+1}^{n+1}(x_j, \xi^n) \), \( j = 0, 1, 2, 3, \cdots , N_x \). (3.114)

Hence, starting from the initial approximations \( f_0(\xi, \eta) \), \( u_0(\xi, \eta) \), \( \theta_0(\xi, \eta) \), \( \phi_0(\xi, \eta) \), given by equations (2.22 - 2.25) and (3.77), equations (3.90 - 3.93) can be solved iteratively to give approximate solutions for \( u_{r+1}(\xi, \eta) \), \( \theta_{r+1}(\xi, \eta) \), \( \phi_{r+1}(\xi, \eta) \), \( r = 0, 1, 2, 3, \cdots \), until a solution that converges to within a given level of accuracy is obtained. The solution \( u_{r+1} \) is used in equation (3.80) which is, in turn, solved for \( f_{r+1} \).

4 Results and discussion

In this section, the spectral perturbation method (SPM) and spectral relaxation method (SRM) results for the set of the governing nonlinear partial differential equations (2.9 - 2.14) are presented. Numerical computation were carried out using the previous spectral perturbation method (SPM) and spectral relaxation method (SRM) discussed in the previous sections for the velocity, temperature and concentration profiles as well as the local skin friction, Nusselt and Sherwood number for different values of the significant physical parameters in this study. Results are displayed in tabular and graphical formats. We remark that all programs for generating solutions were coded in MATLAB 7.12 (R2011a) and the operating system was Windows 7 64-bit operating system running on a Intel (R) Processor Core(TM) i5-3337U CPU 1.80GHz with 4Gb installed memory (RAM). The SPM series was used to generate results and Sherwood number for different values of the significant physical parameters in this study. Results are displayed in tabular and graphical formats. We remark that all programs for generating solutions were coded in MATLAB 7.12 (R2011a) and the operating system was Windows 7 64-bit operating system running on a Intel (R) Processor Core(TM) i5-3337U CPU 1.80GHz with 4Gb installed memory (RAM). The SPM series was used to generate results from the initial analytical solution at \( \xi = 0 \) up to results close to the steady state values at \( \xi = 1 \). The accuracy of the computed SPM results was validated against numerical results obtained using the SRM. The results presented in this work were generated using \( L = 30 \), which was found to give accurate results through numerical experimentation. Increasing the value of \( L \) did not change the results to a significant extent. The number of collocation points used was \( N_x = 100 \) for both SPM and SRM. The values of Prandtl number \( Pr \) used in this work is chosen to be \( Pr = 0.7 \) which represents the \( Pr \) for air, water \((Pr = 1 - 10)\). The values of Schmidt number is chosen to be Hydrogen \((Sc = 0.20)\), Water \((Sc = 0.60)\), Ammonia \((Sc = 0.78)\), Carbon dioxide \((Sc = 0.94)\) and Propyl Benzene \((Sc = 2.62)\). The value of the buoyancy force parameter (which means the ratio of the buoyancy force due to the thermal diffusion) \( N \) takes the values between 0.5 or 1.0 for low concentration. All graphs and tables therefore corresponds to these values except otherwise indicated. The values of all other physical parameters governing the fluid flow are chosen based on values earlier used in literatures. Furthermore, in order to further test the accuracy of the SPM and the SRM, a residual error analysis is conducted. The SPM residual error of the governing partial differential equations (2.9 - 2.11) is defined as;

\[
Res(f) = D^3 U + \frac{n}{2}(1-\xi)\frac{\partial}{\partial \xi}(D^2 U) - (D^2 U) + \lambda(\Theta + N\Phi) - \xi (1-\xi) \frac{\partial (DF)}{\partial \xi},
\]

\[
Res(\theta) = \left( \frac{1}{Pr} \right) \left[ D^2 \Theta + \frac{n}{2}(1-\xi)D\Theta + \xi [[D\Theta](F) - (DF)(\Theta)] - \xi (1-\xi) \frac{\partial (DF)}{\partial \xi} \right],
\]

\[
Res(\phi) = \left( \frac{1}{Sc} \right) \left[ D^2 \Phi + \frac{n}{2}(1-\xi)D\Phi + \xi [[D\Phi](F) - (DF)(\Phi) - \gamma \Phi] - \xi (1-\xi) \frac{\partial (DF)}{\partial \xi} \right].
\]

While the SRM residual error of the governing partial differential equations (2.9 - 2.11) is defined as;

\[
Res(f) = D^2 U + \frac{n}{2}(1-\xi)\frac{\partial}{\partial \xi}(D U) - (D U) + \lambda(\Theta + N\Phi) - \xi (1-\xi) \frac{\partial (DF)}{\partial \xi},
\]

\[
Res(\theta) = \left( \frac{1}{Pr} \right) \left[ D^2 \Theta + \frac{n}{2}(1-\xi)D\Theta + \xi [[D\Theta](F) - (DF)(\Theta)] - \xi (1-\xi) \frac{\partial (DF)}{\partial \xi} \right],
\]
\[
\text{Res}(\phi) = \left( \frac{1}{\text{Sc}} \right) \left[ D^2\Phi + \frac{\eta}{2} (1 - \xi) D\Phi + \xi \left[ (D\Theta)(F) - (U)(\Phi) - \gamma\Phi \right] - \xi (1 - \xi) \frac{\partial(\Phi)}{\partial\xi} \right].
\] (4.120)

In the equations (4.118 - 4.120), \( U = F' \). Also, \( U, F, \Theta \) and \( \Phi \) are defined as
\[
\frac{U_{r+1}^{n+1} + U_r^{n+1}}{2}, \quad \frac{F_{r+1}^{n+1} + F_r^{n+1}}{2}, \quad \frac{\Theta_{r+1}^{n+1} + \Theta_r^{n+1}}{2}, \quad \text{and} \quad \frac{\Phi_{r+1}^{n+1} + \Phi_r^{n+1}}{2}. \] (4.121)

In addition, we define \( \frac{\partial(U)}{\partial\xi}, \frac{\partial(\Theta)}{\partial\xi}, \text{and} \frac{\partial(\Phi)}{\partial\xi} \) as
\[
\frac{U_{r+1}^{n+1} - U_r^{n+1}}{\Delta\xi}, \quad \frac{\Theta_{r+1}^{n+1} - \Theta_r^{n+1}}{\Delta\xi}, \quad \text{and} \quad \frac{\Phi_{r+1}^{n+1} - \Phi_r^{n+1}}{\Delta\xi}. \] (4.122)

Table 4 displays a comparison between the spectral perturbation method (SPM) and the spectral relaxation method (SRM) approximate solution of the skin friction coefficient \( f''(0, \xi) \) at various values of Hartman number \( Ha \). It can be observed from the table that the two sets of results are in excellent agreement. Also, from the table, it was noticed that the effect of the Hartman number was to reduce the skin friction coefficient when \( \xi = 0.5 \). The physical reasoning behind this result, is as a consequence of the presence of transverse magnetic field on the flow. The transverse magnetic field sets in a Lorenz drag force caused by electromagnetism. This Lorenz drag force in turn produces a retarding force on the velocity field and thus, as the Hartman number increase, the retarding force also increases Hence, the boundary layer thickness decreases consequently reducing the shear stress at the sheet. It can be seen from Table 4 that results which are consistent with nine decimal digits were achieved with only four iteration. The number of grid points \( N_t \), used in the \( \xi \) direction is 10000.

Table 1: Comparison of the SPM and SRM approximate solutions of \( f''(0, \xi) \) at different values of \( Ha \), when \( \xi = 0.5, \gamma = 1, \text{Sc} = 0.6, \text{Pr} = 1.5, \lambda = 0.5 \) and \( N = 1 \).

<table>
<thead>
<tr>
<th>( Ha )</th>
<th>Order</th>
<th>SPM</th>
<th>It</th>
<th>( N_t ) (Grid Points)</th>
<th>SRM</th>
<th>Difference (SPM - SRM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>21</td>
<td>-0.5412625054</td>
<td>4</td>
<td>10000</td>
<td>-0.5412625050</td>
<td>-0.0000000004</td>
</tr>
<tr>
<td>0.1</td>
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<td>-0.5687336457</td>
<td>4</td>
<td>10000</td>
<td>-0.5687336453</td>
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<tr>
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<td>-0.6223191104</td>
<td>4</td>
<td>10000</td>
<td>-0.6223191101</td>
<td>-0.0000000003</td>
</tr>
<tr>
<td>0.5</td>
<td>22</td>
<td>-0.6741858388</td>
<td>4</td>
<td>10000</td>
<td>-0.6741858385</td>
<td>-0.0000000003</td>
</tr>
<tr>
<td>0.7</td>
<td>18</td>
<td>-0.7244385083</td>
<td>4</td>
<td>10000</td>
<td>-0.7244385080</td>
<td>-0.0000000003</td>
</tr>
<tr>
<td>0.9</td>
<td>17</td>
<td>-0.7731640397</td>
<td>4</td>
<td>10000</td>
<td>-0.7731640396</td>
<td>-0.0000000002</td>
</tr>
<tr>
<td>1.0</td>
<td>18</td>
<td>-0.7969854223</td>
<td>4</td>
<td>10000</td>
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<td>-0.0000000001</td>
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<td>1.7</td>
<td>21</td>
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<td>4</td>
<td>10000</td>
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<td>0.0000000003</td>
</tr>
<tr>
<td>2.0</td>
<td>11</td>
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<td>4</td>
<td>10000</td>
<td>-1.0176356559</td>
<td>0.0000000003</td>
</tr>
</tbody>
</table>
Table 2: Comparison of the SPM and SRM approximate solutions of $\theta'(0, \xi)$ at different values of $\lambda$, and $Pr$, when $\xi = 0.5$, $\gamma = 1$, $Sc = 0.6$, $Ha = 1$, and $N = 1$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$Pr$</th>
<th>Order</th>
<th>SPM</th>
<th>It</th>
<th>$N_{t}$/ (Grid Points)</th>
<th>SRM</th>
<th>Difference (SPM - SRM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>23</td>
<td>-0.6278318239</td>
<td>4</td>
<td>10000</td>
<td>-0.6278318241</td>
<td>0.0000000002</td>
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<tr>
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<td>4</td>
<td>10000</td>
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<td>0.0000000003</td>
</tr>
<tr>
<td>0</td>
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<td>28</td>
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<td>4</td>
<td>10000</td>
<td>-1.4270081807</td>
<td>0.0000000003</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>20</td>
<td>-1.8845313181</td>
<td>4</td>
<td>10000</td>
<td>-1.8845313184</td>
<td>0.0000000003</td>
</tr>
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</table>

Table 4 gives a comparison of the spectral perturbation method (SPM) and the spectral relaxation method (SRM) solutions of the local Nusselt number for varying Prandtl number $Pr$ and mixed convection parameter $\lambda$. A comparison of the two results indicates that the SPM results are in good agreement with the SRM results for nine to ten decimal places. It can be observed from the table that the heat transfer rate reduces with an increase in $Pr$. The heat transfer rate is decreased by increase in $Pr$ due to the reducing manner of the thermal boundary layer thickness with increment in $Pr$. We remark that the number of iterations needed to give the SRM solutions in Table 4 is four. The number of grid points $N_{t}$ used in generating the results in Table 4 is 10000.
Table 3: Comparison of the SPM and SRM approximate solutions of $\phi'(0, \xi)$ at different values of $\lambda$, $N$, $\gamma$, and $Sc$, when $Ha = 1$, $\xi = 0.5$, and $Pr = 1.5$.

<table>
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<tr>
<th>$N$</th>
<th>$\lambda$</th>
<th>$\gamma$</th>
<th>$Sc$</th>
<th>Order</th>
<th>SPM</th>
<th>It</th>
<th>SRM</th>
<th>Difference (SPM - SRM)</th>
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<td>-0.6683346544</td>
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<td>-1.3244650949</td>
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A comparison of the spectral perturbation method (SPM) and the spectral relaxation method (SRM) approximate solution for the mass transfer coefficient for different values of the concentration to thermal buoyancy ratio $N$, mixed convection parameter $\lambda$, chemical reaction parameter $\gamma$ and Schmidt number $Sc$ is shown in Table 4. The mass transfer coefficient decreased with an increase in $N$, $\lambda$, $\gamma$ and $Sc$. The mass transfer rate is reduce by an increase in $N$, $\lambda$, $\gamma$ and $Sc$ due to the reducing manner of the solutal boundary layer thickness with increment in these parameters. Table 4 shows that the SPM results are in good agreement with the (SRM) solutions. This shows that the SPM is a viable method for solving the model equations. In the SRM results in Table 4, a uniform grid with $N_t = 10000$ was used in the $x$ direction to generate the results that are consistent to at least nine decimal places. It can be observed from the table that only four iterations was required to obtain the SRM solution. This is so because for larger number of grid points $N_t$, only few iterations are required to give converged results.

Table 4: Approximate numerical values of the skin friction $f''(0, \xi)$ for various $\xi$ and $N_t$ (Grid Points) computed using the SRM, when $Pr = 1.5$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 0.5$, $N = 1$ and $Ha = 1$.

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<thead>
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</thead>
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<tr>
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</tr>
<tr>
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<tr>
<td>0.6</td>
<td>-0.84421553</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.89160155</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.93911733</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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<tr>
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</tr>
<tr>
<td>0.9</td>
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<tr>
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Table 5: Approximate numerical values of the heat transfer rate $\theta'(0, \xi)$ for various $\xi$ and grid points $N_t$ computed using the SRM, when $Pr = 1.5, \gamma = 0.6, \lambda = 1, N = 1$ and $Ha = 1$.

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<th>2000</th>
<th>5000</th>
<th>10000</th>
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<td>-1.10646912</td>
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Table 6: Approximate numerical values of the mass transfer rate $\phi'(0, \xi)$ for various $\xi$ and grid points $N_t$ computed using the SRM, when $Pr = 1.5, \gamma = 0.6, \lambda = 1, N = 1$ and $Ha = 1$.

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<th>10000</th>
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Table 7: Comparison of the SPM and SRM numerical values of the skin friction $f''(0, \xi)$ at different values of $\xi$ when $Pr = 1.5, \gamma = 0.6, \lambda = 1, N = 1$ and $Ha = 1$.

<table>
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<th>SPM Time(sec)</th>
<th>SRM</th>
<th>SRM Time(sec)</th>
<th>$N_t$ (Grid Points)</th>
<th>It</th>
<th>SRM Time(sec)</th>
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<td>0.330</td>
<td>-0.98672993</td>
<td>1000</td>
<td>4</td>
<td>10.923</td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>147</td>
<td>-1.01053773</td>
<td>1.103</td>
<td>-1.01053773</td>
<td>2000</td>
<td>4</td>
<td>21.072</td>
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<tr>
<td>0.98</td>
<td>403</td>
<td>-1.02476407</td>
<td>5.076</td>
<td>-1.02476407</td>
<td>5000</td>
<td>4</td>
<td>49.585</td>
<td></td>
</tr>
</tbody>
</table>
Table 8: Comparison of the SPM and SRM numerical values of the heat transfer rate $\theta'(0, \xi)$ at different values of $\xi$ when $Pr = 1.5, Sc = 0.6, \gamma = 1, \lambda = 0.5, N = 1$ and $Ha = 1$.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>Order</th>
<th>SPM</th>
<th>SPM Time(sec)</th>
<th>SRM</th>
<th>$N_f$ (Grid Points)</th>
<th>It</th>
<th>SRM Time(sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>5</td>
<td>-0.75235579</td>
<td>0.008</td>
<td>-0.75235579</td>
<td>2000</td>
<td>2</td>
<td>10.910</td>
</tr>
<tr>
<td>0.3</td>
<td>9</td>
<td>-0.87320800</td>
<td>0.019</td>
<td>-0.87320800</td>
<td>2000</td>
<td>3</td>
<td>15.024</td>
</tr>
<tr>
<td>0.5</td>
<td>19</td>
<td>-0.99135188</td>
<td>0.037</td>
<td>-0.99135188</td>
<td>2000</td>
<td>3</td>
<td>15.433</td>
</tr>
<tr>
<td>0.6</td>
<td>19</td>
<td>-1.04931597</td>
<td>0.042</td>
<td>-1.04931597</td>
<td>2000</td>
<td>3</td>
<td>15.834</td>
</tr>
<tr>
<td>0.7</td>
<td>27</td>
<td>-1.10646912</td>
<td>0.068</td>
<td>-1.10646912</td>
<td>2000</td>
<td>3</td>
<td>16.064</td>
</tr>
<tr>
<td>0.8</td>
<td>43</td>
<td>-1.16271389</td>
<td>0.107</td>
<td>-1.16271389</td>
<td>2000</td>
<td>3</td>
<td>16.091</td>
</tr>
<tr>
<td>0.9</td>
<td>89</td>
<td>-1.21781792</td>
<td>0.330</td>
<td>-1.21781792</td>
<td>2000</td>
<td>3</td>
<td>16.252</td>
</tr>
<tr>
<td>0.95</td>
<td>180</td>
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<td>1.103</td>
<td>-1.24469310</td>
<td>2000</td>
<td>3</td>
<td>16.392</td>
</tr>
<tr>
<td>0.98</td>
<td>429</td>
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<td>5.076</td>
<td>-1.26030625</td>
<td>10000</td>
<td>3</td>
<td>81.792</td>
</tr>
</tbody>
</table>

Table 9: Comparison of the SPM and SRM numerical values of the mass transfer rate $\phi'(0, \xi)$ at different values of $\xi$ when $Pr = 1.5, Sc = 0.6, \gamma = 1, \lambda = 0.5, N = 1$ and $Ha = 1$.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>Order</th>
<th>SPM</th>
<th>SPM Time(sec)</th>
<th>SRM</th>
<th>$N_f$ (Grid Points)</th>
<th>It</th>
<th>SRM Time(sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>5</td>
<td>-0.51175546</td>
<td>0.008</td>
<td>-0.51175546</td>
<td>2000</td>
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<td>10.811</td>
</tr>
<tr>
<td>0.3</td>
<td>10</td>
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<td>0.019</td>
<td>-0.65570413</td>
<td>2000</td>
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<td>10.824</td>
</tr>
<tr>
<td>0.5</td>
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<td>0.037</td>
<td>-0.79217947</td>
<td>2000</td>
<td>3</td>
<td>15.435</td>
</tr>
<tr>
<td>0.6</td>
<td>13</td>
<td>-0.85766610</td>
<td>0.042</td>
<td>-0.85766610</td>
<td>2000</td>
<td>3</td>
<td>15.886</td>
</tr>
<tr>
<td>0.7</td>
<td>21</td>
<td>-0.92101818</td>
<td>0.068</td>
<td>-0.92101818</td>
<td>2000</td>
<td>3</td>
<td>15.941</td>
</tr>
<tr>
<td>0.8</td>
<td>32</td>
<td>-0.98250103</td>
<td>0.107</td>
<td>-0.98250103</td>
<td>2000</td>
<td>3</td>
<td>16.046</td>
</tr>
<tr>
<td>0.9</td>
<td>78</td>
<td>-1.04196816</td>
<td>0.330</td>
<td>-1.04196816</td>
<td>2000</td>
<td>3</td>
<td>16.289</td>
</tr>
<tr>
<td>0.95</td>
<td>152</td>
<td>-1.07093113</td>
<td>1.103</td>
<td>-1.07093113</td>
<td>2000</td>
<td>3</td>
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</tr>
<tr>
<td>0.98</td>
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<td>-1.08807095</td>
<td>5.076</td>
<td>-1.08807095</td>
<td>2000</td>
<td>4</td>
<td>21.619</td>
</tr>
</tbody>
</table>
Tables 1-4 give the approximate numerical values of the skin friction \( f''(0, \xi) \), the heat transfer rate \( \theta'(0, \xi) \), and the mass transfer rate \( \phi'(0, \xi) \) respectively for various values of \( \xi \) and grid points \( N_t \) computed using the SPM. The results presented in these tables were computed using the same number of collocation points \( N_x \) and \( L \). Increasing the grid points improves the accuracy of the results until the results match exactly to within eight decimal places. As observed from Table 4, for small value of \( \xi \), full convergence to at least eight decimal digits was reached with grid points \( N_t = 1000 \). As \( \xi \) approaches 1, more grid points \( N_t \) are required to give the converged results presented in Table 4. In Table 5, when \( \xi \) is small, the number of grid points \( N_t \) required to give converged results that are consistent to at least eight decimal digits was 2000. The number of grid points \( N_t \) was progressively increased as \( \xi \) tends closer to 1. Similarly, in Table 6, convergence to within eight decimal places was reached when the \( N_t \) was at least 2000. We remark that for all the values of grid points \( N_t \) used in Tables 4-6, only four iterations were used to obtain the results. Results given in Tables 7-9 give further validation of the accuracy of the SPM. The tables present a comparison of the SPM and the SRM approximate solutions for the skin friction \( f''(0, \xi) \), heat transfer rate \( \theta'(0, \xi) \), and mass transfer rate \( \phi'(0, \xi) \) at different values of the dimensionless time \( \xi \) and at different order of approximation. The solutions were obtained for Pr = 1.5, Sc = 0.6, \( \gamma = 1 \), \( \lambda = 0.5 \), \( N = 1 \) and \( Ha = 1 \). It can be observed from the tables that as the dimensionless time \( \xi \) increases, the order of the SPM approximation required to give converged results increases. This shows that when \( \xi \) is very small, only few terms of the SPM approximation are needed to give converged results and higher order of approximation are required when \( \xi \) is closer to 1. Furthermore, the column on the run time in (sec) for both SPM and SRM is displayed in Tables 7-9. It can be seen from the table that the desired solution for the SPM was obtained after only a few seconds. This shows the efficiency of the proposed SPM in terms of the amount of time it takes to give desired results. Comparing the SPM and the SRM computational times clearly shows that the SPM is faster than the SRM in the computation of the solution for the governing equation. This computation speed of the SPM may be explained by the fact that discretization was done only in \( \eta \) – direction unlike the SRM, where discretization was done both in \( \eta \) – and \( \xi \) directions. Hence, the numerical results presented in the tables, show that the two methods were in good agreement on comparison. In addition, the table further gives the number of grid points \( N_t \) and iterations \( (H) \) required to give converged SRM results that match with the SPM results to within eight decimal places. It can be observed from the Tables that for the time \( \tau \) closer to 1, both the values of grid points \( N_t \) and iterations required to obtain the results presented in Tables 7-9 increase.

The velocity profile \( f'((\xi, \eta)) \) for different values of \( \xi \) is shown in Fig. 1. It can be seen that increasing the values of \( \xi \) tends to reduce the velocity distribution in the boundary layer. The influence of \( \xi \) on the temperature profile \( \theta((\xi, \eta)) \) is displayed in Figure 2. The influence of \( \xi \) is to reduce the temperature distribution. It can be observed from Figure 3 that the effect of increasing \( \xi \) on the concentration distribution \( \phi((\xi, \eta)) \) in the solutal boundary is to reduce the solutal boundary layer. Similar observations in Figure 1 were made in earlier studies by Aurangzaib et al. [44], while related effect in Figure 2 was observed in similar studies by Ishak et al. [43] and in Figure 3 by Pal and Mondal [2] and Aurangzaib et al. [44].

Figures 4 - 6 illustrates the effect of the Hartman number \( (Ha) \) on the fluid velocity \( f'((\xi, \eta)) \), temperature \( \theta((\xi, \eta)) \), and concentration \( \phi((\xi, \eta)) \) respectively. It was observed from Figure 4 that as \( Ha \) increases, there is a reduction in the fluid velocity. This is due to an increase in the strength of the magnetic field normal to the flow direction in an electrically conducting fluid which result in a drag Lorenz force acting in the opposite direction to that of the flow. Hence, applying moderate magnetic field stabilizes the flow. Figure 5 shows the influence of \( Ha \) on the temperature distribution. It is clear that the thermal boundary layer increases with an increase in \( Ha \). Therefore, an increase in the values of \( Ha \) causes an increase in temperature. Figure 6 presents the effect of \( Ha \) on the concentration distribution. An increase in \( Ha \) leads to an increase in the concentration profiles. This is because application of magnetic field heats up the fluid and thereby decreasing the heat and mass transfers from the wall. This causes an increase in the fluid temperature and concentration profiles. The effect of the Hartman number on the fluid velocity \( f'((\xi, \eta)) \), temperature \( \theta((\xi, \eta)) \), and concentration \( \phi((\xi, \eta)) \) profiles respectively correlate with results obtained by El-Kabeir and Rashad [4] and Chamkha and El-Kabeir[5].
Figure 1: Effects of $\xi$ on velocity distribution $f'(\xi, \eta)$, when $Ha = 1$, $Sc = 0.6$, $Pr = 1.5$, $N = 1$, $\lambda = 0.5$, $L = 30$, $N_x = 100$.

Figure 2: Effect of $\xi$ on temperature distribution $\theta(\xi, \eta)$, when $N = 1$, $Sc = 0.6$, $Ha = 1$, $Pr = 1.5$, $\lambda = 0.5$, $\gamma = 1$, $L = 30$, $N_x = 100$.

Figures 7 - 9 shows the effect of the chemical reaction parameter $\gamma$ on the velocity $f'(\xi, \eta)$, temperature $\theta(\xi, \eta)$ and the solute concentration $\phi(\xi, \eta)$ profiles. We note that both the fluid velocity and concentration distributions are reduced with increasing values of $\gamma$ (which represents severe destructive reactant). A reverse effect was observed for the temperature distribution which in this case increases with an increment in $\gamma$. This result, however, implies that the increase in $\gamma$ makes the concentration of the diffusing species to decrease. The reduction in the concentration of the diffusing species reduces mass diffusion, consequently showing down the fluid motion and increasing the fluid temperature. These findings are consistent with those of Chamkha and El-Kabeir [5] in a similar investigation.
Figure 3: Effect of $\xi$ on concentration distribution $\phi(\xi, \eta)$ when $\lambda = 0.5$, $Pr = 1.5$, $Sc = 0.6$, $\gamma = 1$, $Ha = 1$, $N = 1$, $L = 30$, $N_x = 100$.

Figure 4: Effect of Hartmann number $Ha$ on velocity distribution $f'(\xi \eta)$, when $\xi = 0.5$, $Sc = 0.6$, $\gamma = 1$, $Pr = 1.5$, $N = 1$, $\lambda = 1$, $L = 30$, $N_x = 100$. 
Figure 5: Effect of Hartman number $Ha$ on temperature profile $\theta(\xi, \eta)$ when $\xi = 0.5$, $\lambda = 1$, $Sc = 0.6$, $\gamma = 1$, $Pr = 1.5$, $N = 1$, $L = 30$, $N_x = 100$.

Figure 6: Effect of Hartman number $Ha$ on concentration distribution $\phi(\xi, \eta)$ when $\xi = 0.5$, $\lambda = 1$, $Pr = 1.5$, $Sc = 0.6$, $\gamma = 1$, $N = 1$, $L = 30$, $N_x = 100$. 

Figure 7: Effect of Chemical reaction parameter $\gamma$ on velocity distribution $f'(\eta)$, when $\xi = 0.5$, $Sc = 0.6$, $Ha = 1$, $Pr = 1.5$, $N = 1$, $\lambda = 1$, $L = 30$, $N_x = 100$.

Figure 8: Effect of chemical reaction parameter $\gamma$ on temperature profile $\theta(\xi, \eta)$, when $\xi = 0.5$, $Ha = 1$, $Sc = 0.6$, $N = 1$, $\lambda = 1$, $Pr = 1.5$, $L = 30$, $N_x = 100$. 
Figure 9: Effect of Chemical reaction parameter $\gamma$ on concentration distribution $\phi(\xi, \eta)$ when $\xi = 0.5$, $\lambda = 1$, $Pr = 1.5$, $Sc = 0.6$, $Ha = 1$, $N = 1$, $L = 30$, $N_x = 100$.

Figure 10: Effect of Mixed convection parameter $\lambda$ on velocity distribution $f'(\xi, \eta)$, when $\xi = 0.5$, $Sc = 0.6$, $Ha = 1$, $Pr = 0.7$, $N = 1$, $\gamma = 1$, $L = 30$, $N_x = 100$. 
Figure 11: Effect of Mixed convection parameter $\lambda$ on temperature distribution $\theta(\xi, \eta)$, when $\xi = 0.5$, $Sc = 0.6$, $Ha = 1$, $Pr = 0.7$, $N = 1$, $\gamma = 1$, $L = 30$, $N_x = 100$.

Figure 12: Effect of Mixed convection parameter $\lambda$ on concentration distribution $\phi(\xi, \eta)$ when $\xi = 0.5$, $Pr = 1.5$, $Sc = 0.6$, $\gamma = 1$, $Ha = 1$, $N = 1$, $L = 30$, $N_x = 100$.

The influence of mixed convection parameter $\lambda$ on the velocity $f'(\xi, \eta)$, temperature $\theta(\xi, \eta)$ and concentration $\phi(\xi, \eta)$ profiles are given in Figures 10 - 12 respectively. When the mixed convection parameter value was increased, there was an increase in the boundary layer of the velocity profile while the temperature and concentration profiles decreases. The faster moving fluid removes the heat and species, thereby causing stabilization and reduction in the growth of the thermal and diffusion boundary layers along the vertical walls. This behavior can be clearly seen in Figures 10 - 12. These observations are consistent with those of El-Kabeir [5].
Figure 13: Effect of concentration to thermal buoyancy ratio parameter $N$ on velocity distribution $f'(\xi, \eta)$, when $\xi = 0.5$, $Sc = 0.6$, $Ha = 1$, $Pr = 0.7$, $\gamma = 1$, $L = 30$, $N_x = 100$.

Figures 13 - 15 depicts the impact of the concentration to thermal buoyancy ratio $N$ on the velocity $f'(\xi, \eta)$, temperature $\theta(\xi, \eta)$ and concentration $\phi(\xi, \eta)$ distributions respectively. It was observed that the boundary layer thickness increases with increasing values of $N$ on the velocity distribution while there was a reduction in the temperature and solute concentration distributions.
Figure 14: Effect of concentration to thermal buoyancy ratio $N$ on temperature distribution $\theta(\xi, \eta)$, when $\xi = 0.5$, $Ha = 1$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 1$, $Pr = 1.5$, $L = 30$, $N_x = 100$.

Figure 15: Effect of concentration to thermal buoyancy parameter $N$ on concentration distribution $\phi(\xi, \eta)$ when $\xi = 0.5$, $\lambda = 1$, $Pr = 1.5$, $Sc = 0.6$, $\gamma = 1$, $Ha = 1$, $L = 30$, $N_x = 100$. 
Figure 16: Variation of temperature $\theta(\xi, \eta)$ for different values of Prandtl number $Pr$, when $\xi = 0.5$, $Ha = 1$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 1$, $N = 1$, $L = 30$, $N_x = 100$.

Figure 17: Effect of Schmidt number $Sc$ on concentration distribution $\phi(\xi, \eta)$ when $\xi = 0.5$, $\lambda = 1$, $Pr = 1.5$, $N = 1$, $\gamma = 1$, $Ha = 1$, $L = 30$, $N_x = 100$. 
Figure 18: Residual error curve $Res(f)$ against SRM iterations when $\xi = 0.3, 0.5, 0.7, 0.9$, $Ha = 1$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 1$, $N = 1$, grid points $N_t = 5000$, $L = 30$, $N_x = 100$.

Figure 19: Residual error curve $Res(\theta)$ against SRM iterations when $\xi = 0.3, 0.5, 0.7, 0.9$, $Ha = 1$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 1$, $N = 1$, grid points $N_t = 5000$, $L = 30$, $N_x = 100$. 
Figure 20: Residual error curve $Res(\phi)$ against SRM iterations when $\xi = 0.3, 0.5, 0.7, 0.9$, $Ha = 1$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 1$, $N = 1$, grid points $N_x = 5000$, $L = 30$, and $N_t = 100$.

Figure 21: Residual error curve $Res(f)$ against increasing SPM approximation order when $\xi = 0.3, 0.5, 0.7, 0.9$, $Ha = 1$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 1$, $N = 1$, $L = 30$, and $N_t = 100$. 

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Figure 22: Residual error curve $\text{Res}(\bar{\theta})$ against increasing SPM approximation order when $\bar{\zeta} = 0.3, 0.5, 0.7, 0.9$, $Ha = 1$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 1$, $N = 1$, $L = 30$, and $N_e = 100$.

Figure 23: Residual error curve $\text{Res}(\phi)$ against increasing SPM approximation order when $\bar{\zeta} = 0.3, 0.5, 0.7, 0.9$, $Ha = 1$, $Sc = 0.6$, $\gamma = 1$, $\lambda = 1$, $N = 1$, $L = 30$, and $N_e = 100$. 
Figure 16 illustrate the influence of the Prandtl number \( Pr \) on the temperature \( \theta(\xi, \eta) \) distribution. We note that an increase in \( Pr \) leads to a decrease in the temperature distribution which in turn yields a reduction in the thermal boundary layer thickness. These results are in agreement with studies by Pal and Mondal [2], and Hayat et al. [23]. The impact of Schmidt number \( Sc \) on the concentration profile \( \phi(\xi, \eta) \) is shown in Figure 17. We note that the concentration profile \( \phi(\xi, \eta) \) reduces with an increase in \( Sc \). These findings are consistent with those of Hayat et al. [23].

Figures 18 - 20 displays the variation of the SRM residual errors in \( f \), \( \theta \) and \( \phi \) respectively against the number of iterations. The results are given for different values of the time \( \xi \). From the graphs, it can be seen that for small value of time \( \xi \), convergence to accurate result to within a specific level below \( 10^{-10} \) can be achieved with few iterations of the SRM. In addition, it can be observed from the Figures that the residual error curves for \( f(\xi, \eta) \), \( \theta(\xi, \eta) \), \( \phi(\xi, \eta) \) level at a certain level below \( 10^{-10} \) for all values of \( \xi \) considered.

Figures 21 - 23 shows the residual error in \( f \), \( \theta \) and \( \phi \) respectively, against increasing orders of the SPM approximation at different values of time \( \xi \). It can be observed that the residual error curves tend to plateau at more or less a fixed level for the different values of time \( \xi \) considered. The interpretation of these results is that the SPM will converge up to a specific saturation level which corresponds to the level at which the curves levels off. In the equation for \( f(\xi, \eta) \), the residual error curve levels off at a fixed level below \( 10^{-8} \) for all values of \( \xi \). In the equation for \( \theta(\xi, \eta) \), it can be seen that the residual error curve levels off at a certain level below \( 10^{-10} \) and below \( 10^{-10} \) in the equation for \( \phi(\xi, \eta) \). Furthermore, we note that as \( \xi \) approaches 1, the plateau is reached at higher orders of the SPM approximation. This observation is in line with the results presented in Tables 4 - 4 where it was noted that when \( \xi \) is small, only few terms of the SPM approximation are required to obtain converged results that are accurate up to eight decimal digits and more terms are required as \( \xi \) tends closer to 1. In addition, this observation can be linked with the well known fact that the standard perturbation based methods give accurate results when the series expansion is with respect to a small parameter. It is interesting to note that with the SPM, accurate results can be obtained even when \( \xi \) approaches 1, albeit with a higher orders of the SPM approximation.

5 Conclusion

In this work, we have considered the application of the perturbation technique coupled with the Chebyshev pseudo-spectral collocation method to the solution of unsteady heat and mass transfer by MHD mixed convection flow over an impulsively stretched vertical surface with chemical reaction effect. Approximate numerical results were generated using the spectral perturbation method for the solution of the skin friction coefficient, heat transfer rate, velocity distribution, temperature distribution and concentration distribution at different flow parameter values. The accuracy of the SPM was demonstrated by comparing with results generated using the spectral relaxation (SRM) where a good agreement was achieved between the two set of results up to at least eight decimal places. The computational efficiency of the SPM was confirmed by comparing the computational times of generating the SPM solutions with the computational times of the SRM solutions. It was observed that the SPM is much faster than the SRM. Residual errors were obtained for the SPM and SRM. From the residual errors analysis we observed that in terms of convergence, the SPM converges faster than the SRM but the SRM gives more accurate results than the SPM in terms of accuracy. The study showed that the SPM can be used as an alternative numerical method to the usual perturbation methods to obtain numerical solutions of partial differential equations (PDEs). It was noted that the SPM solves a partial differential equation by applying discretization only in the space direction. This feature combined with integrating using the spectral method results in computation time saving. Unlike standard perturbation methods, the SPM gives higher order approximate solutions which are not possible or very complicated to find with the usual perturbation methods. For problems related to one investigated in this work, the SPM can be used efficiently. The numerical results presented in this study suggests that the proposed SPM has the potential to be utilized for solving complex nonlinear partial differential equations particularly defined using the [3] transformation. In conclusion, the SPM presented in this study adds to a growing body of literature on numerical methods for solving complex nonlinear fluid flow problems. Also, the present study contributes additional evidence that suggests the use of the SPM as a very good numerical approach for solving complex nonlinear PDEs defined using the [3] transformation.
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