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Research Article

Application of the homotopy perturbation method and the homotopy analysis method for the dynamics of tobacco use and relapse

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Abstract

We obtain approximate analytical solutions of two mathematical models of the dynamics of tobacco use and relapse including peer pressure using the homotopy perturbation method (HPM) and the homotopy analysis method (HAM). To enlarge the domain of convergence we apply the Padé approximation to the HPM and HAM series solutions. We show graphically that the results obtained by both methods are very accurate in comparison with the numerical solution for a period of 30 years.

Keywords: Dynamics of tobacco, homotopy perturbation method, homotopy analysis method.

1 Introduction

Epidemic models are a mathematical formulation of the spread and dynamics of a given biological behaviour. Some epidemic models [1, 2, 3, 4] have been successfully studied analytically. Similarly, and to the best of our knowledge for the first time, we study mathematical models for the dynamics of tobacco use and relapse including peer pressure analytically using both HPM and HAM. To enlarge the convergence of the analytical solutions we use the Padé approximations. Moreover, we verify our results with numerical solutions obtained by ode45 (an ode solver in Matlab).

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In recent years the homotopy analysis method proposed by Liao in 1992 and the homotopy perturbation method proposed by He in 1998 have generated a lot of interest in solving different types of nonlinear problems as seen from the literature [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Both HPM and HAM can be applied to nonlinear problems and they do not require the existence of any perturbation parameter in the problem. Therefore, both techniques are useful to obtain approximate analytical solutions of nonlinear problems which do not contain small or large parameters.
This paper is organized as follows. In Section 2, we present the two mathematical models of the dynamics of tobacco use and relapse including peer pressure. In Section 3, we briefly consider the basic concepts of HPM. Section 4 introduce the basics of HAM. In Section 5, the approximate solutions obtained by HPM and HAM and the corresponding Padé approximations are presented. A discussion is presented in Section 6. Finally, we present a brief conclusion in Section 7.

2 Models for the dynamics of tobacco use and relapse with peer pressure

A general model for tobacco use with vital dynamics including peer pressure [30] is described by the following equations

$$\begin{aligned} &\dot{S}-\mu N+\beta S\frac{D}{N}+\mu S=0,\\ &\dot{D}-\beta S\frac{D}{N}+\gamma D-\delta R+\mu D=0,\\ &\dot{R}-\gamma D+\delta R+\mu R=0. \end{aligned} \tag{2.1}$$

A nonlinear relapse model including peer pressure [30] is as follows

$$\begin{aligned} &\dot{S}-\mu N+\beta S\frac{D}{N}+\mu S=0,\\ &\dot{D}-\beta S\frac{D}{N}+\gamma D-\beta' R\frac{D}{N}+\mu D=0,\\ &\dot{R}-\gamma D+\beta' R\frac{D}{N}+\mu R=0. \end{aligned} \tag{2.2}$$

where the dots denotes differentiation with respect to t. S, D, and R are the classes of individuals in a constant population size N, which has been taken as unity [30] through out the analysis. The class S is composed of individuals susceptible to becoming regular drug users, the class D is composed of individuals who are regular drug users. The class R is composed of individuals that are treating themselves in any form or that have recovered from habitual drug use.
The parameter μ denotes the constant mortality rate; β denotes the per capita effective influence rate; γ is the recovery rate per habitual drug user per unit of time; δ denotes the relapse rate per recovered individual per unit of time and β' is the per capita effective influence rate for the relapse drug use.
We keep the parameters as taken in [30] $\beta = 0.625$ $\delta = 0.89, \gamma = 0.15$, $\mu = 0.10$, $\beta' = 1.4$ fixed throughout the discussion. The initial conditions as taken in [30] for both the models are $S(0) = 0.8, D(0) = 0.2$ and $R(0) = 0$.

3 Fundamentals of HPM

For a brief description of HPM, we refer [32, 31, 33, 34, 35, 36]. Consider a general nonlinear equation in the form

$$A(u)-f(r)=0,\ r\in\Omega, \tag{3.3}$$

with the following boundary conditions

$$B(u, \partial u/\partial n) = 0, \ r \in \Gamma, \tag{3.4}$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary operator, Γ is the boundary of the domain Ω and $\partial u/\partial \eta$ denotes differentiation along the normal drawn outwards from Ω [33]. In most cases, the A operator can be split into two operators, namely L and N, which represent the linear and the non-linear operators, respectively. Hence, (3.3) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{3.5}$$

In a broad sense, a homotopy can be constructed in the following form

$$\begin{aligned} H(v,p) &= (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \\ & \quad p \in [0,1], \ r \in \Omega, \end{aligned} \tag{3.6}$$

where p is a homotopy parameter, whose values range between 0 and 1, u_0 is the first approximation to the solution of (3.5) that satisfies the boundary conditions. Assume that the solution for (3.6) can be written as a power series of p

$$v = v_0 + v_1 p + v_2 p^2 + v_3 p^3 + \cdots . \tag{3.7}$$

Substituting (3.7) into (3.6) and equating identical powers of p, it is possible to obtain the values for the sequence $v_0, v_1, v_2, \ldots$

When $p \to 1$, it yields the approximate solution for (3.3) in the form

$$v = v_0 + v_1 + v_2 + v_3 + \cdots . \tag{3.8}$$

4 Fundamentals of HAM

We refer to [15, 18, 29, 37, 38] for more information on HAM. Consider the following nonlinear equation

$$N[u(t)] = 0 \tag{4.9}$$

where N is a nonlinear operator, $u(t)$ is an unknown function, and t denotes the independent variable. The so-called zero order deformation equation is:

$$(1-p)L[\phi(t;p) - u_0(t)] = phN[\phi(t;p)] \tag{4.10}$$

where, L is an auxiliary linear operator, $p \in [0,1]$ is the embedding parameter, h is a nonzero auxiliary linear parameter, $u_0(t)$ is the initial guess of $u(t)$, and $\phi(t;p)$ is an unknown function. Obviously, at $p = 0$ and $p = 1$, we have

$$\phi(t;0) = u_0(t) \quad \phi(t;1) = u(t) \tag{4.11}$$

respectively. Thus, as p increases from 0 to 1, $\phi(t;p)$ varies from the initial guess $u_0(t)$ to the exact solution $u(t)$. By expanding $\phi(t;p)$ in a Taylor's series with respect to p, one has

$$\phi(t;p) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) p^m \tag{4.12}$$

where

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;p)}{\partial p^m} |p = 0 \tag{4.13}$$

If $u_0(t)$, L and h are properly chosen, such that the series (4.12) converges at $p = 1$, then we have

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) \tag{4.14}$$

as the solution expression which would be one of the solutions of (4.9) as proved by Liao [37]. Differentiating equation (4.10) m-times with respect to p and then setting $p = 0$ and finally dividing by $m!$, we have the so-called m-th order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = hH(t)R_m(u_{m-1}) \tag{4.15}$$

where

$$R_m(u_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(t;p)]}{\partial p^{m-1}}|_{p=0} \tag{4.16}$$

and

$$\chi_m = \begin{cases} 0; & \text{for} \quad m = 1 \\ 1; & \text{for} \quad m \neq 1 \end{cases} \tag{4.17}$$

5 Approximate solutions obtained by HPM and HAM

In this section, we present the solution procedure of the tobacco use and relapse models by using HPM and HAM.

5.1 HPM and HPM-Padé solution of the general model with vital dynamics

According to the HPM (relation (3.6)), we can construct a homotopy of the given system as follows

$$\begin{aligned}
&(1-p)(\dot{v}_1 - \dot{S}_0) + p\left(\dot{v}_1 - \mu N + \beta v_1 \frac{v_2}{N} + \mu v_1\right) = 0,\\
&(1-p)(\dot{v}_2 - \dot{D}_0) + p\left(\dot{v}_2 - \beta v_1 \frac{v_2}{N} + \gamma v_2 - \delta v_3 + \mu v_2\right) = 0,\\
&(1-p)(\dot{v}_3 - \dot{R}_0) + p(\dot{v}_3 - \gamma v_2 + \delta v_3 - \mu v_3) = 0,
\end{aligned} \tag{5.18}$$

where the dots denotes differentiation with respect to t, and the initial approximations are

$$\begin{aligned}
v_{1,0}(t) &= S_0(t) = S(0) = r_1,\\
v_{2,0}(t) &= D_0(t) = D(0) = r_2,\\
v_{3,0}(t) &= R_0(t) = R(0) = r_3.
\end{aligned} \tag{5.19}$$

From (3.7), we assume that the solution of (5.18) can be written as a power series of p as follows

$$\begin{aligned}
v_1 &= v_{1,0} + pv_{1,1} + p^2 v_{1,2} + p^3 v_{3,0} + \cdots,\\
v_2 &= v_{2,0} + pv_{2,1} + p^2 v_{2,2} + p^3 v_{2,3} + \cdots,\\
v_3 &= v_{3,0} + pv_{3,1} + p^2 v_{3,2} + p^3 v_{3,3} + \cdots,
\end{aligned} \tag{5.20}$$

where $v_{i,j}$ $i,j = 1,2,3,\ldots$ are functions yet to be determined. Substituting (5.19) and (5.20) into (5.18), and arranging the coefficients of p powers, we have

$$\begin{aligned}
&\dot{v}_{1,0} + (\dot{v}_{1,1} - \mu N + \beta v_{1,0}v_{2,0}/N + \mu v_{1,0})p\\
&\qquad + (\dot{v}_{1,2} + \mu v_{1,1} + \beta v_{1,1}v_{2,0}/N + \beta v_{1,0}v_{2,1}/N)p^2 + \cdots = 0,\\
&\dot{v}_{2,0} + (\dot{v}_{2,1} - \beta v_{1,0}v_{2,0}/N - \delta v_{3,0} + \mu v_{2,0} + \gamma v_{2,0})p\\
&\qquad + (\dot{v}_{2,2} - \delta v_{3,1} + \gamma v_{2,1} + \mu v_{2,1} - \beta v_{1,0}v_{2,1}/N - \beta v_{1,1}v_{2,0}/N)p^2 + \cdots = 0,\\
&\dot{v}_{3,0} + (\dot{v}_{3,1} + \delta v_{3,0} - \gamma v_{2,0} + \mu v_{3,0})p + (\dot{v}_{3,2} - \gamma v_{2,1} + \delta v_{3,1} + \mu v_{3,1})p^2 + \cdots = 0.
\end{aligned} \tag{5.21}$$

In order to obtain the unknown $v_{i,j}(t)$, $i,j=1,2,3,\ldots$, we must construct and solve the following system of equations, considering the initial conditions of $v_{i,j}(0)=0$, $i,j=1,2,3,\ldots$, resulting

$$\begin{aligned}
&\dot{v}_{1,0}=0,\\
&\dot{v}_{1,1}-\mu N+\beta v_{1,0}v_{2,0}/N+\mu v_{1,0}=0,\\
&\dot{v}_{1,2}+\mu v_{1,1}+\beta v_{1,1}v_{2,0}/N+\beta v_{1,0}v_{2,1}/N=0,\\
&\vdots\\
&\dot{v}_{2,0}=0,\\
&\dot{v}_{2,1}-\beta v_{1,0}v_{2,0}/N-\delta v_{3,0}+\mu v_{2,0}+\gamma v_{2,0}=0,\\
&\dot{v}_{2,2}-\delta v_{3,1}+\gamma v_{2,1}+\mu v_{2,1}-\beta v_{1,0}v_{2,1}/N-\beta v_{1,1}v_{2,0}/N=0,\\
&\vdots\\
&\dot{v}_{3,0}=0,\\
&\dot{v}_{3,1}+\delta v_{3,0}-\gamma v_{2,0}+\mu v_{3,0}=0,\\
&\dot{v}_{3,2}-\gamma v_{2,1}+\delta v_{3,1}+\mu v_{3,1}=0,\\
&\vdots
\end{aligned}\tag{5.22}$$

Therefore,

$$\begin{aligned}
v_{1,0}(t)&=r_1,\\
v_{1,1}(t)&=t(N^2\mu-\beta r_1r_2-\mu r_1N)/N,\\
v_{1,2}(t)&=-t^2\big[N^3\mu^2-3N\mu\beta r_1r_2-N^2\mu^2r_1+\beta r_2\mu N^2-\beta^2r_2^2r_1\\
&\quad+\beta^2r_1^2r_2+\beta r_1\delta r_3N-\beta r_1\gamma r_2N\big]/(2N^2),\\
&\vdots\\
v_{2,0}(t)&=r_2,\\
v_{2,1}(t)&=t(\beta r_1r_2+\delta r_3N-\mu r_2N-\gamma r_2N)/N,\\
v_{2,2}(t)&=t^2\big[-\delta^2N^2r_3+\delta N^2\gamma r_2-2\delta N^2\mu r_3-2\beta r_1\gamma r_2N-\gamma N^2\delta r_3\\
&\quad+2\gamma N^2\mu r_2+\gamma^2N^2r_2-3N\mu\beta r_1r_2+N^2\mu^2r_2+\beta^2r_1^2r_2\\
&\quad+\beta r_1\delta r_3N+\beta r_2\mu N^2-\beta^2r_2^2r_1\big]/(2N^2),\\
&\vdots\\
v_{3,0}(t)&=r_3,\\
v_{3,1}(t)&=t(-\delta r_3+\gamma r_2-\mu r_3),\\
v_{3,2}(t)&=t^2\big[\beta r_1\gamma r_2+\gamma\delta r_3N-2\gamma\mu r_2N-\gamma^2r_2N\\
&\quad+\delta^2Nr_3-\delta N\gamma r_2+2\delta N\mu r_3+N\mu^2r_3\big]/(2N),\\
&\vdots
\end{aligned}\tag{5.23}$$

We obtained $v_{1,3}, v_{2,3}$, and $v_{3,3}$ and the succeeding terms, nevertheless, because they were too cumbersome, we skip them and use them only in the final results. Using initial conditions, we consider that $r_1=0.8, r_2=0.2$, and $r_3=0$. Now, from (3.7), we obtain a 15-th order approximation, then setting $p=1$ yields the approximate solution of (2.1) as

$$S(t)=\lim_{p\to 1} v_1(t)=\lim_{p\to 1}\left(\sum_{k=0}^{15} p^k v_{1,k}(t)\right),$$
$$D(t)=\lim_{p\to 1} v_2(t)=\lim_{p\to 1}\left(\sum_{k=0}^{15} p^k v_{2,k}(t)\right), \tag{5.24}$$
$$R(t)=\lim_{p\to 1} v_3(t)=\lim_{p\to 1}\left(\sum_{k=0}^{15} p^k v_{3,k}(t)\right).$$

We obtain the following solution expressions

$$\begin{aligned}
S(t) &= 0.8-0.08t+0.000000020434122t^{10}-0.0000000036187767t^{11}+\\
&+0.00000000056608213t^{12}-9.5872631\times 10^{-11}t^{13}+1.5632632\times 10^{-11}t^{14}+\\
&-2.1791127\times 10^{-12}t^{15}-0.0035000000t^{2}+\\
&-0.0013375000t^{3}+0.00066701562t^{4}-0.000099307140t^{5}+0.000018519042t^{6}\\
&-0.0000042713711t^{7}+0.00000067845594t^{8}-0.00000010372018t^{9},\\
D(t) &= 0.2+0.05t-0.0000000099108402t^{10}+0.0000000028068293t^{11}\\
&-0.00000000053418183t^{12}+9.9606923\times 10^{-11}t^{13}-1.6963917\times 10^{-11}t^{14}\\
&+2.4366167\times 10^{-12}t^{15}+0.014600000t^{2}-0.0030555000t^{3}+0.00053483312t^{4}\\
&-0.00015470391t^{5}+0.000027260378t^{6}-0.0000027872979t^{7}+\\
&+0.00000024731622t^{8}-0.0000000022366983t^{9},\\
R(t) &= 0.03t-0.000000010523281t^{10}+0.00000000081194745t^{11}+\\
&-3.1900298\times 10^{-11}t^{12}-3.7343061\times 10^{-12}t^{13}+1.3312858\times 10^{-12}t^{14}+\\
&-2.5750403\times 10^{-13}t^{15}-0.011100000t^{2}+0.0043930000t^{3}-0.0012018488t^{4}+\\
&+0.00025401106t^{5}-0.000045779423t^{6}+0.0000070586694t^{7}\\
&-0.00000092577218t^{8}+0.00000010595688t^{9}.
\end{aligned} \tag{5.25}$$

In order to enlarge the domain of convergence, we apply a Padé approximant [39] to (5.25) to obtain

$$\begin{aligned}
S(t)_{[7/7]} &= (0.8+0.35718844t+0.059857355t^{2}+0.0034828989t^{3}+1.6858846e(-4)t^{4}\\
&+6.4711304e(-5)t^{5}+8.4303666e(-6)t^{6}+1.1247734e(-7)t^{7})/\Delta_1,\\
\Delta_1 &= 1+0.54648556t+0.13384525t^{2}+0.021800898t^{3}+0.0030562844t^{4}\\
&+0.00037415994t^{5}+0.000030866077t^{6}+0.0000011004193t^{7},\\
D(t)_{[6/6]} &= (0.20000000+0.12121360t+0.046599720t^{2}+0.0078912184t^{3}\\
&+0.0013357217t^{4}+0.000085621900t^{5}+0.0000085878562t^{6})/\Delta_2,\\
\Delta_2 &= 1.0+0.35606797t+0.070981601t^{2}+0.010995230t^{3}+0.0015138068t^{4}\\
&+0.00015276221t^{5}+0.000011527548t^{6}\\
R(t)_{[7/7]} &= (0.030000000t+0.00050770054t^{2}+0.0022605537t^{3}+0.0000022774052t^{4}\\
&+0.000033213115t^{5}-0.00000016562041t^{6}+0.000000052608234t^{7})/\Delta_3,\\
\Delta_3 &= 0.99999998+0.38692334t+0.072080096t^{2}+0.010148698t^{3}\\
&+0.0013409367t^{4}+0.00014205125t^{5}+0.0000093719701t^{6}\\
&+0.00000027042191t^{7}
\end{aligned} \tag{5.26}$$

5.2 HPM and HPM-Padé solution of the nonlinear relapse model

According to the HPM (relation (3.6)), we can construct a homotopy of the system considered as follows

$$
\begin{aligned}
&(1-p)(\dot{v}_1-\dot{S}_0)+p\left(\dot{v}_1-\mu N+\beta v_1\frac{v_2}{N}+\mu v_1\right)=0,\\
&(1-p)(\dot{v}_2-\dot{D}_0)+p\left(\dot{v}_2-\beta v_1\frac{v_2}{N}+\gamma v_2-\frac{\beta'}{N}v_2v_3+\mu v_2\right)=0,\\
&(1-p)(\dot{v}_3-\dot{R}_0)+p\left(\dot{v}_3-\gamma v_2+\frac{\beta'}{N}v_2v_3+\mu v_3\right)=0,
\end{aligned}
\tag{5.27}
$$

where the dots denotes differentiation with respect to t, and initial approximations are taken from (5.19).
From (3.7), we can assume that the solution of (5.27) can be written as a power series of p, just done in (5.20). Now, substituting (5.19) and (5.20) into (5.27), and arranging the coefficients of p, we obtain

$$
\begin{aligned}
&\dot{v}_{1,0}+(\dot{v}_{1,1}-\mu N+\beta v_{1,0}v_{2,0}/N+\mu v_{1,0})p\\
&\quad+(\dot{v}_{1,2}+\mu v_{1,1}+\beta v_{1,1}v_{2,0}/N+\beta v_{1,0}v_{2,1}/N)p^2+\cdots=0,\\
&\dot{v}_{2,0}+(\dot{v}_{2,1}-\beta v_{1,0}v_{2,0}/N-\beta' v_{2,0}v_{3,0}/N+\mu v_{2,0}+\gamma v_{2,0})p\\
&\quad+(\dot{v}_{2,2}-\beta' v_{3,1}v_{2,0}/N-\beta' v_{3,0}v_{2,1}/N+\gamma v_{2,1}+\mu v_{2,1}\\
&\qquad-\beta v_{1,0}v_{2,1}/N-\beta v_{1,1}v_{2,0}/N)p^2+\cdots=0,\\
&\dot{v}_{3,0}+(\dot{v}_{3,1}+\beta' v_{2,0}v_{3,0}/N-\gamma v_{2,0}+\mu v_{3,0})p\\
&\quad+(\dot{v}_{3,2}-\gamma v_{2,1}+\beta' v_{3,1}v_{2,0}/N+\beta' v_{3,0}v_{2,1}/N+\mu v_{3,1})p^2+\cdots=0.
\end{aligned}
\tag{5.28}
$$

In order to obtain the unknown $v_{i,j}(t)$, $i,j=1,2,3,\ldots$, we construct and solve the following system of equations, considering the initial conditions of $v_{i,j}(0)=0$, $i,j=1,2,3,\ldots$, resulting

$$
\begin{aligned}
&\dot{v}_{1,0}=0,\\
&\dot{v}_{1,1}-\mu N+\beta v_{1,0}v_{2,0}/N+\mu v_{1,0}=0,\\
&\dot{v}_{1,2}+\mu v_{1,1}+\beta v_{1,1}v_{2,0}/N+\beta v_{1,0}v_{2,1}/N=0,\\
&\vdots\\
&\dot{v}_{2,0}=0,\\
&\dot{v}_{2,1}-\beta v_{1,0}v_{2,0}/N-\beta' v_{2,0}v_{3,0}/N+\mu v_{2,0}+\gamma v_{2,0}=0,\\
&\dot{v}_{2,2}-\beta' v_{3,1}v_{2,0}/N-\beta' v_{3,0}v_{2,1}/N+\gamma v_{2,1}+\mu v_{2,1}\\
&-\beta v_{1,0}v_{2,1}/N-\beta v_{1,1}v_{2,0}/N)=0,\\
&\vdots\\
&\dot{v}_{3,0}=0,\\
&\dot{v}_{3,1}+\beta' v_{2,0}v_{3,0}/N-\gamma v_{2,0}+\mu v_{3,0}=0,\\
&\dot{v}_{3,2}-\gamma v_{2,1}+\beta' v_{3,1}v_{2,0}/N+\beta' v_{3,0}v_{2,1}/N+\mu v_{3,1}=0.
\end{aligned}
\tag{5.29}
$$

Therefore,

$$
\begin{aligned}
v_{1,0}(t) &= r_1, \\
v_{1,1}(t) &= t(N^2\mu - \beta r_1 r_2 - \mu r_1 N)/N, \\
v_{1,2}(t) &= -t^2\big[N^3\mu^2 - 3N\mu\beta r_1 r_2 - N^2\mu^2 r_1 + \beta r_2 \mu N^2 - \beta^2 r_2^2 r_1 \\
&\qquad + \beta^2 r_1^2 r_2 + \beta r_1 \delta r_3 N - \beta r_1 \gamma r_2 N\big]/(2N^2), \\
&\vdots \\
v_{2,0}(t) &= r_2, \\
v_{2,1}(t) &= t(\beta r_1 r_2/N + \beta' r_2 r_3/N - \mu r_2 - \gamma r_2), \\
v_{2,2}(t) &= -r_2 t^2\big[2\gamma N\beta' r_3 - \beta'^2 r_3^2 - 2\beta r_1 \beta' r_3 + 3\beta' \mu r_3 N - 2\gamma N^2 \mu \\
&\qquad + 3N\mu\beta r_1 - \mu^2 N^2 - \gamma^2 N^2 + 2\beta r_1 \gamma N - \beta\mu N^2 + r_2 \beta^2 r_1 \\
&\qquad - r_2 \beta' \gamma N + r_2 \beta'^2 r_3 - \beta^2 r_1^2\big]/(2N^2), \\
&\vdots \\
v_{3,0}(t) &= r_3, \\
v_{3,1}(t) &= t(\gamma r_2 + (-r_2\beta'/N - \mu) r_3), \\
v_{3,2}(t) &= t^2\big[-r_2\gamma^2 N^2 + 2r_2\gamma N\beta' r_3 + r_2\beta r_1 \gamma N - 2r_2\gamma N^2 \mu + 3r_2\beta' \mu r_3 N \\
&\qquad - r_2^2 \beta' \gamma N + r_2^2 \beta'^2 r_3 - r_2 \beta'^2 r_3^2 - r_2 \beta r_1 \beta' r_3 + N^2 \mu^2 r_3\big]/(2N^2) \\
&\vdots
\end{aligned}
\tag{5.30}
$$

We also obtained $v_{1,3}, v_{2,3}$, and $v_{3,3}$ and the succeeding terms, but because they were too cumbersome, we use them only in the final results. From (3.7), we obtain the 22-nd order approximation, then setting $p = 1$ yields the approximate solution of (2.1) to

$$
\begin{aligned}
S(t) &= \lim_{p\to 1} v_1(t) = \lim_{p\to 1}\left(\sum_{k=0}^{22} p^k v_{1,k}(t)\right), \\
D(t) &= \lim_{p\to 1} v_2(t) = \lim_{p\to 1}\left(\sum_{k=0}^{22} p^k v_{2,k}(t)\right), \\
R(t) &= \lim_{p\to 1} v_3(t) = \lim_{p\to 1}\left(\sum_{k=0}^{22} p^k v_{3,k}(t)\right).
\end{aligned}
\tag{5.31}
$$

We obtain the solution expressions as follows

$$\begin{aligned}
S(t) &= 0.8-0.08t+2.7402147e(-10)t^{10}+3.0923745\times10^{-12}t^{11}\\
&-3.9895245\times10^{-12}t^{12}-4.8586144\times10^{-13}t^{13}+1.0399441\times10^{-14}t^{14}\\
&+8.9424213\times10^{-15}t^{15}+8.1857350\times10^{-16}t^{16}-5.4215999\times10^{-17}t^{17}\\
&-1.9285476\times10^{-17}t^{18}-1.2548444\times10^{-18}t^{19}-0.0035000000t^{2}\\
&+1.7553442\times10^{-19}t^{20}+4.0035644\times10^{-20}t^{21}+1.5742026\times10^{-21}t^{22}\\
&+0.00018750000t^{3}+0.000085796875t^{4}+0.0000052398282t^{5}\\
&-0.00000067501432t^{6}-0.00000015111074t^{7}-0.0000000057858584t^{8}\\
&+0.0000000017007642t^{9}\\
D(t) &= 0.2+0.05t-3.8627581e(-10)t^{10}-2.4721109\times10^{-11}t^{11}\\
&+3.6721367\times10^{-12}t^{12}+8.3407454\times10^{-13}t^{13}+3.2941564\times10^{-14}t^{14}\\
&-1.0358837\times10^{-14}t^{15}-1.7372840\times10^{-15}t^{16}-2.1963555\times10^{-17}t^{17}\\
&+2.7020203\times10^{-17}t^{18}+3.4494006\times10^{-18}t^{19}+0.00545t^{2}\\
&-6.7453985\times10^{-20}t^{20}-6.6466719\times10^{-20}t^{21}-6.4154164\times10^{-21}t^{22}\\
&-0.000007t^{3}-0.000079581875t^{4}-0.0000088862120t^{5}\\
&+0.00000027217498t^{6}+0.00000017480867t^{7}+0.000000015367870t^{8}\\
&-0.0000000011485818t^{9}\\
R(t) &= 0.03t+1.1225434e(-10)t^{10}+2.1628733\times10^{-11}t^{11}+3.1738784\times10^{-13}t^{12}\\
&-3.4821305\times10^{-13}t^{13}-4.3341006\times10^{-14}t^{14}+1.4164151\times10^{-15}t^{15}\\
&+9.1871062\times10^{-16}t^{16}+7.6179553\times10^{-17}t^{17}-7.7347267\times10^{-18}t^{18}\\
&-2.1945563\times10^{-18}t^{19}-0.0019500000t^{2}-1.0808044\times10^{-19}t^{20}\\
&+2.6431071\times10^{-20}t^{21}+4.8412132\times10^{-21}t^{22}-0.00018050000t^{3}\\
&-0.0000062150000t^{4}+0.0000036463838t^{5}+0.00000040283933t^{6}\\
&-0.000000023697947t^{7}-9.5820118e(-9)t^{8}-5.5218244e(-10)t^{9}
\end{aligned} \tag{5.32}$$

In order to enlarge the domain of convergence, we apply a Padé approximant [39] to (5.32) to obtain

$$\begin{aligned}
S(t)_{[4/5]} &= (0.8-0.17015079t+0.021984570t^{2}-0.0013295812t^{3}\\
&+0.000053690243t^{4})/\Delta_1,\\
\Delta_1 &= 1.0-0.11268849t+0.020586863t^{2}-0.00033067737t^{3}\\
&+0.000043277863t^{4}+0.0000035916421t^{5},\\
D(t)_{[10/9]} &= (0.2+0.0028449310t+0.0040329438t^{2}+0.00031975202t^{3}\\
&+0.000046583166t^{4}+0.0000036777852t^{5}+3.5289612e(-7)t^{6}\\
&+2.1414221e(-8)t^{7}+1.2453839e(-9)t^{8}+4.2401556\times10^{-11}t^{9}\\
&+1.2588427\times10^{-12}t^{10})/\Delta_2,\\
\Delta_2 &= 1-0.23577534t+0.051858554t^{2}-0.0049060005t^{3}+0.00043592759t^{4}\\
&-0.0000044755687t^{5}-3.6898248e(-7)t^{6}+1.3533779e(-7)t^{7}\\
&-3.5630121e(-9)t^{8}+1.9930613e(-10)t^{9},\\
R(t)_{[10/9]} &= (0.029999999t-0.0073614826t^{2}+0.0013678991t^{3}-0.00011484554t^{4}\\
&+0.0000075028330t^{5}+0.000000042547736t^{6}-0.000000011999280t^{7}\\
&+0.0000000015887830t^{8}-8.9292275\times10^{-12}t^{9}+7.1307464\times10^{-14}t^{10})/\Delta_3\\
\Delta_3 &= 0.99999998-0.18038276t+0.039888422t^{2}-0.0021135736t^{3}+0.00019379207t^{4}\\
&+0.000018058474t^{5}-0.00000013423095t^{6}+0.000000091219720t^{7}\\
&+0.0000000056923577t^{8}+0.00000000010508626t^{9}
\end{aligned} \tag{5.33}$$

Table 1: Comparison of the 15^{th} order HPM solution and 15^{th} order HAM solution at different values of h with numerical solution for $S(t)$ in the general model.

t	**HPM**	**HAM** $h=-1$	**HAM** $h=-0.90$	**HAM** $h=-0.85$	**Numerical solution**
1	0.715745	0.715745	0.715745	0.715745	0.7157444
2	0.623569	0.623569	0.623569	0.623569	0.623567
3	0.529610	0.52961	0.529618	0.529619	0.529618
4	0.441779	0.441779	0.442554	0.44257	0.44259
5	0.342903	0.342903	0.368402	0.36853	0.369169
6	-0.136843	-0.136843	0.300578	0.30393	0.312008

Table 2: Comparison of the 15^{th} order HPM solution and 15^{th} order HAM solution at different values of h with numerical solution for $D(t)$ in the general model.

t	**HPM**	**HAM** $h=-1$	**HAM** $h=-0.90$	**HAM** $h=-0.85$	**Numerical solution**
1	0.261949	0.261949	0.261949	0.261949	0.2619483
2	0.339007	0.339007	0.339007	0.339007	0.339005
3	0.419726	0.419726	0.419717	0.419716	0.419715
4	0.495327	0.495327	0.494444	0.494429	0.494422
5	0.586570	0.58657	0.557551	0.557407	0.55679
6	1.113703	1.1137	0.616948	0.612426	0.66312

Table 3: Comparison of the 15-th order HPM solution and 15-th order HAM solution at different values of h with numerical solution for $R(t)$ in the general model.

t	**HPM**	**HAM** $h=-1$	**HAM** $h=-0.90$	**HAM** $h=-0.85$	**Numerical solution**
1	0.0223056	0.0223056	0.0223056	0.0223056	0.023073
2	0.0374244	0.0374244	0.0374244	0.0374244	0.0374276
3	0.0506638	0.0506638	0.050665	0.050665	0.0506665
4	0.0628935	0.0628935	0.0630024	0.063006	0.0629875
5	0.0705263	0.0705263	0.0740475	0.074063	0.0740425
6	0.0231399	0.0231398	0.0824747	0.0836414	0.083274

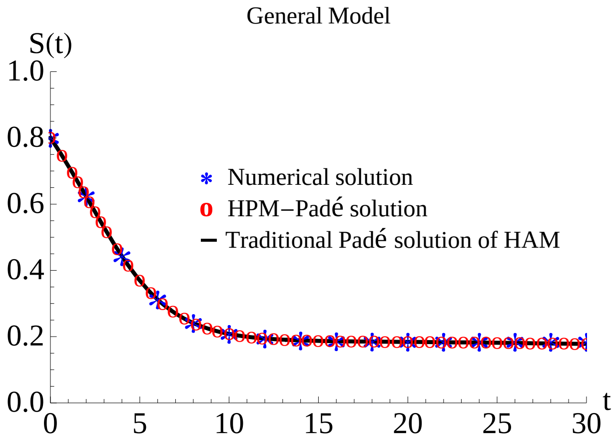

Figure 1: Solution comparison of $S(t)$ for the general model. Star : Numerical solution; Hollow circle : HPM-Padé solution (5.26); Solid line : Traditional Padé solution (5.47) of HAM (5.46).

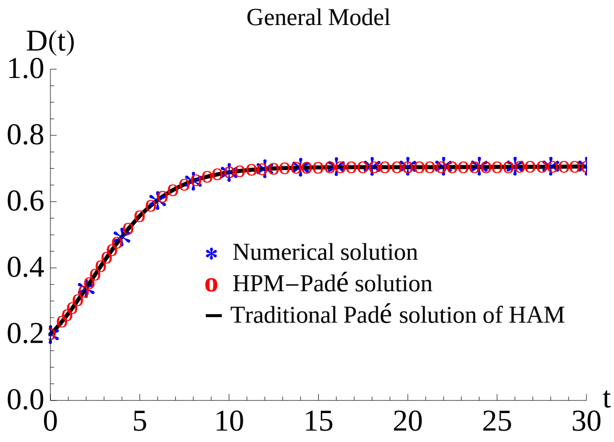

Figure 2: Solution comparison of $D(t)$ for the general model. Star : Numerical solution; Hollow circle : HPM-Padé solution (5.26); Solid line : Traditional Padé solution (5.47) of HAM (5.46).

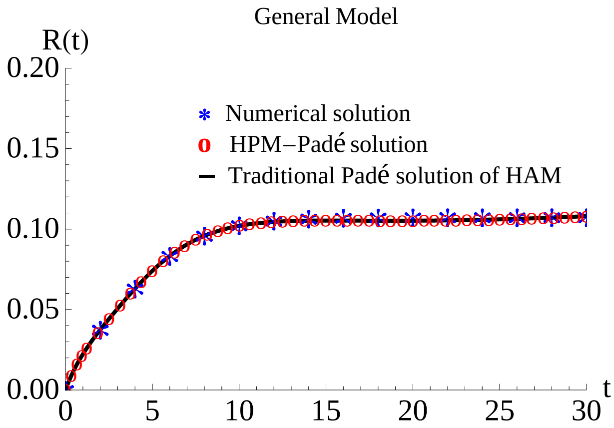

Figure 3: Solution comparison of $R(t)$ for the general model. Star : Numerical solution; Hollow circle : HPM-Padé solution (5.26); Solid line : Traditional Padé solution (5.47) of HAM (5.46).

5.3 HAM and Padé solution of the general model with vital dynamics

To solve (2.1) by HAM, we choose the initial guesses as

$$v_{1,0} = r_1, v_{2,0} = r_2, v_{3,0} = r_3$$

and the auxiliary linear operator

$$L[\phi_i(t;q)] = \frac{\partial[\phi_i(t;p)]}{\partial t}, \quad i = 1,2,3 \tag{5.34}$$

with the property

$$L[c_i] = 0 \tag{5.35}$$

where the c_i's are the constants of integration. The nonlinear operators for (2.1) are

$$\begin{aligned} N_1 &= \phi_1'(t;p) - \mu N + \frac{\beta}{N}\phi_1(t,p)\phi_2(t,p) + \mu\phi_1(t,p) \\ N_2 &= \phi_2'(t;p) - \frac{\beta}{N}\phi_1(t,p)\phi_2(t,p) + \gamma\phi_2(t,p) - \delta\phi_3(t,p) + \mu\phi_2(t,p) \\ N_3 &= \phi_3'(t;p) - \gamma\phi_2(t,p) + \delta\phi_3(t,p) + \mu\phi_3(t,p) \end{aligned} \tag{5.36}$$

The so-called zero order deformation equations are

$$\begin{aligned} (1-p)L[\phi_1(t;p) - v_{1,0}(t)] &= ph_1N_1 \\ (1-p)L[\phi_2(t;p) - v_{2,0}(t)] &= ph_2N_2 \\ (1-p)L[\phi_3(t;p) - v_{3,0}(t)] &= ph_3N_3 \end{aligned} \tag{5.37}$$

The Taylor series expansion gives

$$\begin{aligned} \phi_1(t;p) = v_1(t) &= \sum_{m=0}^{\infty} v_{1,m}(t)p^m \\ \phi_2(t;p) = v_2(t) &= \sum_{m=0}^{\infty} v_{2,m}(t)p^m \\ \phi_3(t;p) = v_3(t) &= \sum_{m=0}^{\infty} v_{3,m}(t)p^m \end{aligned} \tag{5.38}$$

where

$$v_{1,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_1(t;p)}{\partial p^m}|p=0$$
$$v_{2,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_2(t;p)}{\partial p^m}|p=0$$
$$v_{3,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_3(t;p)}{\partial p^m}|p=0 \tag{5.39}$$

The m-th order deformation equations take the following form

$$L[v_{1,m}(t) - \chi_m v_{1,0}(t)] = h_1 R_{1,m}$$
$$L[v_{2,m}(t) - \chi_m v_{2,0}(t)] = h_2 R_{2,m}$$
$$L[v_{3,m}(t) - \chi_m v_{3,0}(t)] = h_3 R_{3,m} \tag{5.40}$$

with the initial conditions

$$v_{1,m}(0) = 0, v_{2,m}(0) = 0, v_{3,m}(0) = 0 \quad for \quad m = 1,2,3,\dots. \tag{5.41}$$

where

$$R_{1,m} = \dot{v}_{1,m-1} - \mu N(1-\chi_m) + \frac{\beta}{N}\sum_{k=0}^{m-1} v_{1,k}v_{2,m-k-1} + \mu v_{2,m-1} \tag{5.42}$$

$$R_{2,m} = \dot{v}_{2,m-1} - \frac{\beta}{N}\sum_{k=0}^{m-1} v_{1,k}v_{2,m-k-1} + \gamma v_{2,m-1} - \delta v_{3,m-1} + \mu v_{2,m-1} \tag{5.43}$$

$$R_{3,m} = \dot{v}_{3,m-1} - \gamma v_{2,m-1} + \delta v_{3,m-1} + \mu v_{3,m-1} \tag{5.44}$$

For our convenience and for comparing HPM solutions we chose $h_1 = h_2 = h_3 = h$. After obtaining the terms like, $v_{i,j}$ for $i, j = 1,2,3$ iteratively, and choosing the proper value of h, the approximate solution of (2.1) from (5.38) at $p = 1$ becomes

$$S(t) = v_1(t) = \sum_{m=0}^{\infty} v_{1,m}(t)$$
$$D(t) = v_2(t) = \sum_{m=0}^{\infty} v_{2,m}(t)$$
$$R(t) = v_3(t) = \sum_{m=0}^{\infty} v_{3,m}(t) \tag{5.45}$$

The 15-th order HAM solutions for $r_1 = 0.8, r_2 = 0.2, r_3 = 0$ at h=-1 are

$$\begin{aligned}
S(t) &= 0.8 - 0.08t - 0.0035t^2 - 0.0013375t^3 - 4.2713710807569734 \times 10^{-6}t^7 \\
&+1.563263021304604 \times 10^{-11}t^{14} - 2.1791126834337404 \times 10^{-12}t^{15} \\
&+6.784559399365888 \times 10^{-7}t^8 - 1.0372017765965167 \times 10^{-7}t^9 \\
&+2.0434121936423668 \times 10^{-8}t^{10} - 3.618776715186458 \times 10^{-9}t^{11} \\
&+5.660821255904755 \times 10^{-10}t^{12} - 9.587262294999274 \times 10^{-11}t^{13} \\
&+0.000667016t^4 - 0.0000993071t^5 + 0.000018519t^6 \\
D(t) &= 0.2 + 0.05t + 0.0146t^2 - 0.0030555t^3 - 2.7872978682629994 \times 10^{-6}t^7 \\
&-0.000154704t^5 + 0.0000272604t^6 + 2.473161775345941 \times 10^{-7}t^8 \\
&-2.2366915544193404 \times 10^{-9}t^9 - 9.9108415109143 \times 10^{-9}t^{10} \\
&+2.806829315675804 \times 10^{-9}t^{11} - 5.34181831576795 \times 10^{-10}t^{12} \\
&+9.96069293856062 \times 10^{-11}t^{13} - 1.696391612583877 \times 10^{-11}t^{14} \\
&+2.4366167149364483 \times 10^{-12}t^{15} + 0.000534833t^4 \\
R(t) &= 0.03t - 0.0111t^2 + 0.004393t^3 + 7.058668949020095 \times 10^{-6}t^7 \\
&+0.000254011t^5 - 0.0000457794t^6 - 9.257721174711655 \times 10^{-7}t^8 \\
&+1.0595686921407175 \times 10^{-7}t^9 - 1.0523280425509374 \times 10^{-8}t^{10} \\
&+8.119473995106502 \times 10^{-10}t^{11} - 3.1900294013681066 \times 10^{-11}t^{12} \\
&-3.734306435613461 \times 10^{-12}t^{13} + 1.331285912792732 \times 10^{-12}t^{14} \\
&-2.575040315027081 \times 10^{-13}t^{15} - 0.00120185t^4
\end{aligned} \tag{5.46}$$

In order to enlarge the domain of convergence, we apply the traditional Padé approximation as discussed in [37] to (5.46) and we obtain

$$\begin{aligned}
S(t)_{[7/7]} &= (0.8 + 0.357189t + 0.0598577t^2 + 0.00348296t^3 \\
&+8.430430473128923 \times 10^{-6}t^6 + 1.1248717247620005 \times 10^{-7}t^7 \\
&+0.000168591t^4 + 0.0000647113t^5)/\Delta_1 \\
\Delta_1 &= 1 + 0.546487t + 0.133846t^2 + 0.021801t^3 + 0.00305631t^4 \\
&+0.000374163t^5 + 0.0000308664t^6 + 1.1004417259586521 \times 10^{-6}t^7 \\
D(t)_{[6/6]} &= (0.2 + 0.121213t + 0.0465995t^2 + 0.00789114t^3 + 0.00133571t^4 \\
&+0.0000856205t^5 + 8.58781964641849 \times 10^{-6}t^6)/\Delta_2 \\
\Delta_2 &= 1 + 0.356066t + 0.070981t^2 + 0.0109951t^3 \\
&+0.0015138t^4 + 0.00015276t^5 + 0.0000115274t^6 \\
R(t)_{[7/7]} &= (0.03t + 0.000508786t^2 + 0.00226059t^3 \\
&+2.330624353126965 \times 10^{-6}t^4 + 0.0000332191t^5 \\
&-1.6600051849898558 \times 10^{-7}t^6 + 5.291878945958253 \times 10^{-8}t^7)/\Delta_3 \\
\Delta_3 &= 1 + 0.38696t + 0.0720948t^2 + 0.0101506t^3 + 0.00134114t^4 + 0.000142117t^5 \\
&+9.383897597403222 \times 10^{-6}t^6 + 2.711709632451674 \times 10^{-7}t^7
\end{aligned} \tag{5.47}$$

5.4 HAM and Padé solution of the nonlinear relapse model

To solve the nonlinear relapse model (2.2) by HAM we follow the above methodology in the same manner. The only difference is in writing $R_{i,m}$ for $i = 2,3$ in equation (5.40), we note that the form of $R_{1,m}$ will be same because the first equation of both the models is same. The other two will take the following forms:

$$R_{2,m} = \dot{v}_{2,m-1} - \frac{\beta}{N}\sum_{k=0}^{m-1} v_{1,k}v_{2,m-k-1} + \gamma v_{2,m-1} - \frac{\beta'}{N}\sum_{k=0}^{m-1} v_{2,k}v_{3,m-k-1} + \mu v_{2,m-1} \tag{5.48}$$

$$R_{3,m} = \dot{v}_{3,m-1} - \gamma v_{2,m-1} + \frac{\beta'}{N}\sum_{k=0}^{m-1} v_{2,k}v_{3,m-k-1} + \mu v_{3,m-1} \tag{5.49}$$

The 22-th order HAM solutions for $r_1 = 0.8, r_2 = 0.2, r_3 = 0$ at h=-1 are

$$\begin{aligned}
S(t) = {} & 0.8 - 0.08t - 0.0035t^2 + 0.0001875t^3 + 5.2398281249997625\times10^{-6}t^5 \\
& -6.750143098958682\times10^{-7}t^6 - 1.5111073196615075\times10^{-7}t^7 \\
& -5.785858983614109\times10^{-9}t^8 + 1.700764155992725\times10^{-9}t^9 \\
& +2.740214576340383\times10^{-10}t^{10} + 3.0923756445350815\times10^{-12}t^{11} \\
& -3.989524219597993\times10^{-12}t^{12} - 4.858614428134729\times10^{-13}t^{13} \\
& +1.0399438379508726\times10^{-14}t^{14} + 8.942420988269827\times10^{-15}t^{15} \\
& +8.18573476958723\times10^{-16}t^{16} - 5.421598384711083\times10^{-17}t^{17} \\
& -1.928547609199037\times10^{-17}t^{18} - 1.254844529304149\times10^{-18}t^{19} \\
& +1.7553440045430126\times10^{-19}t^{20} + 4.003564350150486\times10^{-20}t^{21} \\
& +1.5742028484110186\times10^{-21}t^{22} + 0.0000857969t^4 \\[2ex]
D(t) = {} & 0.2 + 0.05t + 0.00545t^2 - 6.999999999994532\times10^{-6}t^3 - 0.0000795819t^4 \\
& -8.886211874999768\times10^{-6}t^5 + 2.721749526041962\times10^{-7}t^6 \\
& +1.7480868130617826\times10^{-7}t^7 + 1.5367870648414197\times10^{-8}t^8 \\
& -1.148581613644493\times10^{-9}t^9 - 3.8627579879315356\times10^{-10}t^{10} \\
& -2.472111020960178\times10^{-11}t^{11} + 3.672136338841118\times10^{-12}t^{12} \\
& +8.340745111426262\times10^{-13}t^{13} + 3.29415693132888\times10^{-14}t^{14} \\
& -1.035883600994146\times10^{-14}t^{15} - 1.7372841493496612\times10^{-15}t^{16} \\
& -2.196356775431531\times10^{-17}t^{17} + 2.702020334442286\times10^{-17}t^{18} \\
& +3.4494009029824233\times10^{-18}t^{19} - 6.74539425853283\times10^{-20}t^{20} \\
& -6.646671455177506\times10^{-20}t^{21} - 6.415416401855684\times10^{-21}t^{22} \\[2ex]
R(t) = {} & 0.03t - 0.00195t^2 - 0.0001805t^3 - 6.214999999999492\times10^{-6}t^4 \\
& +3.6463837500000373\times10^{-6}t^5 + 4.028393572916645\times10^{-7}t^6 \\
& -2.3697949340030635\times10^{-8}t^7 - 9.582011664800624\times10^{-9}t^8 \\
& -5.521825423482899\times10^{-10}t^9 + 1.1225434115911227\times10^{-10}t^{10} \\
& +2.162873456506671\times10^{-11}t^{11} + 3.173878807568876\times10^{-13}t^{12} \\
& -3.482130683291528\times10^{-13}t^{13} - 4.3341007692797536\times10^{-14}t^{14} \\
& +1.4164150216716325\times10^{-15}t^{15} + 9.187106723909384\times10^{-16}t^{16} \\
& +7.617955160142613\times10^{-17}t^{17} - 7.73472725243249\times10^{-18}t^{18} \\
& -2.194556373678274\times10^{-18}t^{19} - 1.0808045786897294\times10^{-19}t^{20} \\
& +2.643107105027019\times10^{-20}t^{21} + 4.841213553444665\times10^{-21}t^{22}
\end{aligned} \tag{5.50}$$

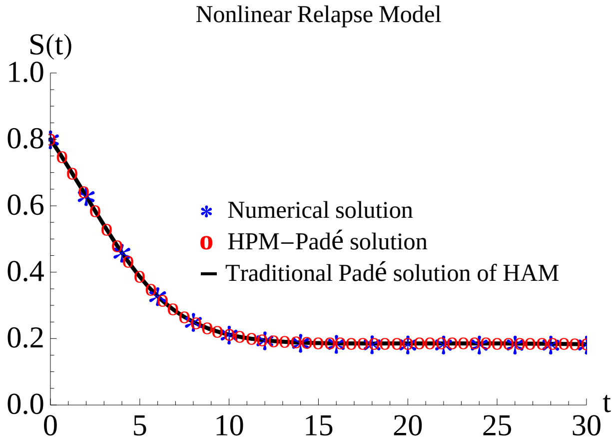

Figure 4: Solution comparison of $S(t)$ for the nonlinear relapse model. Star : Numerical solution; Hollow circle : HPM-Padé solution (5.33); Solid line : Traditional Padé solution (5.51) of HAM (5.50).

In order to enlarge the domain of convergence, we apply the traditional Padé approximation as discussed in [37] to (5.50) and we obtain

$$
\begin{aligned}
&S(t)_{[4/5]} = (0.8-0.170151t+0.0219846t^2-0.00132958t^3+0.0000536904t^4)/\Delta_1\\
&\Delta_1 = 1-0.112689t+0.0205869t^2-0.00033068t^3\\
&\quad +0.000043278t^4+3.591649091528135\times10^{-6}t^5\\
&D(t)_{[10/9]} = (0.2+0.00330628t+0.00402799t^2+0.000325755t^3+0.0000469916t^4\\
&\quad +3.725562951283311\times10^{-6}t^5+3.551243483027385\times10^{-7}t^6\\
&\quad +2.163706551963901\times10^{-8}t^7+1.2565058185451715\times10^{-9}t^8\\
&\quad +4.2986855603044676\times10^{-11}t^9+1.2644123909850545\times10^{-12}t^{10})/\Delta_2\\
&\Delta_2 = 1-0.233469t+0.0512571t^2-0.00478848t^3+0.00042506t^4\\
&\quad -3.825413324149654\times10^{-6}t^5-3.569627150084984\times10^{-7}t^6\\
&\quad +1.3224956737037204\times10^{-7}t^7-3.340530164470739\times10^{-9}t^8\\
&\quad +1.9555452286579906\times10^{-10}t\\
&R(t)_{[10/9]} = (0.03t-0.00734909t^2+0.00136554t^3-0.000114408t^4\\
&\quad +7.475110911686804\times10^{-6}t^5+4.446283222472908\times10^{-8}t^6\\
&\quad -1.1886354575465256\times10^{-8}t^7+1.5791958584759088\times10^{-9}t^8\\
&\quad -7.863021455141622\times10^{-12}t^9+5.141982981661828\times10^{-14}t^{10})/\Delta_3\\
&\Delta_3 = 1-0.17997t+0.0398367t^2-0.00209987t^3+0.000193533t^4+0.000018127t^5\\
&\quad -1.2399205814711002\times10^{-7}t^6+9.127947348697485\times10^{-8}t^7\\
&\quad +5.746144759642487\times10^{-9}t^8+1.0745495835153733\times10^{-10}t^9
\end{aligned}
\tag{5.51}
$$

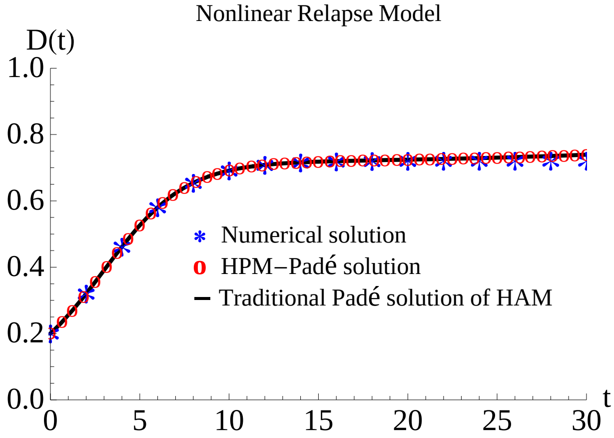

Figure 5: Solution comparison of $D(t)$ for the nonlinear relapse model. Star : Numerical solution; Hollow circle : HPM-Padé solution (5.33); Solid line : Traditional Padé solution (5.51) of HAM (5.50).

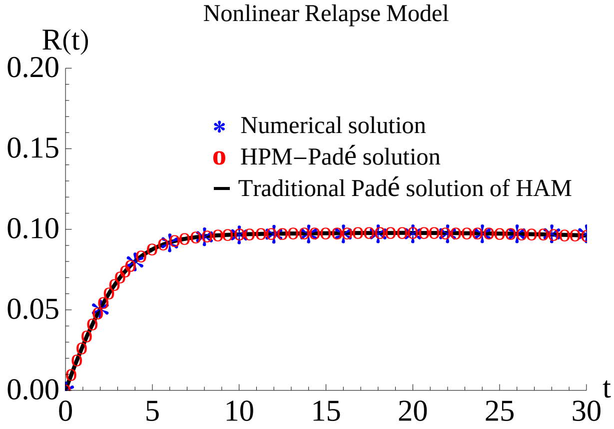

Figure 6: Solution comparison of $R(t)$ for the nonlinear relapse model. Star : Numerical solution; Hollow circle : HPM-Padé solution (5.33); Solid line : Traditional Padé solution (5.51) of HAM (5.50).

6 Discussion

Tables [*5.2-5.2*] compares the solutions of the general model obtained by HPM (Eq.(5.25)), HAM (Eq.(5.45)) at different values of h with numerical techniques. The numerical solutions are developed by matlab solver namely, ode45. This solver implements a Runge-Kutta method with a variable time step for efficient computation. The relative and absolute errors for nemerical solutions are of order 10^{-6} (default values). The HPM solutions and HAM solutions at $h=-1$ are not matching the numerical solutions even at $t=6$. Furthermore, we found similar behaviour for the nonlinear relapse model solutions (5.32) and (5.50). However, in HAM, we have freedom to choose h values different from -1. We obtain the region of convergence for HAM solutions as discussed by Liao in [37] as $-1.5 \leq h \leq -0.4$, approximately, for both models. We can choose any value of h in this region to increase the convergence of the solution series. For this purpose we write HAM solutions (5.45) for the general model at $h=-0.90$ and $h=-0.85$ in Table [*5.2-5.2*]. In the case of the general model we keep the order of approximation in both the techniques at 15 and in the case of the nonlinear relapse model we keep the order at 22.
To enhance the region of convergence of the solution series of models (2.1) and (2.2), we use the Padé approximation for both techniques. On one side, the obtained HPM-Padé solutions (5.26) and (5.33) exhibit a good agreement with numerical results as depicted in figures 1-6. On the other side, for our convenience, we use $h=-1$ for developing the traditional Padé approximations of HAM solutions, but obviously we have the freedom to choose other h values too. Eq. (5.47) and (5.51) give the Padé approximations for (5.46) and (5.50) respectively.
The accuracy at $h=-1$ either by HAM or by HPM will be the same and it will take the same CPU time for computation. But to obtain much accurate results by HAM especially when we search for optimal h [15, 29, 38], then the optimal procedure of HAM will take more CPU time in comparison to HPM or standard HAM procedures.
Based on the numerical results of this work, we can state that the Padé aftertreatment of power series solutions, obtained by using HPM and HAM methods, is a tool that can achieve a remarkable enlargement of the region of convergence of solutions.

7 Conclusions

It is clear from Table [*5.2-5.2*] that the HPM solutions are the same as the HAM solutions at $h=-1$. A proper choice of h increases the convergence of solution series in HAM as is clear from Table [*5.2-5.2*]. For example at $h=-0.90$ and $h=-0.85$, HAM solutions match better than HPM solutions in comparison with the numerical solution.
The Padé approximation greatly increases the convergence region of the solution series for both techniques as depicted in Fig. (1-6). For the general model Fig. (1-3) and for the nonlinear relapse model Figure (4-6) show excellent agreement with the numerical solutions. Finally, we conclude that both HPM and HAM are useful in solving nonlinear systems but HAM is more general than HPM. Nonetheless, HPM method can be easily applied to all kind of nonlinear problems, converting this technique into an attractive mathematical tool for engineers. We also note that using the Padé approximations, the region of agreement with the numerical solutions is remarkable enhanced for both HPM and HAM. For developing the HPM and HAM solutions, we have used Maple symbolic mathematics software and Mathematica numerical mathematics software, respectively.

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References

[1] J. Biazar, Solution of the epidemic model by Adomian decomposition method, Appl. Math. Comput, 173 (2006) 1101-1106.
http://dx.doi.org/10.1016/j.amc.2005.04.036

[2] M. Rafei, D. D. Ganji, H. Daniali, Solution of the epidemic model by homotopy perturbation method, Applied Mathematics and Computation, 187 (2007) 1056-1062.
http://dx.doi.org/10.1016/j.amc.2006.09.019

[3] F. Awawdeh, A. Adawi, Z. Mustafa, Solutions of the SIR models of epidemics using HAM, Chaos Solitons & Fractals, 42 (2009) 3047-3052.
http://dx.doi.org/10.1016/j.chaos.2009.04.012

[4] H. Khan, R. N. Mohapatra, K. Vajravelu, S. J. Liao, The explicit series solution of SIR and SIS epidemic models, Appl. Math. Comput, 215 (2009) 653-669.
http://dx.doi.org/10.1016/j.amc.2009.05.051

[5] M. A. Fariborzi Araghi, M. Sotoodeh, An enhanced modified homotopy perturbation method for solving nonlinear volterra and fredholm integro-differential equation, World Applied Sciences Journal, 20 (2012) 1646-1655.

[6] M. A. Fariborzi Araghi, B. Rezapour, Application of homotopy perturbation method to solve multidimensional schrodinger's equations, International Journal of Mathematical Archive (IJMA), ISSN 2229-5046, 2 (2011) 1-6.

[7] J. Biazar, H. Ghazvini, Exact solutions for non-linear schrodinger equations by he's homotopy perturbation method, Physics Letters A, 366 (2007) 79-84.
http://dx.doi.org/10.1016/j.physleta.2007.01.060

[8] J. Biazar, H. Ghazvini, M. Eslami, He's homotopy perturbation method for systems of integro-differential equations, Chaos Solitons & Fractals, 39 (2009) 1253-1258.
http://dx.doi.org/10.1016/j.chaos.2007.06.001

[9] Y. Khan, H. Vazquez-Leal, L. Hernandez-Martinez, Removal of noise oscillation term appearing in the nonlinear equation solution, Journal of Applied Mathematics, (2012) 1-9.
http://dx.doi.org/10.1155/2012/387365

[10] U. Filobello-Nino, H. Vazquez-Leal, Y. Khan, R. Castaneda-Sheissa, A. Yildirim, L. Hernandez-Martinez, J. Sanchez-Orea, R. Castaneda-Sheissa, F. R. Bernal, HPM applied to solve nonlinear circuits: a study case, Appl. Math. Sci, 6 (2012) 4331-4344.

[11] U. Filobello-Nino, H. Vazquez-Leal, R. Castaneda-Sheissa, A. Yildirim, L. Hernandez-Martinez, D. Pereyra-Diaz, A. Perez-Sesma, C. Hoyos-Reyes, An approximate solution of blasius equation by using hpm method, Asian Journal of Mathematics and Statistics, 5 (2012) 50-59.
http://dx.doi.org/10.3923/ajms.2012.50.59

[12] S. Abbasbandy, The application of the homotopy analysis method to nonlinear equations arising in heat transfer, Physics Letters A, 360 (2006) 109-113.
http://dx.doi.org/10.1016/j.physleta.2006.07.065

[13] X. Hang, S. J. Liao, I. Pop, Series solution of unsteady boundary layer flows of non-newtonian fluid near a forward stagnation point, J. Non-newtonian Fluid Mech, 139 (2006) 31-43.
http://dx.doi.org/10.1016/j.jnnfm.2006.06.003

[14] M. Sajid, M. Awais, S. Nadeem, T. Hayat, The influence of slip condition on thin film flow of a fourth grade fluid by the homotopy analysis method, Computers and Mathematics with Applications, 56 (2008) 2019-2026.
http://dx.doi.org/10.1016/j.camwa.2008.04.022

[15] R. Muthuraj, S. Srinivas, A. K. Shukla, T. R. Ramamohan, Effects of thermal-diffusion, diffusion-thermo, and space porosity on MHD mixed convective flow of micropolar fluid in a vertical channel with viscous dissipation, Heat Transfer–Asian Research, 43 (6) (2014) 561576
http://dx.doi.org/10.1002/htj.21100

[16] Mahdi Bayat, Mahmoud Bayat, Iman Pakar, Nonlinear vibration of an electrostatically actuated microbeam, Latin American Journal of Solids and Structures, 11 (2014) 534-544.
http://dx.doi.org/10.1590/S1679-78252014000300009

[17] M. Bayat, I. Pakar, A. Emadi, Vibration of electrostatically actuated microbeam by means of homotopy perturbation method, Structural Engineering and Mechanics, 48 (2013) 823-831.
http://dx.doi.org/10.12989/sem.2013.48.6.823

[18] S. Xinhui, Z. Liancun, Z. Xinxin, S. Xinyi, Homotopy analysis method for the asymmetric laminar flow and heat transfer of viscous fluid between contracting rotating disks, Applied Mathematical Modelling, 36 (2012) 1806-1820.
http://dx.doi.org/10.1016/j.apm.2011.09.010

[19] B. Raftari, K. Vajravelu, Homotopy analysis method for MHD viscoelastic fluid flow and heat transfer in a channel with a stretching wall, Commun Nonlinear Sci Numer Simulat, 17 (2012) 4149-4162.
http://dx.doi.org/10.1016/j.cnsns.2012.01.032

[20] H. Vazquez-Leal, Y. Khan, Guillermo Fernandez-Anaya, A. Herrera-May, A. Sarmiento-Reyes, U. Filobello-Nino, V. M. Jimenez-Fernandez, D. Pereyra-Diaz, A General Solution for Troesch's Problem, Mathematical Problems in Engineering, 2012 (2012) 1-14.
http://dx.doi.org/10.1155/2012/208375

[21] H. Vazquez-Leal, A. Sarmiento-Reyes, Y. Khan, U. Filobello-Nino, A. Diaz-Sanchez, Rational Biparameter Homotopy Perturbation Method and Laplace-Padé Coupled Version, Journal of Applied Mathematics, 2012 (2012) 1-21.
http://dx.doi.org/10.1155/2012/923975

[22] M. Goodarzi, M. M. Rashidi, A. B. Parsa, Analytical and numerical nolutions of vapor flow in a flat plate heat pipe, Walailak J Sci & Tech, 9 (2012) 65-81.

[23] H. Vazquez-Leal, Rational homotopy perturbation method, Journal of Applied Mathematics, 2012 (2012) 1-14.
http://dx.doi.org/10.1155/2012/490342

[24] H. Vazquez-Leal, Generalized Homotopy Method for Solving Nonlinear Differential Equations, Computational and Applied Mathematics, 23 (2014) 275-288.
http://dx.doi.org/10.1007/s40314-013-0060-4

[25] M. M. Rashidi, M. T. Rastegari, M. Asadi, O. Anwar Beg, A study of non-newtonian flow and heat transfer over a non-isothermal wedge using the homotopy analysis method, Chemical Engineering Communications, 199 (2012) 231-256.
http://dx.doi.org/10.1080/00986445.2011.586756

[26] M. M. Rashidi, S. A. Mohimanian Pour, T. Hayat, S. Obaidat, Analytic approximate solutions for steady flow over a rotating disk in porous medium with heat transfer by homotopy analysis method, Computers & Fluids, 54 (2012) 1-9.
http://dx.doi.org/10.1016/j.compfluid.2011.08.001

[27] A. B. Parsa, M. M. Rashidi, O. Anwar Beg, S. M. Sadri, Semi-computational simulation of magneto-hemodynamic flow in a semi-porous channel using optimal homotopy and differential transform methods, Comp. in Bio. and Med, 43 (2013) 1142-1153.
http://dx.doi.org/10.1016/j.compbiomed.2013.05.019

[28] R. J. Moitsheki, M. M. Rashidi, A. Basiriparsa, A. Mortezaei, Analytical Solution and Numerical Simulation for One-Dimensional Steady Nonlinear Heat Conduction in a Longitudinal Radial Fin with Various Profiles, Heat Trans. Asian Res, (2013).
http://dx.doi.org/10.1002/htj.21104

[29] A. K. Shukla, T. R. Ramamohan, S. Srinivas, A new analytical approach for limit cycles and quasi-periodic solutions of nonlinear oscillators: The example of the forced Van der Pol Duffing oscillator, Physica Scripta, 89 (2014), art. no. 075202.
http://dx.doi.org/10.1088/0031-8949/89/7/075202

[30] Carlos Castillo-Garsow, Guarionex Jordan-Salivia, Ariel Rodriguez-Herrera, Mathematical Models for the Dynamics of Tobacco Use, Recovery, and Relapse, Mathematical and theoretical biology institute, Arizona State University, BU-1505-M, (1997) 17-44.

[31] J. H. He, Comparison of homotopy perturbation method and homotopy analysis method, Applied Mathematics and Computation, 156 (2004) 527-539.
http://dx.doi.org/10.1016/j.amc.2003.08.008

[32] J. H. He, An elementary introduction to the homotopy perturbation method, Computers and Mathematics with Applications, 57 (2009) 410-412.
http://dx.doi.org/10.1016/j.camwa.2008.06.003

[33] H. Vazquez-Leal, A. Sarmiento-Reyes, Y. Khan, U. Filobello-Nino, A. Diaz-Sanchez, Rational biparameter homotopy perturbation method and laplace-padé coupled version, Journal of Applied Mathematics, 2012 (2012), Article ID 923975, 21 pages.
http://dx.doi.org/10.1155/2012/923975

[34] H. Vazquez-Leal, Y. Khan, G. Fernandez-Anaya, A. Herrera-May, A. Sarmiento-Reyes, U. Filobello-Nino, V. M. Jimenez-Fernandez, D. Pereyra-Diaz, A general solution for troesch's problem, Mathematical Problems in Engineering, 2012 (2012), Article ID 208375, 14 pages.
http://dx.doi.org/10.1155/2012/208375

[35] H. Vazquez-Leal, R. Castaneda-Sheissa, U. Filobello-Nino, A. Sarmiento-Reyes, J. Sanchez-Orea, High Accurate Simple Approximation of Normal Distribution Integral, Mathematical Problems in Engineering, 2012 (2012) 22 pages.
http://dx.doi.org/10.1155/2012/124029

[36] H. Vazquez-Leal, U. Filobello-Nino, R. Castaneda-Sheissa, L. Hernandez-Martinez, A. Sarmiento-Reyes, Modified HPMs inspired by homotopy continuation methods, Mathematical Problems in Engineering, 2012 (2012) 19 pages.
http://dx.doi.org/10.1155/2012/309123

[37] S. Liao, Beyond Perturbation: An Introduction to the Homotopy Analysis Method, CRC Press, Boca Raton:Chapman and Hall, (2003).
http://dx.doi.org/10.1201/9780203491164

[38] S. Liao, Homotopy Analysis Method in Nonlinear Differential Equations, Higher Education Press, Springer (2012).
http://dx.doi.org/10.1007/978-3-642-25132-0

[39] G. Baker, Essentials of padé approximations, Academic Express, London, Klebano, (1975).