A modification of He’s variational iteration method by Taylor’s series for solving second order nonlinear partial differential equations

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Abstract
In this work, a modification of He’s variational iteration method by Taylor’s series is used for finding the solution of the second order nonlinear partial differential equations. This modification expands the application of variational iteration method for those nonlinear equations which have logarithmic or trigonometric nonlinear part. To show the efficiency of the method, several examples are presented.

Keywords: Variational iteration method; Nonlinear PDEs; Taylor series; Correction functional; Lagrange multiplier

1 Introduction

Variational iteration method which a useful instrument for solving linear and nonlinear partial differential equations, first proposed by He [1, 2, 3, 4, 5, 6, 7, 8, 9], and has successfully been applied to many situations such as PDE’s with polynomial nonlinearities [10]. In this work we have tried to make a modification on this method which increases its efficiency. This modified method lets us to solve nonlinear PDEs with logarithmic, trigonometric or exponential type nonlinearities. To illustrate the basic concepts of the variational iteration method (VIM) we consider the following general nonlinear system:

$$Lu(x,t) + Nu(x,t) = g(x,t), \quad (1.1)$$

where \(L\) is a linear operator and \(N\) is a nonlinear operator, and \(g(x,t)\) is the inhomogeneous term. In the variational iteration method, a correction function for Eq.(1.1) can be written as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)(Lu_n(x,\tau) + Nu_n(x,\tau) - g(x,\tau))d\tau. \quad (1.2)$$

It is obvious that the successive approximation \(u_n, n \geq 0\) can be established by determining \(\lambda\), a general Lagrange multiplier, which can be identified optimally via the variational theory [11]. The function \(u_0\) is a restricted variation, which means \(\delta u_0 = 0\). Therefore, we first determine the Lagrange multiplier \(\lambda\) that will be identified via integration by parts. The successive approximations \(u_{n+1}(x,t)\) of the solution \(u(x,t)\) will be readily obtained upon using the Lagrange multiplier obtained by using any selective function \(u_0(x,t)\). A modification of variational iteration method is described during Section 2. To show the efficiency of method Section 3 is devoted to numerical examples.

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2 A modification of He’s variational iteration method by Taylor’s series

In this section, we introduce a new method for solving nonlinear PDEs. We will consider, amongst others, the second order nonlinear partial differential equations of the form [12]:

\[ u_t = f(x, t, u, u_x) + F(u), \quad 0 < x < \alpha, \quad t > 0. \] (2.3)

where \( F \) is a nonlinear part of equation that could be logarithmic, trigonometric or exponential type.

**Theorem 2.1. Taylor’s theorem with Lagrange remainder [13]:** Suppose that \( f \in C^n[a, b] \) and if \( f^{(n+1)} \) exists on \((a, b)\), then for any points \( c \) and \( x \) in \([a, b]\),

\[ f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x - c)^k + E_n(x), \] (2.4)

where, for some points \( \xi \) between \( c \) and \( x \),

\[ E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - c)^{n+1} \] (2.5)

In this method, we use Taylor’s polynomial with remainder instead of a nonlinear part \( F(u) \), as follows:

\[ F(u) = P_r(u) + E_r(u), \] (2.6)

such that

\[ P_r(u) = \sum_{i=0}^{r} m_i u^i \] (2.7)

where

\[ m_i = \frac{F^{(i)}(a)}{i!}. \] (2.8)

As a result, we obtain the following formula

\[ u_t = f(x, t, u, u_x) + P_r(u) + E_r(u), \] (2.9)

by omitting the remainder we consider a new equation with polynomial nonlinearities:

\[ u_t = f(x, t, u, u_x) + P_r(u), \] (2.10)

Therefore by means of (VIM) we get to the following correctional function

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\tau) \left( \frac{\partial}{\partial \tau} u_n(x, \tau) 
- f(x, \tau, (u_n(x, \tau))_x), (u_n(x, \tau))_{xx} 
- P_r(u_n(x, \tau)) \right) d\tau, \] (2.11)

by noting that \( \delta \tilde{u}_n = 0 \) and \( \lambda = -1 \), we have

\[ u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left( \frac{\partial}{\partial \tau} u_n(x, \tau) 
- f(x, \tau, (u_n(x, \tau))_x), (u_n(x, \tau))_{xx} 
- P_r(u_n(x, \tau)) \right) d\tau. \] (2.12)

By using \( u_0(x, t) \) as a selective function, an approximation of \( u(x, t) \) could be obtained which is close to the exact solution.

**Theorem 2.2.** If \( u \) be an analytical function, then the error of the approximating function will be tend to zero by increasing the number of iterations.
Proof. It is straightforward, because by this assumption we have

$$\lim_{r \to \infty} E_r(u) = 0.$$ 

The advantages of using the proposed modified VIM in comparison with the usual VIM is that integration of some nonlinear function which must be done in VIM may be impossible or very hard to do, but it may be easier in this method. Also there is no need that the nonlinear types which were concerned, be just polynomial ones, but they can be more complicated. We used Taylor’s polynomial from degree $r$ instead of nonlinear part. It gives us an acceptable solution, even by considering a small amount of $r$.

3 Numerical examples

In this section, we use the modification of He’s variational iteration method by Taylor’s series to nonlinear partial differential equations. In order to solve these equations, we used the commercial package Mathematica 8. In the examples we used $P_3$ from degree 3. For a fixed index $i$, $g_i$ is the error of obtaining function in the $i$-th iteration.

Example 3.1. We consider the following partial differential equation:

$$u_t = t u - t \ln(u+1) - u_x + u_{xx} + e^t(x+1) + t(\ln x + t - xe^t + 1),$$

$$0 < x < 1, \quad t > 0,$$

subject to the boundary conditions:

$$u(0,t) = -1, \quad u(1,t) = -1 + e^t$$

and the initial condition:

$$u(x,0) = -1 + x.$$  

The exact solution is:

$$u(x,t) = -1 + xe^t.$$  

To solve Eq.(3.13), we use iterative formula (2.12) to find the iteration for Eq.(3.13) given by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left( \frac{\partial u_n(x,\tau)}{\partial \tau} - \tau u_n(x,\tau) + \tau P_3(u_n(x,\tau)) + \frac{\partial}{\partial x} u_n(x,\tau) - \frac{\partial^2}{\partial x^2} u_n(x,\tau) 
- e^\tau (x+1) - \tau (\ln x + \tau - xe^\tau + 1) \right) d\tau.$$  

To get the iteration, we start with an initial approximation $u_0(x,t) = -1 + x$ and we obtain the following successive approximations as follows:

$$u_0(x,t) = -1 + x,$$

$$u_1(x,t) = -1 + xe^t + g_1(x,t),$$

$$u_2(x,t) = -1 + xe^t + g_2(x,t),$$

$$\vdots$$

graphs of $g_1(x,t)$ and $g_2(x,t)$ are shown in Figure 1 and Figure 2 respectively.
Figure 1: $g_1(x,t)$ in $u_t = tu - t \ln(u+1) - u_x + u_{xx} + e^t(x+1) + t(\ln x + t - xe^t + 1)$

Figure 2: $g_2(x,t)$ in $u_t = tu - t \ln(u+1) - u_x + u_{xx} + e^t(x+1) + t(\ln x + t - xe^t + 1)$

Example 3.2. We consider the following partial differential equation:

$$u_t = -2^{u-1} - u_x - xu_{xx} + \frac{1}{2} + x^2 + \frac{1}{2}(2x^2 - 1) + 4xt,$$
subject to the boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = t.$$ (3.20)

and the initial condition:

$$u(x,0) = 0.$$ (3.21)

The exact solution is:

$$u(x,t) = tx^2.$$ (3.22)

To solve Eq.(3.19), we use iterative formula (2.12) to find the iteration for Eq.(3.19) given by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left( \frac{\partial}{\partial \tau} u_n(x,\tau) + Pu_n(x,\tau) - \frac{\partial u_n(x,\tau)}{\partial x} - \frac{1}{2} \right) + x \frac{\partial^2}{\partial \tau^2} u_n(x,\tau) - x^2 - \frac{1}{2}(2x^2 - 1) - 4xt) d\tau.$$ (3.23)
To get the iteration, we start with an initial approximation $u_0(x,t) = 0$ and we obtain the successive approximations as follows:

$$
\begin{align*}
  u_0(x,t) &= 0, \\
  u_1(x,t) &= tx^2 + g_1(x,t), \\
  u_2(x,t) &= tx^2 + g_2(x,t), \\
  &\vdots \\

\end{align*}
$$

(3.24)

graphs of $g_1(x,t)$ and $g_2(x,t)$ are shown in Figure 3 and Figure 4 respectively.

Figure 3: $g_1(x,t)$ in $u_t = -2(u - 1) - u_x - xu_{xx} + \frac{1}{4} + x^2 + \frac{1}{4}(2tx^2 - 1) + 4xt$

Figure 4: $g_2(x,t)$ in $u_t = -2(u - 1) - u_x - xu_{xx} + \frac{1}{4} + x^2 + \frac{1}{4}(2tx^2 - 1) + 4xt$

**Example 3.3.** We consider the following partial differential equation:

$$
\begin{align*}
  u_t &= \cos(xt) - \cos u - \frac{u^2}{2} + x(1 + \frac{u^2}{2}), \\
  &0 < x < 1, \; t > 0,
\end{align*}
$$

(3.25)

subject to the boundary conditions:

$$
\begin{align*}
  u(0,t) &= 0, &u(1,t) &= t,
\end{align*}
$$

(3.26)

and the initial condition:

$$
\begin{align*}
  u(x,0) &= 0.
\end{align*}
$$

(3.27)
The exact solution is:

\[ u(x,t) = tx. \]  

(3.28)

To solve Eq.(3.25), we use iterative formula (2.12) to find the iteration for Eq.(3.25) given by:

\[ u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left( \frac{\partial}{\partial \tau} u_n(x,\tau) - \cos(\tau x) + P_3(u_n(x,\tau)) \right) + \frac{u_n(x,\tau)^2}{2} - x(1 + \frac{x^2}{2}) \ d\tau. \]  

(3.29)

To get the iteration, we start with an initial approximation \( u_0(x,t) = 0 \) and we obtain the successive approximations as follows:

\[ u_0(x,t) = 0, \]
\[ u_1(x,t) = tx + g_1(x,t), \]
\[ u_2(x,t) = tx + g_2(x,t), \]
\[ \vdots \]  

(3.30)

graphs of \( g_1(x,t) \) and \( g_2(x,t) \) are shown in Figure 5 and Figure 6 respectively.

Figure 5: \( g_1(x,t) \) in \( u_t = \cos(\pi x) - \cos u - \frac{u^2}{2} + x(1 + \frac{x^2}{2}) \)

Figure 6: \( g_2(x,t) \) in \( u_t = \cos(\pi x) - \cos u - \frac{u^2}{2} + x(1 + \frac{x^2}{2}) \)
4 Conclusion

In this work, we employed variational iteration method and Taylor’s polynomial with remainder to find a solution for nonlinear partial differential equations. There is no need that the nonlinear types which were concerned, be just polynomial ones, but they can be more complicated. We used Taylor’s polynomial from degree $r$ instead of nonlinear part. It gives us an acceptable solution, even by considering a small amount of $r$. This method could be used as an efficient instrument for many kinds of nonlinear partial differential equations.

References


