A correction note on “A family of optimal iterative methods with fifth and tenth order convergence for solving nonlinear equations”

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Abstract
In M. Matinfar, M. Aminzadeh (2012) [M. Matinfar, M. Aminzadeh, A family of optimal iterative methods with fifth and tenth order convergence for solving nonlinear equations, J. of Interpolation and Approximation in Scientific Computing, volume 2012, year 2012, article id jiasc-00012, 11 pages, doi:10.5899/2012/jiasc-00012.], an iterative method with fifth and tenth order of convergence has been proposed. In this note, we show that the both proofs of proposed iterative schemes are incorrect. The both iterative schemes are quadratically convergent.

Keywords: Nonlinear equations, Iterative methods, Convergence order, Computational order of convergence

1 Introduction
In (2012), M. Matinfar, M. Aminzadeh [1] proposed an iterative scheme but unfortunately the developed iterative scheme does not have claimed convergence-order. The efficiency index \(EI = \frac{p^m}{b}\) where \(p\) is the convergence order of iterative scheme and \(b\) is total number of function evaluations per one complete iteration of iterative method. According to Kung-Traub conjecture[3], a multipoint without memory iterative scheme for solving nonlinear equations has the optimal efficiency index \(2^{(\beta-1)/\beta}\) and optimal rate of convergence \(2^{\beta-1}\). It is stated in [1], the efficiency index for iterative scheme (2.7) on page 3 found to be \(5^{1/3}\) and similarly efficiency index for iterative scheme (2.20) on page 5 is \(10^{1/4}\). Notice that according to [3], if an iterative scheme uses three function iterations its optimal convergence order is \(2^{3-1} = 4\) and for four function evaluations is \(2^{4-1} = 8\). The claimed orders of convergence in [1] are contrary to the Kung-Traub conjecture. Authors in [1], have provided proofs of order of convergence and shown the computational order of convergence in the favor of their claimed orders of convergence. We show that if the both iterative schemes namely (2.12) on page 3 and (2.31) on page 7 converge then both are quadratically convergent. We generate Maple software based proofs to show quadratic convergence for both proposed schemes. We also report the computational order of convergence in the favor of our claim.

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2 Convergence analysis and computational order of convergence

For completeness, we restate the iterative schemes from [1].

\[
\begin{align*}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \frac{f(x_n)^2}{x_n} + f'(x_n), \\
x_{n+1} &= z_n - \frac{\theta_n}{\Psi_f(x_n,y_n,z_n)} + \frac{q}{n} e_n,
\end{align*}
\]  
(2.1)

\[
\begin{align*}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \frac{f(x_n)^2}{x_n} + f'(x_n), \\
x_{n+1} &= z_n - \frac{\theta_n}{\Psi_f(x_n,y_n,z_n)} + \frac{q}{n} e_n,
\end{align*}
\]  
(2.2)

where \( \theta_n = \frac{1}{\Psi_f(x_n,y_n,z_n)} \) and \( \Psi_f(x_n,y_n,z_n) = -\frac{x_n + 2y_n - 3z_n}{(x_n - y_n)(y_n - z_n)} f(z_n) + \frac{(x_n - z_n)^2}{(x_n - y_n)^2(y_n - z_n)} f(y_n) + \frac{y_n - z_n f(x_n)}{(x_n - y_n)f(z_n) - (x_n - z_n)(y_n - z_n)^2 f(x_n)} \).

The iterative schemes (2.1) and (2.2) both are quadratically convergent. In figure 1 and figure 2, a Maple based program shows the following error equations.

\[
e_{n+1} = c_2 e_n^2 + (2c_3 - 2c_2^2 - c_1 c_2^2) e_n^3 + O(e_n^4), \quad \text{For scheme (2.1).}
\]

\[
e_{n+1} = -\frac{1}{2} \left( \frac{3c_2^2 + 1 - 4c_1}{c_1^2} \right) e_n^2 + O(e_n^3), \quad \text{For scheme (2.2).}
\]

**Definition 2.1.** Let \( x_{n-1}, x_n \) and \( x_{n+1} \) be successive iterations in the vicinity of root \( \alpha \) of \( f(x) = 0 \), the computational order of convergence (COC) [2], can be approximated by

\[
\text{COC} \approx \frac{\ln|\alpha - x_n - x_{n-1}|}{\ln|\alpha - x_n - \alpha|^{-1}}.
\]

A set of ten functions [1] is listed in Table 1.

<table>
<thead>
<tr>
<th>Functions</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1(x) = x^4 + 4x^2 - 15 )</td>
<td>( \alpha = 1.93198055660636 )</td>
</tr>
<tr>
<td>( f_2(x) = x \exp(x^2) - \sin(x)^2 + 3 \cos(x) + 5 )</td>
<td>( \alpha = 1.207647827130919 )</td>
</tr>
<tr>
<td>( f_3(x) = 10 \exp(-x^2) - 1 )</td>
<td>( \alpha = 1.67963061042845 )</td>
</tr>
<tr>
<td>( f_4(x) = \sin(x)^2 - x^2 + 1 )</td>
<td>( \alpha = 1.4044916482153411 )</td>
</tr>
<tr>
<td>( f_5(x) = x^3 + x^4 + 4x^2 - 15 )</td>
<td>( \alpha = 1.347 )</td>
</tr>
<tr>
<td>( f_6(x) = \log(x) + \sqrt{x} - 5 )</td>
<td>( \alpha = 8.3094326942315718 )</td>
</tr>
<tr>
<td>( f_7(x) = \exp(x^2) + 3x - 30 - 1 )</td>
<td>( \alpha = 3.0 )</td>
</tr>
<tr>
<td>( f_8(x) = \sin(x) \exp(x) - 2x - 5 )</td>
<td>( \alpha = 2.5232452307325549 )</td>
</tr>
<tr>
<td>( f_9(x) = \sqrt{x} - 1/x - 3 )</td>
<td>( \alpha = 9.6335955628326952 )</td>
</tr>
<tr>
<td>( f_{10}(x) = \sqrt{x^2 + 2x + 5} - 2 \sin(x) - x^2 + 3 )</td>
<td>( \alpha = 2.331967658839640 )</td>
</tr>
</tbody>
</table>

The absolute error in numerically calculated root is depicted in Table 2. Table 2 also shows absolute error in root, total number of iterations and computational order of convergence.
Table 2: Absolute errors ($|x_n - \alpha|$) and computational order of convergence for iterative schemes (2.1),(2.2).

| $f_n(x), x_0$ | Iter | $|x_n - \alpha|$ (2.1) | $|f(x_n)|$ (2.1) | COC (2.1) | $|x_n - \alpha|$ (2.2) | $|f(x_n)|$ (2.2) | COC (2.2) |
|--------------|------|-----------------|----------------|---------|-----------------|----------------|---------|
| $f_1, 2$     | 4    | 5.5e-8          | 1.159e-6       | 1.590   | 8.4e-23         | 1.778e-21     | 2.000   |
| $f_2, -1.5$  | 4    | divergent       | -              | -       | divergent       | -              | -       |
| $f_3, 1.5$   | 4    | 1.8e-15         | -              | -       | 5.5e-12         | 1.525e-11     | 2.001   |
| $f_4, 2$     | 4    | 6.3e-7          | 1.555e-6       | 2.003   | 5.5e-12         | 1.525e-11     | 2.001   |
| $f_5, 1.4$   | 4    | 6.2e-24         | 2.308e-22      | 2.000   | 3.4e12          | 1.130e25      | 3.001   |
| $f_6, 8$     | 4    | 2.3e-29         | 6.802e-30      | 2.000   | 1.1e-25         | 3.207e-26     | 2.000   |
| $f_7, 3.5$   | 4    | divergent       | -              | -       | divergent       | -              | -       |
| $f_8, -2.4$  | 4    | 2.8e-38         | 5.923e-38      | 2.000   | 3.5e-44         | 7.299e-44     | 2.000   |
| $f_9, 9$     | 4    | 2.2e-26         | 3.866e-27      | 2.000   | 7.5e-23         | 1.285e-23     | 2.000   |
| $f_{10}, 2$  | 4    | 5.7e-28         | 1.389e-27      | 2.000   | 1.5e-23         | 3.585e-23     | 2.000   |
\[
\begin{align*}
\text{\#} f(\alpha) &= 0; \quad e = x_n - \alpha; \quad c_1 = f'(\alpha); \quad c_k = \frac{(e)^k}{k!} f^{(k)}(\alpha); \quad k = 2, 3, \ldots \quad \text{\#} \\
n &:= 8; \\
\text{fxn} &= c_1^* (e + c_2 e^2 + c_3 e^3 + c_4 e^4 + c_5 e^5 + c_6 e^6 + c_7 e^7 + c_8 e^8 + c_9 e^9 + c_{10} e^{10} + c_{11} e^{11}) \\
&= c_1 (e + c_2 e^2 + c_3 e^3 + c_4 e^4 + c_5 e^5 + c_6 e^6 + c_7 e^7 + c_8 e^8 + c_9 e^9 + c_{10} e^{10} + c_{11} e^{11}) \\
d\text{fxn} &= \text{diff}(\text{fxn}, e) \\
c_1 l (1 + 2 c_2 e + 3 c_3 e^2 + 4 c_4 e^3 + 5 c_5 e^4 + 6 c_6 e^5 + 7 c_7 e^6 + 8 c_8 e^7 + 9 c_9 e^8 + 10 c_{10} e^9 + 11 c_{11} e^{10}) \\
ye &:= \text{simplify}(\text{taylor}(e - \frac{\text{fxn}}{d\text{fxn}}, e = 0, n)) \\
c_2 e^2 + (2 c_3 - 2 c_2^2) e^3 + (3 c_4 - 7 c_2 c_3 + 4 c_2^2) e^4 + (4 c_5 - 10 c_2 c_4 - 6 c_3 c_2^2 + 20 c_3 c_2^2 - 8 c_2^2) e^5 + (13 c_5 c_2 c_3 + 5 c_6 - 17 c_4 e^4 + 28 c_4 c_2^2 + 33 c_3 c_2^3 - 52 c_3 c_2^3 + 16 c_2^3) e^6 + (22 c_5 c_2 c_3 + 36 c_2^2 c_3 + 5 c_7 - 16 c_6 e^5 - 12 c_2 c_4 - 92 c_2 c_4 c_3 - 72 c_4 e^5 + 18 c_3 c_2^2 - 126 c_2 c_3 c_2^2 + 128 c_3 c_2^3 - 32 c_2^3) e^7 + O(e^8) \\
\text{fyn} &= c_1^* (ye + c_2 ye^2 + c_3 ye^3 + c_4 ye^4 + c_5 ye^5 + c_6 ye^6 + c_7 ye^7 + c_8 ye^8 + c_9 ye^9 + c_{10} ye^{10} + c_{11} ye^{11}) \\
\text{fyn} &= \text{simplify}(\text{taylor}(\text{fyn}, e = 0, n)) \\
c_1 c_2 e^2 - 2 c_1 (-c_3 + c_2^2) e^2 + 1 (3 c_4 - 7 c_2 c_3 + 5 c_2^2) e^3 - 2 c_1 (-2 c_5 + 5 c_2 c_4 + 3 c_3 c_2^2 + 6 c_4 c_2^2) e^4 + c_1 (-13 c_5 c_2 c_3 + 5 c_6 - 17 c_4 c_3 + 34 c_4 c_2^2 + 37 c_2 c_3^2 - 73 c_3 c_2^3 + 28 c_2^3) e^5 - 2 c_1 (-22 c_5 c_2 c_3 + 52 c_4 c_2^2 + 80 c_2^2 c_3^2 - 103 c_2 c_3^2 + 32 c_2 c_3^2 - 72 c_2 c_4 c_3 + 11 c_5 c_3 - 3 c_7 + 8 c_2 c_6 + 6 c_4^2 - 9 c_3 c_3^2) e^6 + O(e^7) \\
m &= \text{simplify}(\text{taylor}(\frac{\text{fyn}}{e}, e = 0, n)) \\
c_1 c_2 e - 2 c_1 (-c_3 + c_2^2) e^2 + c_1 (3 c_4 - 7 c_2 c_3 + 5 c_2^2) e^3 - 2 c_1 (-2 c_5 + 5 c_2 c_4 + 3 c_3 c_2^2 + 6 c_4 c_2^2) e^4 + c_1 (-13 c_5 c_2 c_3 + 5 c_6 - 17 c_4 c_3 + 34 c_4 c_2^2 + 37 c_2 c_3^2 - 73 c_3 c_2^3 + 28 c_2^3) e^5 - 2 c_1 (-22 c_5 c_2 c_3 + 52 c_4 c_2^2 + 80 c_2^2 c_3^2 - 103 c_2 c_3^2 + 32 c_2 c_3^2 - 72 c_2 c_4 c_3 + 11 c_5 c_3 - 3 c_7 + 8 c_2 c_6 + 6 c_4^2 - 9 c_3 c_3^2) e^6 + O(e^7) \\
ze &= \text{simplify}(\text{taylor}(ye - (m + m^2) \frac{\text{fyn}}{d\text{fxn}}, e = 0, n)) \\
c_2 e^2 + (2 c_3 - 2 c_2^2 - c_1 c_2^2) e^3 + (3 c_4 - 7 c_2 c_3 + 4 c_2^2 - 4 c_1 c_2 c_3 + 6 c_1 c_2^2 c_3 - 6 c_1^2 c_2^2 c_3 c_3 - 26 c_1 c_2^2 - 6 c_2 c_2^2 c_3 + 8 c_2^2 c_2^2 - 4 c_1 c_2^3 c_3 + 5 c_6 - 12 c_1 c_2 c_4 + 43 c_1^2 c_2^5 c_3 + 48 c_1 c_2^2 c_4 - 8 c_1 c_2 c_5 + 60 c_1 c_2 c_3^2 - 180 c_1 c_2 c_3 + 9 c_1^2 c_2^3 c_4 + 60 c_1^2 c_2^3 c_3 c_3 - 12 c_2 c_2^2 c_3 c_3 - 72 c_1 c_2 c_3^2 + 28 c_2 c_3^2 + 33 c_2 c_3^2 + 52 c_3 c_3^2 + 96 c_1 c_2^5 + 16 c_2^2) e^6 + (6 c_7 - 12 c_2 c_2^2 c_3 - 321 c_1 c_2 c_5 + 190 c_1^2 c_2^2 - 72 c_4 c_2^2 + 36 c_2^2 c_5) e^7 + O(e^8)
\end{align*}
\]

Figure 1: Maple program to verify the order of convergence of iterative schemes (2.1) and (2.2).
Figure 2: Maple program to verify the order of convergence of iterative schemes (2.1) and (2.2).

\[
\text{se} := \text{simplify}\left(\frac{(e - ze)^2 \cdot (ye - ze) + fyn \cdot (ye - ze) \cdot dfyn}{(e - ze) \cdot (ye - ze) \cdot (e - ye)}\right)
\]

\[
\text{thetae} := \text{simplify}\left(\frac{1}{ze - se}, e = 0, n\right)
\]

\[
x_{n+1} := \text{simplify}\left(\frac{1}{c1} + \frac{2 \cdot (-1 + 2 \cdot c1 \cdot c2)}{c1^2}, e \cdot 0, 4\right)
\]
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