Another Method for Deriving two Results Contiguous to Kummer’s Second Theorem

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Abstract

The aim of this research paper is to derive two results closely related to the well known classical and useful Kummer’s second theorem obtained earlier by Kim et al. [Comput. Math. & Math. Phys., 50 (3) (2010), 387 - 402] by employing classical Gauss’s summation theorem for the series _2F_1.

MSC (2010): Primary 33C05; Secondary 33C20.

Keywords: Hypergeometric Transformations; Gauss’s Summation Theorem; Kummer’s Second Summation Theorem.

1 Introduction

The generalized hypergeometric function with p numerator and q denominator parameters is defined by (see [9] or [11, Section 1.5])

\[ pF_q \left[ \begin{array}{c} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{array} \right] z = \frac{\sum_{n=0}^{\infty} (a_1)_n \ldots (a_p)_n z^n}{(b_1)_n \ldots (b_q)_n n!} \]

(1.1)

where \((a)_n\) denotes the Pochhammer’s symbol (or the shifted or raised factorial, since \((1)_n = n!\)) defined by

\[ (a)_n = \begin{cases} 1, & n = 0 \\ a(a+1) \ldots (a+n-1), & n \in \mathbb{N} := \{1, 2, 3, \ldots\} \end{cases} \]

(1.2)

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By using the fundamental property of the Gamma function \( \Gamma(a + 1) = a \Gamma(a) \), \((a)\) can be written in the form
\[
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}.
\]
It is not out of place to mention here that whenever the generalized hypergeometric function \( _pF_q \) or the hypergeometric function \( _2F_1 \) are summed to be expressed in terms of Gamma functions, the results are very important from the application point of view. Thus the classical summation theorems for the series \( _2F_1 \) such as those of Gauss, Gauss second, Kummer and Bailey play an important role in the theory of those hypergeometric series. For generalizations and extensions of those classical summation theorems, we refer to [3] and [8]. For useful applications of the above mentioned classical summation theorem, we refer a very interesting, useful and popular paper by Bailey [1].

We start with:

**Gauss’s summation theorem** [9]

\[
_2F_1 \left[ \begin{array}{c}
\frac{a}{c}; \\
\frac{b}{c}; \\
1
\end{array} \right] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}
\]

provided \( \Re(c - a - b) > 0 \).

**Gauss’s second summation theorem** [9]

\[
_2F_1 \left[ \begin{array}{c}
\frac{a}{\frac{1}{2}(a+b+1)}; \\
\frac{b}{\frac{1}{2}}; \\
\frac{1}{2}(a+b+1); \\
\frac{1}{2}
\end{array} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})}
\]

In (1992), Lavoie et al. [6] have generalized Gauss’s second summation theorem (1.4) and obtained an explicit expression as follows:

\[
_2F_1 \left[ \begin{array}{c}
\frac{a}{\frac{1}{2}(a+b+i+1)}; \\
\frac{b}{\frac{1}{2}}; \\
\frac{1}{2}(a+b+i+1); \\
\frac{1}{2}
\end{array} \right]
\]

for \( i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5 \).

On the other hand, from the theory of differential equations, Kummer [5] derived the following very interesting and useful result which is now in the literature known as Kummer’s second theorem given by

\[
e^{-\frac{x}{2}} _1F_1 \left[ \begin{array}{c}
\frac{\alpha}{2\alpha}; \\
\frac{x}{\alpha + \frac{1}{2}}
\end{array} \right] = _0F_1 \left[ \begin{array}{c}
-; \\
\frac{\alpha}{\alpha + \frac{1}{2}}
\end{array} \right],
\]

which can also be written in the following equivalent form:

\[
e^{x} _0F_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{1}{2}; \\
\frac{x^2}{4}
\end{array} \right] = _1F_1 \left[ \begin{array}{c}
\frac{\alpha}{2\alpha}; \\
2x
\end{array} \right].
\]

We remark in passing that Bailey [1] derived the formula (1.6) by employing Gauss’s second theorem (1.4) and Choi and Rathie [2] derived (1.7) by employing Gauss’s summation theorem (1.3).

In (1995), Rathie and Nagar [10] obtained two results closely related to Kummer’s second theorem (1.6) which are recalled here:
\[ e^{-\frac{x}{2}} \binom{\alpha}{2\alpha + 1} x \] 
\[ = \binom{\alpha}{\alpha + \frac{1}{2}} - \frac{x}{2(2\alpha + 1)} \binom{\alpha}{\alpha + \frac{3}{2}} \] (1.8)

and

\[ e^{-\frac{x}{2}} \binom{\alpha}{2\alpha - 1} x \] 
\[ = \binom{\alpha}{\alpha - \frac{1}{2}} + \frac{x}{2(2\alpha - 1)} \binom{\alpha}{\alpha + \frac{3}{2}} \] (1.9)

In (2001), Malani and Choi [7] established the results (1.8) and (1.9) in their equivalent forms by employing Gauss’s summation theorem (1.3).

In (2010), Kim et al. [4] have generalized the Kummer’s second theorem (1.6) and obtained an explicit expression as follows:

\[ e^{-\frac{x}{2}} \binom{\alpha}{2\alpha + j} x \] (1.10)

for \( j = 0, \pm 1, \cdots, \pm 5 \), by employing the result (1.5), [4, P. 391, Eq. 2.1].

For \( j = \pm 2 \), we mention the results as

\[ e^{-\frac{x}{2}} \binom{\alpha}{2\alpha + 2} x \] 
\[ = \binom{\alpha}{\alpha + 2\alpha + \frac{1}{2}} - \frac{x}{2(2\alpha + 1)} \binom{\alpha}{\alpha + 2\alpha + \frac{3}{2}} \] (1.11)

and

\[ e^{-\frac{x}{2}} \binom{\alpha}{2\alpha - 2} x \] 
\[ = \binom{\alpha}{\alpha - 2\alpha + \frac{1}{2}} + \frac{x}{2(2\alpha - 1)} \binom{\alpha}{\alpha - 2\alpha + \frac{3}{2}} \] (1.12)
In its equivalent forms, they can be written in the following manner

\[
\begin{align*}
\text{1F}_1 \left[ \begin{array}{c}
\alpha; \\
2\alpha + 2;
\end{array} \right] 2x \\
= e^x \left\{ \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{1}{2};
\end{array} \right] \right\} - \frac{x}{(\alpha + 1)} \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{3}{2};
\end{array} \right] \\
+ \frac{x^2}{(\alpha + 1)(2\alpha + 3)} \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{5}{2};
\end{array} \right] \right) \\
\tag{1.13}
\end{align*}
\]

and

\[
\begin{align*}
\text{1F}_1 \left[ \begin{array}{c}
\alpha; \\
2\alpha - 2;
\end{array} \right] 2x \\
= e^x \left\{ \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha - \frac{1}{2};
\end{array} \right] \right\} + \frac{x}{(\alpha - 1)} \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha - \frac{3}{2};
\end{array} \right] \\
+ \frac{x^2}{(\alpha - 1)(2\alpha - 1)} \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{1}{2};
\end{array} \right] \right) \\
\tag{1.14}
\end{align*}
\]

The aim of this research paper is to establish the results (1.13) and (1.14) by employing Gauss’s summation theorem (1.3).

2 Derivation of (1.13) and (1.14)

In order to derive (1.13), we proceed as follows. Denote the right-hand-side of (1.13) by \( S \), we have

\[
S = e^x \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{1}{2};
\end{array} \right] \right\} + \frac{x^2}{(\alpha + 1)(2\alpha + 3)} e^x \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{5}{2};
\end{array} \right] \\
- \frac{x}{(\alpha + 1)} e^x \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{3}{2};
\end{array} \right] \\
= A + B - C, \tag{2.15}
\]

where \( A, B \) and \( C \) are given below.

Now

\[
A = e^x \text{0F}_1 \left[ \begin{array}{c}
-; \\
\alpha + \frac{1}{2};
\end{array} \right] \right\}. 
\]

Express both functions as series, after some simplification, we have

\[
A = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^{m+2n}}{2^{2n} (\alpha + \frac{1}{2})_n n! m!}. 
\]
Now changing \( m \) by \( m - 2n \), we have

\[
A = \sum_{m=0}^{\infty} \sum_{n=0}^{\left\lfloor \frac{m}{2n} \right\rfloor} (-1)^{n} \left( \frac{m}{2n} \right)^{n} \frac{\binom{m}{n}}{n!} \frac{\alpha^m}{m!} \frac{\beta^{m}}{(n+1)!} \frac{1}{(n+1)!}.
\]

Using the identity

\[
(m - 2n)! = \frac{m!}{2^{2n} (-\frac{1}{2} m \cdot (-\frac{1}{2} m + \frac{1}{2})^{n}},
\]

we have, after some algebra,

\[
A = \sum_{m=0}^{\infty} \sum_{n=0}^{\left\lfloor \frac{m}{2n} \right\rfloor} \frac{\binom{m}{n}}{n!} \left( \frac{m}{2n} \right)^{n} \frac{\alpha^m}{m!} \frac{\beta^{m}}{(n+1)!} \frac{1}{(n+1)!}.
\]

Expressing the inner series as \(_2F_1\), we have

\[
A = \sum_{m=0}^{\infty} \frac{\binom{m}{n}}{2^{m} n!} \left( \frac{-\frac{1}{2} m}{\alpha + \frac{1}{2}} \right)^{m} \frac{\beta^{m}}{(n+1)!} \frac{1}{(n+1)!}.
\]

Finally, using Gauss’s summation theorem and making use of some elementary identities, we have

\[
A = \sum_{m=0}^{\infty} \frac{\binom{m}{n}}{(2\alpha + 2)^{m}} \left( \frac{2\alpha}{\alpha^m} \right) \left( \frac{\alpha + m}{\alpha^m} \right).
\]

In the same manner, it is easy to arrive at the following results

\[
B = \sum_{m=0}^{\infty} \frac{\binom{m}{n}}{(2\alpha + 2)^{m}} \left( \frac{2\alpha}{\alpha^m} \right) \left( \frac{m(m-1)}{2\alpha(\alpha + 1)} \right)
\]

and

\[
C = \sum_{m=0}^{\infty} \frac{\binom{m}{n}}{(2\alpha + 2)^{m}} \left( \frac{2\alpha}{\alpha^m} \right) \left( \frac{m(2\alpha + m + 1)}{2\alpha(\alpha + 1)} \right).
\]

On substituting the values of \( A, B \) and \( C \) from (2.16), (2.17) and (2.18) in (2.15), we see that

\[
S = \sum_{m=0}^{\infty} \frac{\binom{m}{n}}{(2\alpha + 2)^{m}} \left( \frac{2\alpha}{\alpha^m} \right) \left\{ \frac{\alpha + m}{\alpha} + \frac{m(m-1)}{2\alpha(\alpha + 1)} - \frac{m(2\alpha + m + 1)}{2\alpha(\alpha + 1)} \right\}
\]

which, upon simplification, gives

\[
S = \sum_{m=0}^{\infty} \frac{\binom{m}{n}}{(2\alpha + 2)^{m}} \frac{2\alpha}{m!} \frac{m^m}{m!}
\]

summing up the series, we arrive at the left-hand-side of the result (1.13). This completes the proof of our main result (1.13). A similar argument as in the proof of (1.13) will establish (1.14). □

**Remark 2.1.** It is interesting to mention here that in view of the following relation involving Bessel’s function [9, P. 108] and modified Bessel’s function [9, P. 116] with \(_0F_1\) viz.

\[
J_n(z) = \frac{1}{\Gamma(1+n)} \left( \frac{z}{2} \right)^n \left[ \frac{-z}{4} \right] ,
\]

and

\[
I_n(z) = \frac{1}{\Gamma(1+n)} \left( \frac{z}{2} \right)^n \left[ \frac{-z}{4} \right] ,
\]

the results (1.13) and (1.14) can be written in terms of Bessel’s function and modified Bessel’s function.
Acknowledgements

1. The authors would like to express their sincere gratitude to the referees for their valuable comments and suggestions.

2. All authors contributed equally in this paper. They read and approved the final manuscript.

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