A note on an interpolation formula

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Abstract
We study an elegant and useful interpolation formula arising from the work of Frappier, Rahman and Ruscheweyh. We give a new and simple proof of that formula based on elementary properties of bound-preserving operators on classes of polynomials and we relate it to other known interpolation formulas.

Keywords: Polynomials, interpolation, discrete Bernstein-type inequalities.

1 Introduction
In this note, we shall study some aspects of an interpolation formula due to Frappier, Rahman and Ruscheweyh. The formula has been introduced in the thesis of Frappier [6] and later also appeared in [7] and [10, p. 524]. Let \( P_n \) denote the class of polynomials of degree at most \( n \) with complex coefficients. The formula reads as, for any \( p \in P_n \) and \( q, y \) real,

\[
e^{i(q+\theta)} p'(e^{i\theta}) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k \frac{\sin^2 \left( \frac{k\pi q}{2n} \right)}{\sin^2 \left( \frac{k\pi y}{2n} \right)} p(e^{i\theta + \frac{k\pi q}{n} + \frac{k\pi y}{n}}).
\]

(1.1)

It follows in particular that (choose \( p(z) \equiv z^n \))

\[
\frac{1}{2n} \sum_{k=1}^{2n} \sin^2 \left( \frac{k\pi y}{2n} \right) = n
\]

and for any \( p \in P_n \) (choose \( \psi = -n\theta \))

\[
|p'(e^{i\theta})| \leq n \max_{1 \leq k \leq 2n} |p(e^{ik\pi/n})|, \quad \theta \in \mathbb{R},
\]

(1.2)

which represents a striking extension of the classical Bernstein inequality.

Undoubtedly, interpolation formulas can play an important role in approximation theory. We first mention the formula of Shapiro (see [11, chapter 2] and [13, chapter 3]). Let \( L \) be a linear functional over \( P_n \). There exist complex numbers \( \{ \alpha_j \}_{j=1}^{n+1} \) and distinct nodes \( \{ e^{i\theta_j} \}_{j=1}^{n+1} \) on the unit circle such that

\[
L(p) = \sum_{k=1}^{n+1} \alpha_k p(e^{i\theta_k})
\]

(1.3)

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and
\[
\max_{p \in \mathcal{P}_n} \sum_{1 \leq k \leq n+1} |p(e^{i\theta_k})| \leq \sum_{k=0}^{n+1} |\alpha_k|.
\]

We also mention a more recent formula due to Dryanov, Fournier and Ruscheweyh (see [4] for references): Let be given \( n + 1 \) angles \( 0 < \theta_0 < \theta_1 < \cdots < \theta_n \leq \pi \) and a linear functional \( L \) over \( \mathcal{P}_n \). There exist complex numbers \( \{\alpha_j\}_{j=0}^n \) such that
\[
L(p) = \sum_{k=0}^{n} \alpha_k p(e^{i\theta_k}) + p(e^{-i\theta_k})
\]
(1.4)
and
\[
\max_{p \in \mathcal{P}_n} \sum_{0 \leq k \leq n} \left| \frac{\alpha_k}{2} \left| L(p) \right| \right| \leq \sum_{k=0}^{n} |\alpha_k|.
\]

It is also not so hard to see that (1.1) with \( \psi = -n \theta \) admits an extension similar to (1.3) and (1.4): given any \( L \in \mathcal{P}_n^* \), there exist \( 2n \) complex numbers \( \{\alpha_k\}_{k=0}^{2n-1} \) such that
\[
L(p) = \sum_{k=0}^{2n-1} \alpha_k p(e^{-ik\pi/n})
\]
(1.5)
with
\[
\max_{p \in \mathcal{P}_n} \sum_{0 \leq k \leq n} \left| \frac{\alpha_k}{2} \left| L(p) \right| \right| \leq \sum_{k=0}^{n} |\alpha_k|.
\]

It is not clear whether or not, given an \( L \in \mathcal{P}_n^* \), equality shall hold in (1.6). We have however at our disposal an explicit representation for the numbers \( \{\alpha_k\}_{k=0}^{2n-1} \) in (1.5). Let \( q(z) := \frac{1}{2n} \frac{1-z^n}{1-z} \) and \( q_k(z) = q(z^{1/n^k}) \). Then,
\[
\alpha_k = L(q_k), \quad 0 \leq k \leq 2n - 1.
\]

A similar representation is also available for (1.3) and (1.4).

The formula (1.3) admits extensions to more abstract spaces and its proof is based on non-trivial but basic principles of functional analysis, as seen from [13, chapter 3]. It is also known [4] that (1.4) follows from the Lagrange interpolation formula.

Here we shall present a new proof of (1.1) based on a general bound-preservation result; as a consequence, we shall recover an improvement of (1.2) due to Mohapatra, O’Hara and Rodriguez [9] (see also [1] for a related result) as well as the corresponding cases of equality.

2 Another proof of (1.1)

Our main ingredient is

Lemma 2.1. Let \( |\zeta| = 1 \), \( w_j = e^{2j\pi/n} \) and \( \ell_j(\zeta) = -\frac{|1-\zeta|^2}{n^2} \frac{w_j\zeta^{1/n}}{(1-w_j\zeta^{1/n})^2} \). Then for any \( p \in \mathcal{P}_n \) and any complex number \( z \),
\[
p(z) + \frac{\zeta - z}{n} p'(z) = \sum_{j=1}^{n} \ell_j(\zeta) p(z^{1/n} w_j z)
\]
where \( \ell_j(\zeta) = 0 \) and \( \sum_{j=1}^{n} \ell_j(\zeta) = 1 \).

This is a known result [5] but for the sake of completeness we shall sketch its proof. By the well-known result of de Bruijn [2]
\[
|p(z) + \frac{\zeta - z}{n} p'(z)| \leq \max_{|z|=1} |p(z)|, \quad |\zeta|, |z| \leq 1,
\]
and this is seen to be equivalent with
\[ |p(z) + \frac{\zeta - 1}{n}zp'(z)| \leq \max_{|z|=1} |p(z)|, \quad |\zeta|, |z| \leq 1. \]

In other words we have
\[ |p(z) \star \sum_{k=0}^{n} \left( 1 + \frac{k(\zeta - 1)}{n} \right) \zeta^k| \leq \max_{|z|=1} |p(z)|, \quad |z| \leq 1 \]
(here \( \star \) denotes the Hadamard product of analytic functions) and
\[ \Re \left( \sum_{k=0}^{n} \left( 1 + \frac{k(\zeta - 1)}{n} \right) \zeta^k + o(z^n) \right) > \frac{1}{2}, \quad |z| < 1, \]
(details can be found in [12, chapter 4]). Now since
\[ 1 + \frac{k(\zeta - 1)}{n} = |\zeta| = 1 \quad \text{if} \quad k = n, \]
we have a representation ([18, chapter 7])
\[ \sum_{k=0}^{n} \left( 1 + \frac{k(\zeta - 1)}{n} \right) \zeta^k + o(z^n) = \sum_{j=1}^{n} \frac{\ell_j}{1 - w_j \zeta^{1/n} z^j}, \quad |z| < 1, \]
and for any \( p \in \mathcal{P}_n \), with \( \ell_j \geq 0 \),
\[ p(z) + \frac{\zeta - 1}{n}zp'(z) = \sum_{j=1}^{n} \ell_j p(w_j \zeta^{1/n} z). \]
The value of \( \ell_j \) can be obtained by solving the linear system
\[ 1 + \frac{k(\zeta - 1)}{n} = \sum_{j=1}^{n} w_j ^{k/n} \ell_j, \quad 1 \leq k \leq n. \]
The following consequence of Lemma 2.1 shall be useful; let \( p \in \mathcal{P}_n, |\zeta| = 1 \neq \zeta \) and \( 0 \neq z \). Then
\[ |p(z) + \frac{\zeta - 1}{n}zp'(z)| \leq \max_{1 \leq j \leq n} |p(z^{1/n} w_j z)| \tag{2.7} \]
with equality if and only if \( p \) is a binomial of the type \( p(u) \equiv A + Bu^n \).

We are now ready to give our proof of (1.1). We have by Lemma 2.1,
\[ \frac{2\zeta z p'(z)}{n} = \left( p(z) + \frac{\zeta - 1}{n} z p'(z) \right) - \left( p(z) + \frac{-\zeta - 1}{n} z p'(z) \right) \]
\[ = \sum_{j=1}^{n} \ell_j(\zeta) p(\zeta^{1/n} w_j z) - \ell_j(-\zeta) p((-\zeta)^{1/n} w_j z) \]
where for \( \zeta = e^{i\psi} \)
\[ \ell_j(\zeta) = \frac{1 - |\zeta|^2}{n^2} \frac{w_j \zeta^{1/n}}{(1 - w_j \zeta^{1/n})^2} \]
\[ = \sin^2 \left( \frac{\psi}{2} \right) / n^2 \sin^2 \left( \frac{j\pi}{n} + \frac{\psi}{2n} \right) \]
\[ = \frac{n^2 \sin^2 \left( \frac{2\pi}{n} + \frac{\psi}{2n} \right)}{n^2 \sin^2 \left( \frac{2\pi}{n} + \frac{\psi}{2n} \right)} \]
and

\[ \ell_j(-\zeta) = \frac{\sin^2\left(n \left( \frac{2j\pi}{2n} + \frac{\psi + \pi}{2n} \right) \right)}{n^2 \sin^2\left(n \left( \frac{2j\pi}{2n} + \frac{\psi + \pi}{2n} \right) \right)} \]

Therefore for any complex number \( z \),

\[ e^{i\psi}z^p'(z) = \frac{1}{2n} \left( \sum_{k=1}^{2n} \sin^2\left( \frac{ki\pi}{2n} + \frac{\psi + \pi}{2n} \right) \right) p(e^{i\psi}z^p(z)) - \sum_{k=1}^{2n} \sin^2\left( \frac{ki\pi}{2n} + \frac{\psi + \pi}{2n} \right) \]

and this is of course equivalent with (1.1).

Now let \( m \geq 2 \) and consider the \( m \) distinct roots of unity \( r_k = e^{\frac{2\pi k}{m}}, 1 \leq k \leq m \). We have for any complex number \( z, |\zeta| = 1 \) and by (2.7),

\[ \left| z^p(z) \right| \leq \frac{1}{m} \left| \sum_{k=1}^{m} \left( p(z) + \frac{r_k\zeta - 1}{n} z^p(z) \right) \right| \leq \frac{1}{m} \left| \sum_{k=1}^{m} \max_{1 \leq j \leq n} |p(\zeta^{1/n}r_k^{1/n}w_jz)| \right| \]

and since \( \zeta \) is arbitrary

\[ \left| z^p(z) \right| \leq \frac{1}{m} \sum_{k=1}^{m} \max_{1 \leq j \leq n} |p(r_k^{1/n}w_jz)|. \]

The case \( m = 2 \) of (2.8) is the inequality of Mohapatra \textit{et al.} alluded to above. It is clear that two consequences of (2.8) are

\[ \left| z^p(z) \right| \leq \max_{1 \leq j \leq n} \left| p\left( e^{\frac{2\pi j}{m}} |z| \right) \right| \]

and (as \( m \to \infty \))

\[ \left| z^p(z) \right| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \max_{1 \leq j \leq n} \left| p(e^{j\theta}/w_jz) \right| d\theta. \]

We finally claim, as a consequence of the equality case in (2.7), that, for a given non-zero complex number \( z \), equality shall hold in (2.8) only for polynomials of the type \( p(u) \equiv A u^q \); the proof of this is left to the reader.

3 Concluding remarks

It may be interesting (and we are unable to settle this question) to decide if equality holds in (1.6) for all functionals \( L \)? This is easily seen to be true for the functional \( L_0(p) = p(0) \) and also, as a consequence of (1.1), for the functional \( L_\theta(p) = e^{i\theta} p(e^{i\theta}) \). What makes this question challenging is the number 2n of sample points in (1.5) as
when compared to the number $n + 1$ of sample points in Shapiro’s formula (1.3). We do not really understand why such a relatively big number of sample points is needed for a formula like (1.5) to be valid; partial answers were given in [13] and particularly in [3, Theorem 1.1] where an extension of (1.2) was given for polynomials $p$ in $\mathcal{P}_{2n}$ (as opposed to $\mathcal{P}_n$ in (1.2)).

It is finally worth mentioning that in the case where $\theta_k = \frac{k\pi}{n}$, $0 \leq k \leq n$, the interpolation formulas (1.4) and (1.5) shall not be the same, even if they coincide for certain functionals, for example $L(p) = p'(1)$!

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