Collocation-homotopy method to initial-boundary value problems

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Abstract

In this paper, an algorithm based on the collocation and homotopy analysis methods, for solving initial-boundary value problems, is introduced. The application of this algorithm is based on the approximation and interpolation of the dependent variables by using suitable functions or polynomials according to their values in the collocation points corresponding to a suitable discretization of the space variable. Then the space derivatives are approximated using interpolation. Replacing them in the equation transforms the initial-boundary value problem into an initial value problem for ordinary differential equations. The obtained initial value problem is solved by homotopy analysis method. In the frame of the homotopy analysis method, the optimum value of convergence-parameter corresponding to each point is computed by a simple stochastic function minimizer, namely differential evolution method. Lagrange polynomials are usually adopted for the interpolation. In this framework, the Burgers model is considered as a prototype example.

Keywords: Initial-boundary value problems, Collocation methods, Lagrange polynomials, Homotopy analysis method; Differential evolution method, Burgers model.

1 Introduction

The interplay between engineering sciences, technology, and applied mathematics often leads to the analysis of initial and/or boundary value problems for nonlinear partial differential equations. A large variety of solution methods can be developed to help solve the above class of problems. The selection of the proper method is, in fact, one of the most difficult tasks of applied mathematics and mainly depends on the structure of the problem to be solved and, to a minor extent, on the aims of the simulation. A well-known solution technique of nonlinear initial-boundary value problems for nonlinear partial differential equations is the generalized collocation method originally called the differential quadrature method. This method discretizes the original continuous model (and problem) into a discrete (in space) and continuous (in time) model, with a finite number of degrees of freedom, while the initial-boundary value problem is transformed into an initial value problem for ordinary differential equations. This method is well documented in the literature on applied mathematics, it was first proposed by Bellman and Casti [1] and developed by several authors in deterministic [2] and stochastic frameworks [3].

On the other hand, one of the very powerful methods for handling the initial value problems is the homotopy analysis

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method. This method transforms a nonlinear problem into a system of linear problems. This method is well documented in the literature on applied mathematics and engineering, it was first proposed by Liao in 1992 [4, 5] and developed by several authors in deterministic [6, 7, 8] and inverse frameworks [9, 10].

Further, mainly initial and boundary value problems [11, 12] have been investigated in the past, and the purpose of this paper is to extend the application of the HAM for solving initial-boundary value problems. Recently, Motsa et al. proposed an algorithm for handling boundary value problems [13, 14], namely spectral–homotopy analysis method (SHAM). They first applied standard HAM to problem and then they suggested using the Chebyshev pseudospectral method to solve the higher order deformation equations. It is clear that by means of SHAM, we can only solve initial or boundary value problems. In fact, SHAM is not applied to initial-boundary value problems.

In this paper, we introduce an algorithm based on the collocation and homotopy analysis methods for solving initial-boundary value problems. In the present paper, we consider the following general class of scalar second-order PDEs

\[
\frac{\partial u}{\partial t} = f(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}),
\]

(1.1)

where the dimensionless dependent variable

\[ u = u(x, t) : [0, 1] \times [0, 1] \rightarrow [0, 1] \]

describes, in the mathematical model, the state of a real physical system, and \( f \) is assumed to be a given function of its arguments.

The initial-boundary value problem for Eq. (1.1) is stated with initial condition

\[ u(x, 0) = \phi(x), \]

(1.2)

and with Dirichlet boundary conditions

\[ u(0, t) = \alpha(t), \ u(1, t) = \beta(t), \]

(1.3)

where \( \phi \), is a given function of space, and \( \alpha \) and \( \beta \) are given smooth functions of time, consistent with the initial condition (1.2). The homogeneous boundary conditions are more convenient for the our algorithm. Thus, without loss of generality, the following change of variable as

\[ u(x, t) = u(x, t) + x(\beta(t) - \alpha(t)) + \alpha(t), \]

(1.4)

is used. Having made this transformation, the dependent variable will again be labeled \( u \) and the conditions will be obtained as

\[ u(x, 0) = \phi(x) + x(\alpha(0) - \beta(0)) - \alpha(0), \]
\[ u(0, t) = 0, \ u(1, t) = 0. \]

(1.5)

2 Collocation methods in one space dimension

This section deals with interpolation techniques of functions of time and one dependent space variable using Lagrange polynomials. The method is presented in view of the solution to initial-boundary value problems such that the dependent variables depend on time and one space variable.

2.1 Interpolation by Lagrange polynomials

In general, \( u = u(x, t) \) can be interpolated and approximated by means of Lagrange polynomials as follows

\[ u(x, t) \approx u_{LNP[n]}(x, t) = \sum_{k=1}^{n} L_k(x) u_k(t), \]

(2.6)
where \( u_k(t) = u(x_k, t) \), and \( L_k(x) \), \( k = 1, \ldots, n \), are Lagrange polynomials. This interpolation can be used to approximate the partial derivatives of the function \( u \) in the nodal points of the discretization

\[
\frac{\partial^r u(x_j; t)}{\partial x^r} \approx \sum_{k=1}^{n} a_{kj}^{(r)} u_k(t), \quad r = 1, 2, \ldots ,
\]

(2.7)

where

\[
a_{kj}^{(r)} = \frac{d^r}{dx^r} L_k(x_j).
\]

Technical calculations provide the following result [15]

\[
a_{kj}^{(1)} = \frac{\prod (x_j)}{(x_j - x_k) \prod (x_k)}, \quad k \neq j,
\]

(2.8)

and

\[
a_{jj}^{(1)} = \sum_{k \neq j} \frac{1}{(x_j - x_k)},
\]

(2.9)

where

\[
\prod (x_i) = \prod_{p \neq i} (x_i - x_p), \quad i = k, j.
\]

Higher-order coefficients may be computed exploiting the following recurrence formula

\[
a_{kj}^{(r)} = r \left( a_{kj}^{(1)} a_{jj}^{(r-1)} + a_{kj}^{(r-1)} \frac{a_{jj}^{(r-1)}}{x_k - x_j} \right), \quad a_{jj}^{(r+1)} = - \sum_{k \neq j} a_{kj}^{(r+1)}.
\]

(2.10)

It is well known that accuracy of the interpolation can be obtained by the selection of the proper collocation points to be related to the selection of the interpolation functions [16]. In the case of Lagrange interpolation, a Chebychev collocation is needed.

3 The collocation-homotopy analysis method (CHAM)

This section presents an algorithm based on the collocation and homotopy analysis methods, to solve Eqs. (1.1) and (1.5). Our algorithm contains two parts. In the first one, an initial-boundary value problem, say Eqs. (1.1) and (1.5), is transformed into an initial value problem for a system of ordinary differential equations by using the generalized collocation method, and in the second part, the solution of the initial value problem is obtained by using the homotopy analysis method. Bearing this in mind, the algorithm can be applied as follows.

Part I. Transforming an initial-boundary value problem into an initial value problem for a system of ODEs.

- The space variable, say \( x \), is discretized into a suitable number of collocation points \( x_j \),
- The dependent variable \( u = u(x, t) \) is approximated by interpolating polynomials or functions through the values \( u_j(t) = u(x_j, t) \) of the dependent variable in the collocation points, and then, the space derivatives are approximated using this interpolation,
- Boundary conditions are imposed at the collocation points corresponding to the boundary of the domain of the independent variables.

Part II. Solving an initial value problem for a system of ODEs by HAM.
3.1 Generalized collocation method to initial-boundary value problem

Applying the Part I to Eqs. (1.1) and (1.5) yields the following initial value problem for a system of ODEs

\[
\frac{du_j}{dt} = f(t, x_j, u_j, u_j^{(1)}, u_j^{(2)}) \quad j = 2, \ldots, n-1, \tag{3.11}
\]

with initial conditions

\[
u_j(0) = \varphi(x_j) + x_j(\alpha(0) - \beta(0)) - \alpha(0), \quad j = 2, \ldots, n-1. \tag{3.12}
\]

where \(x_j\) is a Chebychev collocation.

We prefer to write Eqs. (3.11) and (3.12) in the matrix form

\[
\begin{cases}
\frac{du}{dt} = f(t, u), \\
u(t_0) = u_0.
\end{cases} \tag{3.13}
\]

Therefore, the output of the Part I and then the input of the Part II is an initial value problem for a system of ODEs denoted by (3.13).

3.2 HAM solution to an initial value problem for a system of ODEs

Using the HAM, it is possible to find the exact solution or an approximate solution of the problem in the form of a series. This technique provides a series of functions which converges rapidly to the exact solution of the problem. Some advantages of the HAM are as follows

- It is independent to any small parameters, and hence it is applicable to both weakly and strongly nonlinear problems.
- It provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods.
- It is a unified method for some other non-perturbation methods, such as the Lyapunov artificial small parameter method, the \(\delta\)-expansion method and the Adomian decomposition method
- It can be combined with many other mathematical methods, such as the Pade method, series expansion methods, integral transforms and numerical methods, etc.

Two important steps of the HAM include properly choosing of the auxiliary linear operator and initial guess. Here, we first outline a definition as already given in [7, 9].

**Definition 3.1.** Let \(\phi\) be a function of the homotopy-parameter \(q\), then

\[
D_m(\phi) = \left. \frac{1}{m!} \frac{\partial^m \phi}{\partial q^m} \right|_{q=0}, \tag{3.14}
\]

is called the \(m\)th-order homotopy-derivative of \(\phi\), where \(m \geq 0\) is an integer.

For applying the Part II, we use the following algorithm.

I. Select an auxiliary linear operator, such as

\[
L[\Phi] = \Phi_t, \tag{3.15}
\]

which has the property \(L[C] = 0\), where \(C\) is a constant.

II. According to (3.13), define the nonlinear operator

\[
N[\Phi] = \Phi_t - f(t, \Phi). \tag{3.16}
\]
III. To obtain $\Psi^*_m(t)$ as a special solution, solve the equation

$$L[\Psi^*_m(t)] = \bar{h}H(t)R_m[\Psi_{m-1}(t)],$$

(3.17)

where

$$R_m[\Psi_{m-1}(t)] = D_{m-1}\{N[\Phi(t;q)]\},$$

(3.18)

and

$$\Phi(t;q) = \Psi_0(t) + \sum_{m=1}^{+\infty} \Psi_m(t)q^m,$$

(3.19)

and $q \in [0,1]$ is the embedding parameter, $\bar{h}$ is a diagonal matrix of nonzero convergence-parameters, $H(t)$ is a diagonal matrix of auxiliary functions, $\Psi_0(t)$ is an initial guess of the exact solution $\Psi(t)$ and $\Phi(t;q)$, defined by (3.19), is an unknown function which depends also on convergence-parameters and auxiliary functions, and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2. \end{cases}$$

IV. Iterate the following iterative equation to produce the terms of the series solution

$$\begin{cases} \Psi_m(t) = \chi_m\Psi_{m-1}(t) + \Psi^*_m(t) - \Psi^*_m(t_0), \\ \Psi_0(t) = u(t_0). \end{cases}$$

(3.20)

V. Set

$$\Psi_{approx}[m](t;\bar{h}) = \Psi_0(t) + \sum_{k=1}^{m} \Psi_k(t;\bar{h}).$$

VI. To convergence analysis of the solution series, plot the so-called $\bar{h}$-curves to discover the valid region of $\bar{h}$, which corresponds to the line segment nearly parallel to the horizontal axis.

VII. To analysis and control of error, apply a global optimization algorithm to find the optimum value of the $\bar{h}$ corresponding to the given point. In this framework, we use a simple stochastic function minimizer, namely differential evolution method.

Remark 3.1. It should be emphasized that the values $\bar{h} = -1$ and $H(t) = 1$ reduce HAM to HPM (homotopy perturbation method) and ADM (Adomian decomposition method). Thus, in this case, CHAM reduces to CHPM (collocation-homotopy perturbation method) and CADM (collocation-Adomian decomposition method).

Remark 3.2. It is important to notice that HAM may need significant more computation time and computer hardware requirements, in this case, for the second part of our algorithm, for an alternative way refer to [17].

It should be pointed out that the homotopy analysis method is based on the following assumptions [5]

- There exists the solution of the zero-order deformation equation, defined by

$$\left(1-q\right)\ell[\Phi(t;q) - \Psi_0(t)] = q\bar{h}H(t)N[\Phi(t;q)],$$

(3.21)

in the whole region of the embedding parameter $q \in [0,1]$.

- All of the high-order deformation equations, obtained by operating on both sides of Eq. (3.21) with $D_m$, have solutions.

- All Taylor series expanded in the embedding parameter $q$, similar to (3.19), converge at $q = 1$. 
4 Test examples

To show the efficiency of the CHAM described in the previous part, two examples are presented. For nonhomogeneous boundary conditions, we first use the transformation (1.4) to obtain homogeneous boundary conditions. We use \( n+1 \) terms in evaluating approximate solution \( u_{\text{approx}}[m,n](x,t;\bar{h}) = \sum_{k=1}^{n} L_k(x)u_{k,n}(t;\bar{h}) \). Here, we only choose three collocation points \( (m = 3) \). Thus, \( u_{\text{approx}}[3,n](x,t;\bar{h}) = u_{2,n}(t;\bar{h})L_2(x) \). For comparison the solution series given by CHAM with the exact solution, we report the absolute error which is defined by

\[
\|u_{m,n}(x,t;\bar{h})\| = |u(x,t) - u_{\text{approx}}[m,n](x,t;\bar{h})|.
\]

For an alternative way to report error refer to [18]. Here, for all examples, we choose \( H(t) = 1 \).

Example 4.1. Consider one-dimensional variable-coefficient Burgers problem [19]

\[
\begin{align*}
\frac{∂u}{∂t} + (x+1)\frac{∂u}{∂x} + x^2u + (x+1)u_x - t\sin(x)uu_x &= g(x,t), \\
\begin{align*}
u(x,0) &= 0, \\
u(0,t) &= 0, \ u(1,t) &= 0,
\end{align*}
\end{align*}
\]

(4.22)

where \( 0 \leq x \leq 1, \ 0 \leq t \leq 1 \) and \( g(x,t) = 2(x+1) + x^2 - x - t^3(2x-1)(x^2-x)\sin(x) + x^3(x-1)t + (x+1)(2x-1)t \). The exact solution is \((x^2-x)t\). Results are presented in Figs. 1-4 and Table 1.
Figure 1: The $h$-curve for Example 4.1.

Figure 2: Comparison of the exact solution with analytic approximation given by CHPM and CADM for Example 4.1. Left: Exact solution and Right: $u_{approx}(3.15)(x,t; -1)$. 
Figure 3: Error versus $\bar{h}$ of the analytic approximation $u_{\text{approx}}[3, 15](0.5, t; \bar{h})$ given by CHAM for Example 4.1.

Figure 4: Comparison of the Lagrange interpolation with a Chebychev collocation and analytic approximation given by CHAM. Left: $u_{\text{LINP}}[3](x, t)$ and Right: $u_{\text{approx}}[3, 15](x, t; -1.325)$ for Example 4.1.
Example 4.2. Consider one-dimensional Burgers problem [20]

\[
\begin{align*}
  u_t + uu_x &= u_{xx}, \\
  u(x,0) &= 2x, \\
  u(0,t) &= 0, \quad u(1,t) = \frac{2}{1 + 2t},
\end{align*}
\]

where \(0 \leq x \leq 1, 0 \leq t \leq 1\). The exact solution is \(u(x,t) = \frac{2x}{1 + 2t}\). Having made the transformation (1.4), the dependent variable again labels with \(u\) and the (4.23) becomes

\[
\begin{align*}
  u_t + uu_x + \frac{2}{1 + 2t} (u + xu_x) &= u_{xx}, \\
  u(x,0) &= 0, \\
  u(0,t) &= 0, \quad u(1,t) = 0.
\end{align*}
\]

Now by using our algorithm for (4.24) and choosing only three collocation points, for any order of approximation, we find

\[u_{2,n}(t,h) = 0.\]

Thus, considering the transformation (1.4), the CHAM gives the exact solution of (4.23)

\[u(x,t) = 0 + x\left(\frac{2}{1 + 2t} - 0\right) + 0 = \frac{2x}{1 + 2t}.
\]

5 Comparison and discussion

In this part, a comparison between exact and approximate solutions, to check the accuracy of the method, is given. For the approximate solutions, the \(h\)-curves are plotted to discover the convergent region. According to the \(h\)-curves, it is easy to discover the valid region of \(h\) which corresponds to the line segment nearly parallel to the horizontal axis. For an alternative way to obtain the valid region of \(h\) and to show the convergence of the obtained series refer to [18]. The convergent region of the solution series given by CHAM for Example 4.1 is shown by Fig. 1 where the prime denotes differentiation with respect to the similarity variable \(t\). In Fig. 1, the value -1 is in the convergent region. However, Figs. 2 and 3 show that the CHPM (collocation-homotopy perturbation method) and CADM (collocation-Adomian decomposition method), for the linear operator (3.15), can not give accurate solutions corresponding to the large values of the \(t\) for the mentioned order of the approximation. Fig. 4 shows a comparison between Lagrange interpolation with a Chebychev collocation and analytic approximation solution given by CHAM. Table 1 shows the absolute errors of the approximate solutions given by CHAM, CHPM and CADM for Example 4.1. This table shows that more accurate solutions need the corresponding optimum values of the convergence-parameter. To achieve this purpose, we obtain the optimum values of the convergence-parameter numerically by using a simple stochastic function minimizer, namely differential evolution method. In the frame of the differential evolution method, considering the value of "RandomSeed", starting value for the random number generator, the default values of the Mathematica are used for the rest of values.

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<th>(t_i)</th>
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<th>(h_i)</th>
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<th>(|u_{3,15}(0.5,t_i;1)|)</th>
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6 Concluding remarks

In this paper, an algorithm based on the collocation and homotopy analysis methods (CHAM) was made applicable to initial-boundary value problems. In this respect the CHAM is shown to have practical potential. Dirichlet boundary conditions were considered. Similar results may be obtained when the boundary conditions are replaced by other kinds of boundary conditions. The choice of Lagrange polynomials is related to the fact that the first part of our algorithm is based on approximation generated by interpolation. The alternative of using Bernstein polynomials and sinc functions are also possible. On the other hand, if the approximation is obtained by orthogonal functions, the first part of our algorithm needs to be technically developed as reported in [15]. In this case, the use of wavelets can be considered. The same analysis is also applicable for the wave type equations. Finally, extensions of the method to higher order and dimensional can be accommodated. In other words, the present paper is only an introduction to the topic, and there remains a lot of work to do. We pointed out that the corresponding analytical and numerical solutions are obtained using Mathematica.

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