A new computational method for solving the first order linear fuzzy Fredholm integro-differential equations

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Abstract
Recently, fuzzy integro-differential equations (FIDEs) have attracted some interest. In this paper, we focus on linear fuzzy Fredholm integro-differential equation and propose a new method for solving it. In fact, using parametric form of fuzzy numbers we convert a linear fuzzy Fredholm integro-differential equation to a linear system of Fredholm integro-differential equations in crisp case. We use variational iteration method (VIM) to obtain solution of this system and hence obtain a fuzzy solution of the linear fuzzy Fredholm integro-differential equation. Finally, using the proposed method, we give some illustrative examples.

Keywords: Fuzzy functions; Fuzzy integro-differential equations; Fuzzy numbers; System of linear Fredholm integro-differential equations; Variational iteration method

1 Introduction

Prior to discussing fuzzy integro-differential equations and their solving, it is necessary to present an appropriate brief introduction to preliminary topics such as fuzzy numbers and fuzzy calculus. The concept of fuzzy sets, was originally introduced by Zadeh [26], led to the definition of fuzzy numbers and its implementation in fuzzy control [3] and approximate reasoning problems [27]. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [16], Nahmias [17], Dubois and Prade [4], all of them observed fuzzy numbers as a collection of $\alpha$-levels, $0 \leq \alpha \leq 1$.

Goetschel and Voxman [8] suggested a new approach. They represented fuzzy number as a parameterized triple (see Section 2) and then embedded the set of fuzzy numbers into a topological vector space. This enabled them to design the basics of a fuzzy calculus. The subject of embedding fuzzy numbers in either a topological or a Banach space was investigated also by Puri and Ralescu [21, 22], Kaleva [12] and Ouyang [20]. Solving integro-differential equations requires appropriate and applicable definitions of fuzzy function, the fuzzy derivative and the fuzzy integral of a fuzzy function. The fuzzy mapping function was introduced by Chang and Zadeh [3]. Later, Dubois and Prade [5] presented an elementary fuzzy calculus based on the extension principle [26]. Puri and Ralescu [21] suggested two definitions for fuzzy derivative of fuzzy functions. The first method was based on the $H$-difference notation and was further investigated by Kaleva [12]. The second method was derived from the embedding technique and was followed by Goetschel and Voxman [8] who gave it a more applicable representation. The concept of integration
of fuzzy functions was first introduced by Dubois and Prade [5]. Alternative approaches were later suggested by Goetschel and Voxman [8], Kaleva [12], Nanda [18] and others. While Goetschel and Voxman [8] preferred a Riemann integral type approach, Kaleva [12] chose to define the integral of fuzzy function, using the Lebesgue-type concept for integration. In this work we concentrate on solving fuzzy Fredholm integro-differential equation (FFIDE). In Section 2, we briefly present the basic notations of fuzzy number, fuzzy function, fuzzy derivative and fuzzy integral. FFIDE and parametric form of the FFIDE are discussed in Section 3. We observe parametric form of the FFIDE is a system of Fredholm integro-differential equations in crisp case. In Section 4, we state the basic concepts of the variational iteration method [7, 13, 14, 15]. In Section 5, we apply variational iteration method on the linear system of Fredholm integro-differential equations. In Section 6, we use variational iteration method for solving of linear system of Fredholm integro-differential equations produced by the FFIDE. We shall give some examples to illustrate our method in Section 7. We conclude in Section 8.

2 Preliminaries

In this section, we review the fundamental notations of fuzzy set theory to be used throughout this paper.

Definition 2.1. A fuzzy number is a fuzzy set \( u: \mathbb{R}^1 \rightarrow I = [0, 1] \) which satisfies

i. \( u \) is upper semicontinuous.

ii. \( u(x) = 0 \) outside some interval \([c, d]\).

iii. There are real numbers \( a, b : c \leq a \leq b \leq d \) for which

1. \( u(x) \) is monotonic increasing on \([c, a]\),
2. \( u(x) \) is monotonic decreasing on \([b, d]\),
3. \( u(x) = 1, a \leq x \leq b \).

The set of all fuzzy numbers (as given by Definition 2.1) is denoted by \( E^1 \). An alternative definition or parametric form of a fuzzy number which yields the same \( E^1 \) is given by Kaleva [12].

Definition 2.2. A fuzzy number \( u \) is a pair \((u, \pi)\) of functions \( u(r), \pi(r) ; 0 \leq r \leq 1 \) which satisfying the following requirements:

i. \( u(r) \) is a bounded monotonic increasing left continuous function,

ii. \( \pi(r) \) is a bounded monotonic decreasing left continuous function,

iii. \( u(r) \leq \pi(r), 0 \leq r \leq 1 \).

For arbitrary \( u = (u, \pi) \), \( v = (v, \gamma) \) and \( k > 0 \) we define addition \((u + v)\) and multiplication by \( k \) as

\[
(u + v)(r) = u(r) + v(r), \quad (u + v)(r) = \pi(r) + \gamma(r),
\]

\[
(ku)(r) = ku(r), \quad (ku)(r) = k\pi(r).
\]

The collection of all the fuzzy numbers with addition and multiplication as defined by Eqs. (2.1) and (2.2) is denoted by \( E^1 \) and is a convex cone. It can be shown that Eqs. (2.1) and (2.2) are equivalent to the addition and multiplication as defined by using the \( \alpha \)-cut approach [8] and the extension principles [19].

We will next define the fuzzy function notation and a metric \( D \) in \( E^1 \) [8].

Definition 2.3. For arbitrary fuzzy numbers \( u = (u, \pi) \) and \( v = (v, \gamma) \) the quantity

\[
D(u, v) = \sup_{0 \leq r \leq 1} \left\{ \max\{|u(r) - v(r)|, |\pi(r) - \gamma(r)|\} \right\}
\]

is the distance between \( u \) and \( v \).
This metric is equivalent to the one used by Puri and Ralescu [21] and Kaleva [12]. It is shown [23] that \((E^1, D)\) is a complete metric space.

**Definition 2.4.** A function \(f : \mathbb{R}^1 \rightarrow E^1\) is called a fuzzy function. If for arbitrary fixed \(t_0 \in \mathbb{R}^1\) and \(e > 0\), a \(\xi > 0\) such that
\[
|t - t_0| < \xi \implies D[f(t), f(t_0)] < \varepsilon
\]
exists, \(f\) is said to be continuous.

The concept of fuzzy differentiation was initially introduced by Dubois and Prade [5]. Later Puri and Ralescu [21] proposed two approaches for defining the fuzzy derivative. The first, based on the \(H\)-difference notation regards \(E^1\) as a universe. The second approach suggested also by Goetschel and Voxman [8], considers \(E^1\) (via embedding) as a subset of a larger Banach space with the metric \(D\) and yields the following definition.

**Definition 2.5.** Let \(f : \mathbb{R}^1 \rightarrow E^1\) be a fuzzy function (where \(E^1\) is a subset of a Banach space) and let \(t_0 \in \mathbb{R}^1\). The derivative \(f'(t_0)\) of \(f\) at the point \(t_0\) is defined by
\[
f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}
\]
provided that this limit taken with respect to the metric \(D\), exists.

This metric is equivalent to the one used by Puri and Ralescu [21] and Kaleva [12]. It is shown [23] that \((E^1, D)\) is a complete metric space.

**Definition 2.6.** A function \(f : \mathbb{R}^1 \rightarrow E^1\) is called a fuzzy function. If for arbitrary fixed \(t_0 \in \mathbb{R}^1\) and \(e > 0\), a \(\xi > 0\) such that
\[
|t - t_0| < \xi \implies D[f(t), f(t_0)] < \varepsilon
\]
exists, \(f\) is said to be continuous.

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\]
provided that this limit taken with respect to the metric \(D\), exists.

The elements \(f(t_0 + h), f(t_0)\) at the right-hand side of Eq. (2.5) are observed as elements in the Banach space \(B = \overline{C}[0,1] \times \overline{C}[0,1]\) [25]. Thus if \(f(t_0 + h) = (a, \overline{a})\) and \(f(t_0) = (\overline{b}, \overline{b})\), the difference is simply
\[
f(t_0 + h) - f(t_0) = (a - \overline{b}, \overline{a} - \overline{b}).
\]
Clearly \([f(t_0 + h) - f(t_0)] / h\) may not be a fuzzy number for all \(h\). However, if it approaches \(f'(t_0)\) in \(B\) and \(f'(t_0)\) is also a fuzzy number (i.e., in \(E^1\)), this number is the fuzzy derivative of \(f\) at \(t_0\). In this case if \(f = (\underline{f}, \overline{f})\) it can be easily shown that
\[
f'(t_0) = (\underline{f}'(t_0), \overline{f}'(t_0)),
\]
where \(\underline{f}'\) and \(\overline{f}'\) are the classic derivatives of \(\underline{f}\) and \(\overline{f}\), respectively. For example, we consider the fuzzy function
\[
f(t; r) = (rt^2, (2 - r^2)t), \quad 0 \leq t \leq 1,
\]
for which
\[
\lim_{h \to 0} \frac{f(t_0 + h; r) - f(t_0; r)}{h} = (2rt_0, (2 - r^2)),
\]
is a fuzzy number for \(t : 0 \leq t \leq \frac{1}{r}\). If \(\frac{1}{r} < t \leq 1\) the right-hand side of Eq. (2.7) is not a fuzzy number, because for \(\frac{1}{r} < t \leq 1\) and \(r = 1\) we conclude that \(2rt_0 > (2 - r^2)\). Consequently, \(f'(t)\) does not exist for \(\frac{1}{r} < t \leq 1\).

Throughout this work we also consider fuzzy functions which are defined and differentiable only over a finite interval \([a, b]\) (we simply replace \(\mathbb{R}^1\) by \([a, b]\) in Definitions 2.4 and 2.5).

We now follow Goetschel and Voxman [8] and define the integral of a fuzzy function using the Riemann integral concept.

**Definition 2.6.** Let \(f : [a, b] \rightarrow E^1\). For each partition \(p = \{t_0, t_1, \ldots, t_n\}\) of \([a, b]\) and for arbitrary \(\xi_i : t_{i-1} \leq \xi_i \leq t_i, 1 \leq i \leq n\) let
\[
R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).
\]
The definite integral of \(f(t)\) over \([a, b]\) is
\[
\int_a^b f(t)dt = \lim_{\max_{1 \leq i \leq n}|(t_i - t_{i-1})| \to 0} R_p,
\]
provided that this limit exists in the metric \(D\).
If the fuzzy function \( f(t) \) is continuous in the metric \( D \), its definite integral exists [8]. Furthermore,
\[
\left( \int_a^b f(t;r) dt \right) = \int_a^b f(t;r) dt, \quad \left( \int_a^b f(t;r) dt \right) = \int_a^b f(t;r) dt.
\]
(2.10)

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [12]. However, if \( f(t) \) is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eqs. (2.8) and (2.9) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [8, 12].

3 Fuzzy integro-differential equation

We consider the first order Fredholm integro-differential equation as follows
\[
F'(t) = f(t) + \beta \int_a^b K(s,t)F(s)ds,
\]
(3.11)
where \( \beta > 0 \), \( K(s,t) \) is an arbitrary kernel function over the square \( a \leq s, t \leq b \) and \( f(t) \) is a function of \( t : a \leq t \leq b \). If \( f(t) \) is a crisp function then the solutions of Eq. (3.11) are crisp as well. However, if \( f(t) \) is a fuzzy function this equation may only possess fuzzy solution. Also, the kind of used fuzzy derivative in Eq. (3.11) is presented in Definition 2.5.

Now, we introduce parametric form of a fuzzy Fredholm integro-differential equation (FFIDE) with respect to Definition 2.2. Let \( \left( f(t;\tau), f(t;\tau) \right), (F(t;\tau), F(t;\tau)) \) and \( (F'(t;\tau), F'(t;\tau)) \), \( 0 \leq \tau \leq 1 \) and \( t \in [a,b] \) are parametric form of \( f(t) : F(t) \) and \( F'(t) \), respectively. Then, parametric form of FFIDE is as follows:
\[
\begin{align*}
F'(t;r) &= f(t;r) + \int_a^b K(s,t)F(s;r)ds, \\
F'(t;r) &= \bar{f}(t;r) + \int_a^b K(s,t)\bar{F}(s;r)ds,
\end{align*}
\]
(3.12)
where
\[
K(s,t)F(s;r) = \begin{cases} K(s,t)F(s;r), & K(s,t) \geq 0, \\
K(s,t)\bar{F}(s;r), & K(s,t) < 0,
\end{cases}
\]
(3.13)
and
\[
K(s,t)\bar{F}(s;r) = \begin{cases} K(s,t)\bar{F}(s;r), & K(s,t) \geq 0, \\
K(s,t)\bar{F}(s;r), & K(s,t) < 0,
\end{cases}
\]
(3.14)
for each \( 0 \leq \tau \leq 1 \) and \( a \leq t \leq b \). We can see that (3.12) is a system of linear Fredholm integro-differential equations in crisp case for each \( 0 \leq \tau \leq 1 \) and \( a \leq t \leq b \).

In the next section, we state the basic concepts of the variational iteration method.

4 Variational iteration method

The Variational iteration method (VIM) is proposed by He [9, 10] as a modification of a general Lagrange multiplier method [11]. This method has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions [1, 2, 24]. To illustrate its basic idea of the technique, we consider following general nonlinear system:
\[
L[u(t)] + N[u(t)] = g(t),
\]
(4.15)
where \( L \) is linear operator, \( N \) is a nonlinear operator, and \( g(t) \) is a given continuous function. The basic character of the method is to construct a correction functional for system (4.15), which reads
\[
u_{n+1}(t) = u_n(t) + \int_a^t \lambda(\tau)\{Lu_n(\tau) + Nu_n(\tau) - g(\tau)\}d\tau,
\]
(4.16)
where \( \lambda (\tau) \) is a general Lagrangian multiplier \([9, 10, 11]\) which can be identified optimally via variational theory, the subscript \( n \) denotes the \( n \)th-order approximation and \( \tilde{u}_n \) is considered as a restricted variation \([6]\), i.e. \( \delta \tilde{u}_n = 0 \).

For example, when \( L \equiv \frac{d}{dt} \), we can construct the following correction functional

\[
\begin{align*}
u_{n+1}(t) &= u_n(t) + \int_0^t \lambda(\tau) \left( u'_n(\tau) + Nu_n(\tau) - g(\tau) \right) d\tau,
\end{align*}
\]

(4.17)
calculating variation with respect to \( u_n \), noticing that \( \delta u_n(0) = 0 \), yields

\[
\begin{align*}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(\tau) \left( u'_n(\tau) + Nu_n(\tau) - g(\tau) \right) d\tau \\
&= \delta u_n(t) + \delta \int_0^t \lambda(\tau) u'_n(\tau) d\tau \\
&= (1 + \lambda(\tau)) \delta u_n(t) |_{\tau=t} - \int_0^t \lambda'(\tau) \delta u_n(\tau) d\tau = 0.
\end{align*}
\]

(4.18)

Therefore, we have the following stationary conditions:

\[
\begin{align*}
\lambda'(\tau) &= 0, \quad (4.19) \\
1 + \lambda(\tau) &= 0 |_{\tau=t}.
\end{align*}
\]

(4.20)

So, the Lagrange multiplier can be readily identified

\[
\lambda(\tau) = -1.
\]

(4.21)

Substituting this value of the Lagrange multiplier into functional (4.17) gives the iteration formula:

\[
u_{n+1}(t) = u_n(t) - \int_0^t \left( u'_n(\tau) + Nu_n(\tau) - g(\tau) \right) d\tau.
\]

(4.22)

Iteration formula (4.22) will give several approximations, and the exact solution is obtained at the limit of the resulting successive approximations.

## 5 VIM for linear system of Fredholm integro-differential equations

In this section, we consider the variational iteration method for linear system of Fredholm integro-differential equations of the form

\[
U'(t) = F(t) + \int_a^b K(s,t)U(s)ds,
\]

(5.23)

where

\[
U(t) = (u_1(t), u_2(t), \ldots, u_m(t))^T,
\]

\[
F(t) = (f_1(t), f_2(t), \ldots, f_m(t))^T,
\]

\[
K(s,t) = [k_{ij}(s,t)], \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, m.
\]

Consider the \( i \)th equation of (5.23):

\[
u'_i(t) = f_i(t) + \int_a^b \sum_{j=1}^m k_{ij}(s,t)u_j(s)ds.
\]

(5.24)

In view of the variational iteration method, we construct a correction functional in the following form

\[
u_{m+1}(t) = u_m(t) + \int_0^t \lambda_i(\tau) \left( u'_m(\tau) - f_i(\tau) - \int_a^b \sum_{j=1}^m k_{ij}(s,\tau)\tilde{u}_j(s)ds \right) d\tau,
\]

(5.25)
where $\tilde{u}_{j\alpha}$ is considered as restricted variational, i.e. $\delta \tilde{u}_{j\alpha} = 0$.

To find optimal value of $\lambda_i(\tau)$, we have

$$
\delta u_{i\alpha + 1}(t) = \delta u_{i\alpha}(t) + \delta \int_0^t \lambda_i(\tau)\{u'_i(\tau) - f_i(\tau) - \int_a^b k_{ij}(s, \tau)\tilde{u}_{j\alpha}(s)ds\}d\tau
= \delta u_{i\alpha}(t) + \delta \int_0^t \lambda_i(\tau)\{u'_i(\tau)\}d\tau
= (1 + \lambda_i(\tau))\delta u_{i\alpha}(t)|_{\tau=t} - \int_0^t \lambda'_i(\tau)\delta u_{i\alpha}(\tau)d\tau = 0.
$$

(5.26)

The stationary conditions can be obtained as follows:

$$
\lambda'_i(\tau) = 0,
$$

(5.27)

$$
1 + \lambda_i(\tau) = 0|_{\tau=t}.
$$

(5.28)

Therefore, the Lagrange multiplier can be identified as

$$
\lambda_i(\tau) = -1.
$$

(5.29)

Substituting this value of the Lagrange multiplier into functional (5.25) gives the iteration formula:

$$
u_{i\alpha + 1}(t) = u_{i\alpha}(t) - \int_0^t \{u'_i(\tau) - f_i(\tau) - \int_a^b k_{ij}(s, \tau)u_{j\alpha}(s)ds\}d\tau.
$$

(5.30)

Therefore, we can write the following iteration formulas

$$
\begin{align*}
{u_{i\alpha + 1}(t) &= u_{i\alpha}(t) - \int_0^t \{u'_i(\tau) - f_i(\tau) - \int_a^b k_{ij}(s, \tau)u_{j\alpha}(s)ds\}d\tau, \\
{u_{2\alpha + 1}(t) &= u_{2\alpha}(t) - \int_0^t \{u'_i(\tau) - f_2(\tau) - \int_a^b k_{ij}(s, \tau)u_{j\alpha}(s)ds\}d\tau, \\
&\vdots \\
u_{m\alpha + 1}(t) &= u_{m\alpha}(t) - \int_0^t \{u'_i(\tau) - f_m(\tau) - \int_a^b k_{ij}(s, \tau)u_{j\alpha}(s)ds\}d\tau.
\end{align*}
$$

(5.31)

Iteration formulas (5.31) will give several approximations, and the exact solution is obtained at the limit of the resulting successive approximations.

6 VIM for system produced by the FFIDE

In this section, we apply variational iteration method on the system of Fredholm integro-differential equations produced by the FFIDE, i.e. system of Fredholm integro-differential equations (3.12), and obtain iteration formulas for it.

Prior to applying VIM for system (3.12), we suppose that the kernel $K(s, t)$ is nonnegative for $a \leq s \leq c$ and negative for $c \leq s \leq b$. Therefore, we rewrite system (3.12) in the following form

$$
\begin{align*}
F'(t; r) &= f(t; r) + \beta \int_a^c K(s, t)F(s; r)ds + \beta \int_a^b K(s, t)F(s; r)ds, \\
F'(t; r) &= f(t; r) + \beta \int_a^c K(s, t)F(s; r)ds + \beta \int_a^b K(s, t)F(s; r)ds.
\end{align*}
$$

(6.32)
Eq. (6.32) is a system of linear Fredholm integro-differential equations in crisp case for each $0 \leq r \leq 1$ and $a \leq t \leq b$, therefore using VIM and Eq. (5.31), we can obtain the following iteration formulas:

$$
E_{n+1}(t;r) = E_n(t;r)
$$
$$
- \int_0^t \left( E'_n(t;r) - f(t;r) - \beta \int_a^r K(s,t)E_n(s;r)ds - \beta \int_a^b K(s,t)F_n(s;r)ds \right) d\tau,
$$
(6.33)

$$
F_{n+1}(t;r) = F_n(t;r)
$$
$$
- \int_0^t \left( F'_n(t;r) - \overline{f}(t;r) - \beta \int_a^r K(s,t)F_n(s;r)ds - \beta \int_a^b K(s,t)F_n(s;r)ds \right) d\tau.
$$

Using the current iteration formulas, we can find a solution of system (6.32) and hence obtain a fuzzy solution of the linear fuzzy Fredholm integro-differential equation.

### 7 Test problems

In this section, we apply variational iteration method for solving four fuzzy Fredholm integro-differential equations.

**Example 7.1.** Consider the fuzzy Fredholm integro-differential equation with

$$
f(t;r) = \frac{r}{6}(4 - 3t),
$$
(7.34)

$$
\overline{f}(t;r) = \frac{2 - r}{6}(4 - 3t),
$$
(7.35)

and kernel

$$
K(s,t) = s + t, \quad 0 \leq s, t \leq 1,
$$
(7.36)

and $a = 0$, $b = 1$, $\beta = 1$ and $E(0;r) = \overline{F}(0;r) = 0$.

In this example, $K(s,t) \geq 0$ for each $0 \leq s \leq 1$. Therefore using Eq. (6.33), we can write the following iteration formulas

$$
E_{n+1}(t;r) = E_n(t;r) - \int_0^t \left( E'_n(t;r) - \frac{r}{6}(4 - 3t) - \int_0^r (s + t)E_n(s;r)ds \right) d\tau,
$$
(7.37)

$$
F_{n+1}(t;r) = F_n(t;r) - \int_0^t \left( F'_n(t;r) - \frac{2 - r}{6}(4 - 3t) - \int_0^r (s + t)F_n(s;r)ds \right) d\tau,
$$

with the initial approximations $E_0(t;r) = 0$ and $\overline{F}_0(t;r) = 0$.

By using iteration formulas (7.37) and MATLAB software, we obtain

$$
E_1(t;r) = \frac{5}{11} rt - \left[ \frac{1}{4}rt^2 \right],
$$

$$
\overline{F}_1(t;r) = \left( \frac{4}{3} - \frac{5}{7} \right) rt + \left[ \frac{5}{2} rt^2 - \frac{1}{2}t^2 \right],
$$

$$
E_2(t;r) = \frac{119}{119} rt - \left[ \frac{1}{8}rt^2 \right],
$$

$$
\overline{F}_2(t;r) = \left( \frac{119}{17} - \frac{119}{17} \right) rt + \left[ \frac{1}{8} rt^2 - \frac{1}{2}t^2 \right].
$$
\[ F_3(t; r) = \frac{787}{576}rt - \left[ \frac{37}{576}rt^2 \right], \]
\[ \mathcal{F}_3(t; r) = \left( \frac{287}{452} - \frac{787}{576} \right)t + \left[ \frac{37}{576}rt^2 - \frac{37}{256}rt^2 \right], \]
\[ \vdots \]
\[ F_{10}(t; r) = 0.9984 rt - 0.0012 rt^2, \]
\[ \mathcal{F}_{10}(t; r) = (1.9967 - 0.9984 r)t + 0.0024 rt^2 - 0.0012 t^2, \]
\[ \vdots \]

and so on. Therefore, it is obvious that this solution is convergent to the exact solution
\[ E(t; r) = rt, \quad (7.38) \]
\[ \mathcal{F}(t; r) = (2 - r)t. \quad (7.39) \]

**Example 7.2.** Consider the fuzzy Fredholm integro-differential equation with
\[ f(t; r) = 2t[r^5 + 3r^3] + \frac{1}{4}t^2[6 - 2r - r^5 - 3r^3], \quad (7.40) \]
\[ \mathcal{f}(t; r) = 4t[3 - r] + \frac{1}{4}t^2[r^5 + 3r^3 + 2r - 6], \quad (7.41) \]

and kernel
\[ K(s, t) = t^2s, \quad -1 \leq s, t \leq 1, \quad (7.42) \]

and \( a = -1, b = 1, \beta = 1 \) and \( E(0; r) = \mathcal{F}(0; r) = 0. \)

In this example, \( K(s, t) \leq 0 \) for each \(-1 \leq s \leq 0 \) and \( K(s, t) \geq 0 \) for each \( 0 \leq s \leq 1 \). Therefore using Eq. (6.33), we can write the following iteration formulas
\[
\begin{cases}
E_{n+1}(t; r) = E_n(t; r) - \int_0^t \left( E_n'(\tau; r) - 2\tau[r^5 + 3r^3] - \frac{1}{3}\tau^2[6 - 2r - r^5 - 3r^3] \right) d\tau \\
\quad - \int_0^t \tau^2 s \mathcal{F}_n(s; r) ds - \int_0^t \tau^2 s E_n(s; r) ds d\tau,
\end{cases}
\]
\[ \mathcal{F}_{n+1}(t; r) = \mathcal{F}_n(t; r) - \int_0^t \left( \mathcal{F}_n'(\tau; r) - 4\tau[3 - r] - \frac{1}{3}\tau^2[r^5 + 3r^3 + 2r - 6] \right) d\tau \\
\quad - \int_0^t \tau^2 s \mathcal{F}_n(s; r) ds - \int_0^t \tau^2 s \mathcal{F}_n(s; r) ds d\tau, \quad (7.43) \]

with the initial approximations \( E_0(t; r) = 0 \) and \( \mathcal{F}_0(t; r) = 0 \).

By using iteration formulas (7.43) and MATLAB software, we can obtain
\[ E_1(t; r) = (r^5 + 3r^3)t^2 - t^3[\frac{1}{12}r^5 + \frac{1}{5}r^3 + \frac{1}{8}r - \frac{1}{2}], \]
\[ \mathcal{F}_1(t; r) = (6 - 2r)t^2 + t^3[\frac{1}{12}r^5 + \frac{1}{4}r^3 + \frac{1}{8}r - \frac{1}{2}], \]
\[ F_2(t; r) = (r^5 + 3r^3)t^2, \]
\[ F_2(t; r) = (6 - 2r)t^2, \]
\[ F_3(t; r) = (r^5 + 3r^3)t^2, \]
\[ F_3(t; r) = (6 - 2r)t^2, \]

Here, with two iterations, the exact solution is obtained as follows
\[ F(t; r) = (r^5 + 3r^3)t^2, \] (7.44)
\[ F(t; r) = (6 - 2r)t^2. \] (7.45)

**Example 7.3.** Consider the fuzzy integro-differential equation with
\[ f(t; r) = (r^2 + r)(15t^2 - 2)/5, \] (7.46)
\[ f(t; r) = (4 - r - r^3)(15t^2 - 2)/5, \] (7.47)
and kernel
\[ K(s, t) = s + 1, \quad -1 \leq s, t \leq 1, \] (7.48)
and \( a = -1, b = 1, \beta = 1 \) and \( E(0; r) = \overline{F}(0; r) = 0. \)

In this example, \( K(s, t) \geq 0 \) for each \(-1 \leq s, t \leq 1\). Therefore using Eq. (6.33), we can write the following iteration formulas
\[
\begin{cases}
F_{n+1}(t; r) = F_n(t; r) \\
- \int_0^1 \{f_n'(\tau; r) - (r^2 + r)(15\tau^2 - 2)/5 - \int_1^\beta (s + 1)F_n(s; r)ds\}d\tau,
\end{cases}
\]
(7.49)
\[
\begin{cases}
\overline{F}_{n+1}(t; r) = \overline{F}_n(t; r) \\
- \int_0^1 \{\overline{F}_n'(\tau; r) - (4 - r - r^3)(15\tau^2 - 2)/5 - \int_1^\beta (s + 1)\overline{F}_n(s; r)ds\}d\tau,
\end{cases}
\]
with the initial approximations \( F_0(t; r) = 0 \) and \( \overline{F}_0(t; r) = 0. \)

By using iteration formulas (7.49) and MATLAB software, we obtain
\[ E_1(t; r) = (r^2 + r)t^3 - [\frac{2}{3}r^2 t + \frac{2}{3}rt], \]
\[ F_1(t; r) = (4 - r - r^3)t^3 + [\frac{2}{3}r^3 t + \frac{3}{2}rt - \frac{8}{3}t], \]
write the following iteration formulas 
and so on. Therefore, it is obvious that this solution is convergent to the exact solution

\[ F(t; r) = (r^2 + r)t^3 - \left[ \frac{1}{15} r^2 t + \frac{2}{15} rt \right], \]

\[ F_2(t; r) = (4 - r - r^3)t^3 + \left[ \frac{4}{15} r^2 t + \frac{2}{15} rt - \frac{16}{15} t \right], \]

\[ F_3(t; r) = (r^2 + r)t^3 - \left[ \frac{8}{45} r^2 t + \frac{8}{45} rt \right], \]

\[ F_4(t; r) = (4 - r - r^3)t^3 + \left[ \frac{8}{45} r^2 t + \frac{8}{45} rt - \frac{32}{45} t \right], \]

\[ \vdots \]

\[ F_{10}(t; r) = (r^2 + r)t^3 - \left[ \frac{512}{32805} r^2 t + \frac{512}{32805} rt \right], \]

\[ F_{10}(t; r) = (4 - r - r^3)t^3 + \left[ \frac{512}{32805} r^2 t + \frac{512}{32805} rt - \frac{2048}{32805} t \right], \]

\[ \vdots \]

and so on. Therefore, it is obvious that this solution is convergent to the exact solution

\[ E(t; r) = (r^2 + r)t^3, \quad (7.50) \]

\[ F(t; r) = (4 - r - r^3)t^3. \quad (7.51) \]

**Example 7.4.** Consider the fuzzy Fredholm integro-differential equation with

\[ f(t; r) = \frac{5 - 3r}{12} (2t - 1)^2 + r, \quad (7.52) \]

and kernel

\[ K(s, t) = (2t - 1)^2(1 - 2s), \quad 0 \leq s, t \leq 1, \quad (7.54) \]

and \( a = 0, b = 1, \beta = 1 \) and \( E(0; r) = F(0; r) = 0. \)

In this example, \( K(s, t) \geq 0 \) for each \( 0 \leq s \leq \frac{1}{2} \) and \( K(s, t) \leq 0 \) for each \( \frac{1}{2} \leq s \leq 1. \) Therefore using Eq. (6.33), we can write the following iteration formulas

\[
\begin{align*}
E_{n+1}(t; r) &= E_n(t; r) - \int_0^1 \{ E'_n(\tau; r) - \frac{5-3r}{12} (2\tau - 1)^2 - r \} \, d\tau \\
&- \int_0^1 (2\tau - 1)^2 (1 - 2s) E_n(s; r) \, ds - \int_0^1 \frac{1}{2} (2\tau - 1)^2 (1 - 2s) F_n(s; r) \, ds \, d\tau, \\
\end{align*}
\]

\[
\begin{align*}
F_{n+1}(t; r) &= F_n(t; r) - \int_0^1 \{ F'_n(\tau; r) - \frac{3r-1}{12} (2\tau - 1)^2 + r \} \, d\tau \\
&- \int_0^1 (2\tau - 1)^2 (1 - 2s) F_n(s; r) \, ds - \int_0^1 \frac{1}{2} (2\tau - 1)^2 (1 - 2s) E_n(s; r) \, ds \, d\tau, \\
\end{align*}
\]

with the initial approximations \( E_0(t; r) = 0 \) and \( F_0(t; r) = 0. \)

By using iteration formulas (7.55) and MATLAB software, we obtain
$ F_1(t; r) = \frac{3}{8} rt + \left[ \frac{5}{9} t^3 - \frac{1}{12} t^2 r - \frac{7}{8} t^2 r + \frac{5}{12} t r + \frac{5}{12} t \right], $  \\
$ F_1(t; r) = \left( \frac{15}{12} - \frac{3}{4} r \right) t + \left[ \frac{1}{9} t^3 r - \frac{1}{6} t^2 r + \frac{1}{2} t r + \frac{1}{6} t \right], $  \\
$ F_2(t; r) = \frac{47}{48} rt + \left[ \frac{51}{50} t^3 - \frac{123}{80} t^2 r + \frac{1}{24} t^2 r + \frac{123}{80} t r + \frac{123}{80} t \right], $  \\
$ F_2(t; r) = \left( \frac{1421}{720} - \frac{47}{48} r \right) t + \left[ \frac{1}{12} t^3 r - \frac{123}{80} t^3 - \frac{1}{24} t^2 r + \frac{123}{80} t^2 \right], $  \\
$ F_3(t; r) = \frac{575}{576} rt + \left[ \frac{83}{3200} t^3 - \frac{1}{24} t^3 r - \frac{83}{3200} t^2 r + \frac{1}{24} t^2 r + \frac{83}{3200} t \right], $  \\
$ F_3(t; r) = \left( \frac{86333}{83200} - \frac{575}{576} r \right) t + \left[ \frac{1}{24} t^3 r - \frac{83}{3200} t^3 - \frac{1}{24} t^2 r + \frac{83}{3200} t^2 \right], $  \\
$ F_6(t; r) = \frac{995327}{995328} rt + \left[ \frac{9311}{4998400} t^3 - \frac{1}{24} t^3 r - \frac{9311}{4998400} t^2 r + \frac{1}{24} t^2 r + \frac{9311}{4998400} r \right], $  \\
$ F_6(t; r) = \left( \frac{18662390561}{9931200000} - \frac{995327}{995328} r \right) t + \left[ \frac{1}{24} t^3 r - \frac{9439}{4998400} t^3 - \frac{1}{24} t^2 r + \frac{9439}{4998400} t^2 \right], $  \\
$ \vdots $  \\
and so on. Therefore, the exact solution is obtained as follows

\[ F(t; r) = rt, \]

\[ F(t; r) = (2 - r)t. \quad (7.56) \]

\[ F(t; r) = (2 - r)t. \quad (7.57) \]

8 Conclusion

In this paper, we used variational iteration method (VIM) to obtain solution of the first order linear fuzzy Fredholm integro-differential equation of the second kind. We emphasize that this work which presents applicable computational methods, may help to narrow the existing gap between the theoretical research on fuzzy integro-differential equations (FIDEs).

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