Positive bi-exponential interpolation

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Abstract
A function $b \exp(cx) + d \exp(fx)$ is called positive bi-exponential if $b > 0$, $d > 0$, and $c \neq f$. A positive bi-exponential function interpolates a function at equidistant nodes if and only if the function is strictly log-convex. A positive bi-exponential function is itself strictly log-convex. The sum of two not necessarily strictly log-convex functions is strictly log-convex if their derivatives are distinct.

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1 Introduction
The real motivation for this paper came from physics. We could study an example of a mixing problem when water with salt in a tank is replaced by plain water which is described by a differential equation leading to an exponential function $b \exp(cx)$, $b > 0$. Replacement of a solution by water in two tanks leads to a positive bi-exponential function. The radioactive decay of one element also leads to an exponential function while two distinct elements would lead to a positive bi-exponential function. Another application may be found in evaluation of an investment by comparing it to the interest rate if we had put the money in the bank. This yields an exponential function. If we had put the money in two banks, we would obtain a positive bi-exponential function. Many similar examples could be found.

We may be asked to find the parameters of a bi-exponential function fitting experimental data. Instead of getting the starting values for iterative processes by a fluke, we prefer interpolation.

A general case of interpolation at equidistant points by a sum of exponential functions \( \sum \alpha_j \exp(\sigma_j x) \) with no restrictions to the signs of coefficients is presented in Sidi [3] and includes further references.

We study the special case in which the sum consists of two terms with positive \( \alpha_1, \alpha_2 \) and we show how and when we can calculate the coefficients.

Definition 1.1. A function $b \exp(cx) + d \exp(fx)$ is called positive bi-exponential if $c \neq f$, $b > 0$, and $d > 0$.

We include $0 = c \neq f$, even though the first term is constant. In the same way $c \neq f = 0$ yields a positive bi-exponential function even if the second term is constant, which means $b \exp(cx) + d$ is positive bi-exponential if $b$ and $d$ are positive and $c \neq 0$. Either $c$ or $f$ may be zero but not both.
We do not consider $b \exp(cx) + d \exp(fx)$ with $c = f$ acceptable as positive bi-exponential, the two terms could be written as one, $(b + d) \exp(cx)$.

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Definition 1.2. Let $D \subset \mathbb{R}$, $D$ finite or not, and let $F$ be a real function defined on $D$. We say that $F$ is convex if for all $x_1, x_2, x_3 \in D$ with $x_1 < x_2 < x_3$ we have $F(x_2) \leq F(x_1)(x_3 - x_2)/(x_3 - x_1) + F(x_2)(x_2 - x_1)/(x_3 - x_1)$. If $F(x_2) < F(x_1)(x_3 - x_2)/(x_3 - x_1) + F(x_2)(x_2 - x_1)/(x_3 - x_1)$ for all $x_1, x_2, x_3 \in D$ with $x_1 < x_2 < x_3$, we say $F$ is strictly convex.

Definition 1.3. A positive real function is called log-convex on a set $D \subset \mathbb{R}$ if its logarithm is convex. It is called strictly log-convex if its logarithm is strictly convex.

Since a bi-exponential function has four parameters, we use four interpolation nodes, $D = \{x_1, x_2, x_3, x_4\}$, where $x_1 < x_2 < x_3 < x_4$. The assumption of equidistant points, $x_2 - x_1 = x_3 - x_2 = x_4 - x_3$, enables us to find the solution of the interpolation problem or determine that the solution does not exist. The value of the function at a point $x_i$ will be denoted by $y_i$ to simplify the notation.

We will use the fact that $\alpha \exp(\beta x) = \alpha \exp(\beta y) \exp(\beta (x - y))$ for any $\gamma$ and we may simplify the notation by assuming, without loss of generality, that $x_1 = 0 < x_2 < x_3 < x_4$ by taking $\gamma = x_1$. It is obvious that when an interpolation problem has a solution for $x_1, x_2, x_3, x_4$ then it has a solution for $x_1 = 0, x_2 - x_1, x_3 - x_1, x_4 - x_1$ and vice versa, the values of $y_1, y_2, y_3, y_4$ being the same. The parameters of one problem may be easily calculated once the parameters of the other one are known.

2 Interpolation by positive bi-exponential functions

We want to show that strict log-convexity is a sufficient condition for interpolation with a positive bi-exponential function. The proof of the following theorem is elementary and also shows an elementary way to calculate the coefficients $b, c, d, f$ of the positive bi-exponential interpolating function.

Theorem 2.1. Let $D = \{x_1, x_2, x_3, x_4\} \subset \mathbb{R}$, where $x_1 < x_2 < x_3 < x_4$ are equidistant. Let $F$ be a real strictly log-convex function defined on $D$. Then there are unique $b, c, d, f$, where $b > 0, d > 0, c \neq c$, such that $y_i = b \exp(c x_i) + d \exp(f x_i)$ for $i = 1, 2, 3, 4$.

Proof. Without loss of generality we assume a special choice of $x_1, x_2, x_3, x_4$ in which $x_1 = 0, 0 < x_2, x_3 = 2x_2, x_4 = 3x_2$. We recall what follows from strict log-convexity: $y_i > 0$ for $i = 1, 2, 3, 4, y_1 y_3 - y_2^2 > 0, y_2 y_4 - y_3^2 > 0$.

Step 1. We show the equivalence to a quadratic equation. The equations are:

$$b + d = y_1, \quad (2.1)$$

$$b \exp(c x_2) + d \exp(f x_2) = y_2, \quad (2.2)$$

$$b \exp(c x_2) + d \exp(f x_2) = y_3, \quad (2.3)$$

$$b \exp(c x_2) + d \exp(f x_2) = y_4, \quad (2.4)$$

We substitute $u = \exp(c x_2)$ and $z = \exp(f x_2)$ and write the equations as:

$$b + d = y_1, \quad (2.5)$$

$$bu + dz = y_2, \quad (2.6)$$

$$bu^2 + dz^2 = y_3, \quad (2.7)$$

$$bu^3 + dz^3 = y_4, \quad (2.8)$$

We eliminate $b$ by substituting $b = y_1 - d$.

$$y_1 y_3 - y_2^2 > 0, \quad (2.9)$$

$$y_2 y_4 - y_3^2 > 0. \quad (2.10)$$

$$y_1 y_3 - y_2^2 > 0. \quad (2.11)$$
If we assumed \( y_2 - y_1 u = 0 \), we would get \( u = y_2 / y_1 \) and substitute in (2.9) to obtain

\[
y_1 \frac{y_2}{y_1} - d \frac{y_2}{y_1} + dz = y_2
\]

which would be written as \( d(z - y_2 / y_1) = 0 \). We want to show this is impossible.

If we assume \( d = 0 \), from (2.6) and (2.7) we get \( bu = y_2 \) and \( bu^2 = y_3 \), from (2.5) we get \( b = y_1 \) and obtain \( y_1 u = y_2 \) and \( y_1 u^2 = y_3 \). We eliminate \( u \) and obtain \( y_1 y_3 = y_2^2 \), a contradiction.

Since \( d \neq 0 \), we must have \( z - y_2 / y_1 = 0 \) for \( d(z - y_2 / y_1) = 0 \) to be zero. Thus \( z = u = y_2 / y_1 \). If we assume \( z = u \), we write (2.6), (2.7) as \( (b + d)u = y_2 \), \( (b + d)u^2 = y_3 \), use (2.5) to replace \( b + d \) by \( y_1 \) and eliminate \( u \) to get \( y_1 y_3 = y_2^2 \), a contradiction.

Elimination of \( d \) follows from (2.9) as \( d(z - u) = y_2 - y_1 u \) is applied

\[
y_1 u^2 + d(z^2 - u^2) = y_3, \tag{2.12}
\]

\[
y_1 u^3 + d(z^3 - u^3) = y_4, \tag{2.13}
\]

As \( d(z^2 - u^2) = d(z - u)(z + u) \) and \( d(z^3 - u^3) = d(z - u)(z^2 + zu + u^2) \) we get

\[
y_1 u^2 + (y_2 - y_1 u)(z + u) = y_3, \tag{2.14}
\]

\[
y_1 u^3 + (y_2 - y_1 u)(z^2 + zu + u^2) = y_4, \tag{2.15}
\]

From (2.14) we obtain

\[
z = \frac{y_3 - y_2 u}{y_2 - y_1 u}, \tag{2.16}
\]

and immediately substitute in (2.15)

\[
y_1 u^3 + (y_2 - y_1 u)((y_3 - y_2 u)^2 + y_2 - y_1 u + u^2) = y_4. \tag{2.17}
\]

Now we multiply (2.17) by \( y_2 - y_1 u \) and rearrange the terms

\[
(y_1 y_3 - y_2^2)u^2 + (y_2 y_3 - y_1 y_4)u + y_2 y_4 - y_3^2 = 0 \tag{2.18}
\]

The discriminant is

\[
D = (y_2 y_3 - y_1 y_4)^2 - 4(y_1 y_3 - y_2^2)(y_2 y_4 - y_3^2). \tag{2.19}
\]

**Step 2.** We show the discriminant (2.19) is positive. We replace \( y_4 \) by \( y \) and define \( D_4(y) = (y_2 y_3 - y_1 y)^2 - 4(y_1 y_3 - y_2^2)(y_2 y - y_3^2) \). \( D_4(y) \) is a quadratic function and takes on its minimum at

\[
y_{\text{min}} = \frac{y_2 (3y_1 y_3 - 2y_2^2)}{y_1^2}. \tag{2.20}
\]

The minimum is

\[
D_4(y_{\text{min}}) = (y_2 y_3 - y_1 y_{\text{min}})^2 - 4(y_1 y_3 - y_2^2)(y_2 y_{\text{min}} - y_3^2)
\]

and after substituting (2.20) and simplification we see that

\[
D_4(y_{\text{min}}) = \frac{4}{y_1^2} (y_1 y_3 - y_2^2)^3
\]

is positive due to the strict log-convexity.

**Step 3.** We show both solutions are positive. Since, in the definition (2.19) of the discriminant, we have \( (y_2 y_3 - y_1 y_4)^2 > D \), we also have \( |y_{\text{min}} - y_2 y_4| > \sqrt{D} \). To show that \( y_1 y_4 - y_2 y_3 > 0 \), we multiply the inequality \( y_1 y_3 > y_2^2 \) by
well known that the sum of log-convex functions is log-convex. We rewrite the equation as

\[ b = \frac{(y_2 y_3 - y_1 y_4) - \sqrt{D}}{2(y_1 y_3 - y_2^2)}, \quad u_2 = \frac{(y_2 y_3 - y_1 y_4) + \sqrt{D}}{2(y_1 y_3 - y_2^2)}. \]

**Step 4.** There are precisely two solutions, not more. Let \( u \) be a solution of the quadratic equation. We substitute

\[ z = \frac{y_3 - y_2u}{y_2 - y_1u} \]

in (2.18) and obtain

\[(y_1 y_3 - y_2^2)(\frac{y_3 - y_2u}{y_2 - y_1u})^2 + (y_2 y_3 - y_1 y_4)\frac{y_3 - y_2u}{y_2 - y_1u} + y_2 y_4 - y_3^2 = 0.\]

After multiplying out, simplifying, collecting terms with like powers of \( u \), and, finally, after factoring the coefficients we rewrite the equation as

\[(y_1 y_3 - y_2^2)(\frac{y_3 - y_2u}{y_2 - y_1u})^2 + (y_2 y_3 - y_1 y_4)u + y_2 y_4 - y_3^2 = 0.\]

Since \( y_1 y_3 - y_2^2 \) is positive, we see that \( z = (y_3 - y_2u)/(y_2 - y_1u) \) satisfies the quadratic equation if and only if \( u \) does. It means that if \( u \) is the root, then so is \( z \). Since the quadratic equation has precisely two solutions, \( u_1 \) and \( u_2 \), they must be the same as \( z_1 = (y_3 - y_2u_1)/(y_2 - y_1u_1) \) and \( z_2 = (y_3 - y_2u_2)/(y_2 - y_1u_2) \) when the order of the roots does not matter. Or, equivalently, we can say that \( u_1 = z_2 \) and \( u_2 = z_1 \). It also means that when the two roots, \( u_1 \) and \( u_2 \), of the quadratic have been calculated, we do not have to calculate \( z_1 \) or \( z_2 \).

**Step 5.** We show that \( d > 0 \). If we substitute any real solution of (2.18) in the formula for \( z \) and subtract \( u \), we obtain

\[ z - u = \frac{y_3 - y_2u}{y_2 - y_1u} - u = \frac{y_3 - 2y_2u + y_1u^2}{y_2 - y_1u} \]

We can now write

\[ d = \frac{y_2 - y_1u}{z - u} = \frac{(y_2 - y_1u)^2}{y_3 - 2y_2u + y_1u^2}. \]

Knowing that \((y_2 - y_1u)^2 > 0\), we want to check whether the denominator could be zero or negative. We investigate the quadratic equation \( y_3 - 2y_2u + y_1u^2 = 0 \). Its discriminant, \( 4y_2^2 - 4y_1 y_3 = 4(y_2^2 - y_1 y_3) \), is clearly negative because of strict log-convexity. It means that \( y_3 - 2y_2u + y_1u^2 \) is positive for all \( u \) because at \( u = 0 \) it takes on the value \( y_3 > 0 \).

**Step 6.** We show that \( b > 0 \).

We now evaluate \( b = y_1 - d \) as

\[ b = y_1 - \frac{(y_2 - y_1u)^2}{y_3 - 2y_2u + y_1u^2} = \frac{y_1 y_3 - y_2^2}{y_3 - 2y_2u + y_1u^2} \]

and we already know from step 5 that the denominator is positive and so is the numerator because of strict log-convexity.

3 Strict log-convexity of positive bi-exponential function

It is easy to calculate the logarithm \( \ln b + cx \) of \( b \exp(cx) \) and see that \( b \exp(cx) \) is log-convex but not strictly. It is well known that the sum of log-convex functions is log-convex.

**Theorem 3.1.** A positive bi-exponential function is strictly log-convex.

**Proof.** Let \( f(x) = be^{cx} + de^{fx} \), \( F(x) = \ln f(x) \). We assume \( b > 0 \), \( d > 0 \), and, without loss of generality, \( c > f \). Then

\[ F'(x) = \frac{b e^{cx} + d e^{fx}}{b e^{cx} + d e^{fx}} = \frac{b(c - f)}{b + d e^{(f-c)x}} + f \]

is increasing because \( b + d e^{(f-c)x} \) is decreasing and \( b(c - f) \) is positive.
4 Midpoint convexity

A function $F$ defined on an interval $I$ is called midpoint convex or midconvex if for every $x_1, x_2 \in I$, $x_1 < x_2$, we have $F((x_1 + x_2)/2) \leq (F(x_1) + F(x_2))/2$. This inequality is called the midpoint inequality. It is known that if a function is also continuous on $I$, the midconvexity implies convexity, Roberts [2]. Here we are interested in the strict midconvexity, the inequality takes the form $F((x_1 + x_2)/2) < (F(x_1) + F(x_2))/2$. The proof of our results is a modification of the proof of a theorem on midconvexity implying convexity if the function is continuous. It is presented here to show that from interpolation of a function by positive bi-exponential functions at any four equidistant points we can deduce strict log-convexity of that function.

In our case, to obtain four equidistant points, we divide the interval $(x_1,x_4)$ into three parts of equal width. This is the reason why we define the midconvexity in a slightly more general manner.

**Definition 4.1.** A function $F$ defined on an interval $I$ is called $\alpha$-midconvex for fixed $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, if for every $x_1, x_2 \in I$, $x_1 < x_2$, we have $F(\alpha x_1 + \beta x_2) \leq \alpha F(x_1) + \beta F(x_2)$. This inequality is called an $\alpha$-midpoint inequality. If we have $F(\alpha x_1 + \beta x_2) < \alpha F(x_1) + \beta F(x_2)$ for all such $x_1, x_2$, the function $F(x)$ is called strictly $\alpha$-midconvex.

The usual midconvexity would be called a $1/2$-midconvexity by our definition. If the interval is divided into three parts, we talk about strict $1/3$-midconvexity. There are several ways of proving that midconvexity together with continuity imply convexity but here we have to use the idea we can extend for our purpose, such as the one in Niculescu [1].

**Theorem 4.1.** Let a continuous function $F$ defined on an interval $I$ be $\alpha$-midconvex for some fixed $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$. Then it is convex.

**Proof.** We assume the contrary that there are two points in $a_1, a_2 \in I$, with $a_1 < a_2$ for which there is an $x \in (a_1,a_2)$, $x = \lambda a_1 + (1-\lambda)a_2$ such that $F(x) > \lambda F(a_1) + (1-\lambda)F(a_2)$.

The idea is to subtract the straight line connecting $(a_1,F(a_1))$ with $(a_2,F(a_2))$ from $F(x)$. We define

$$
\phi(x) = F(x) - \frac{F(a_2) - F(a_1)}{a_2 - a_1}(x - a_1) - F(a_1).
$$

$\phi(x)$ is continuous on $I$ and $\phi(a_1) = \phi(a_2) = 0$. $\phi$ is convex if and only if $F$ is. We can also check that $\phi$ is $\alpha$-midconvex if and only if it is. If $\phi$ is not convex, there is a $\gamma > 0$ such that $\gamma = \max\{\phi(x), x \in I\}$. Let $a_0 = \inf\{x \in I, \phi(x) = \gamma\}$. We examine the interval $[b_1,b_2] = [a_0 - \beta h, a_0 + \alpha h]$ for $h > 0$ so small that $[b_1,b_2] \subset I$. Then $\alpha b_1 + \beta b_2 = a_0$.

Since we have $\phi(a_0) > \phi(b_1)$ and $\phi(a_0) \geq \phi(b_2)$, we get a contradiction,

$$
\phi(\alpha b_1 + \beta b_2) = \phi(a_0) = \alpha \phi(a_0) + \beta \phi(a_0) > \alpha \phi(b_1) + \beta \phi(b_2).
$$

We prove the following theorem here for the sake of completeness.

**Theorem 4.2.** If $F$ is convex on an interval $I$ but not strictly, then there is an interval $(a_1,a_2) \subset I$ such that $F(\lambda a_1 + (1-\lambda)a_2) = \lambda F(a_1) + (1-\lambda)F(a_2)$ for all $\lambda \in [0,1]$.

**Proof.** We assume the contrary that there exists for each interval $(a_1,a_2) \subset I$ such that $F(\lambda a_1 + (1-\lambda)a_2) < \lambda F(a_1) + (1-\lambda)F(a_2)$. Let $x_1, x_4 \in I, x_1 < x_4$.

We study the function $\phi(x) = F(x) - (F(x_4) - F(x_1))(x - x_1)/(x_4 - x_1) - F(x_1)$ in the same manner as in the previous Theorem 4.1. It is easy to see that $\phi$ is convex if and only if $F$ is convex, $\phi$ is strictly convex if and only if $F$ is strictly convex.

Let $x_3 = \lambda x_1 + (1-\lambda)x_4 \in (x_1,x_4)$, $0 < \lambda < 1$. There is a $\lambda_2 \in (0,1)$ such that $x_2 = \lambda_2 x_1 + (1-\lambda_2)x_3$, then $\phi(x_2) < 0$.

From the convexity of $\phi$ we get for $x_3 = \lambda_3 x_1 + (1-\lambda_3)x_4$, where $0 < \lambda_3 < 1$ a contradiction

$$
\phi(x_3) \leq \lambda_3 \phi(x_2) + (1-\lambda_3)\phi(x_4) = \lambda_3 \phi(x_2) < 0.
$$
5 Sum of log-convex functions

In our attempt to generalize Theorem 4.3 on the strict log-convexity of a positive bi-exponential function we recall the equivalence of convexity to the existence of a supporting line, A. W. Roberts, [2].

Theorem 5.1. Let $F(x)$ and $G(x)$ be log-convex on an open interval $I$. Let the inequalities

$$[(\ln F)’(x), (\ln F)’(x)] \neq [(\ln G)’(x), (\ln G)’(x)]$$

hold for closed intervals formed by the left and right derivatives at any $x \in I$. Then $F(x) + G(x)$ is strictly log-convex on $I$.

Proof. Let $F(x)$ and $G(x)$ be log-convex on an interval $I$. For any $x_0 \in I$ there are $b, c, d, f$ such that $\ln F(x_0) = b + cx_0$, $\ln G(x_0) = d + fx_0$, and for any $x \in I$ we have $\ln F(x) \geq b + cx$, $\ln G(x) \geq d + fx$ because $[(\ln F)’(x), (\ln F)’(x)] \neq [(\ln G)’(x), (\ln G)’(x)]$ we can pick $c \neq f$ such that $c \in [(\ln F)’(x), (\ln F)’(x)]$ and $f \in [(\ln G)’(x), (\ln G)’(x)]$. It means that $b + cx$ is the line of support at $x_0$ of $\ln F(x)$ and $d + fx$ is the line of support at $x_0$ of $\ln G(x)$ with $c \neq f$. We write it as $F(x_0) = e^b e^{cx_0}$, $G(x_0) = e^d e^{fx_0}$, $F(x) \geq e^b e^{cx}$, $G(x) \geq e^d e^{fx}$. We consider $F(x) + G(x)$. We obviously have $F(x_0) + G(x_0) = e^b e^{cx_0} + e^d e^{fx_0}$, $F(x) + G(x) \geq e^b e^{cy} + e^d e^{fy}$ for any $x \in I$. We know that $e^b e^{cy} + e^d e^{fy}$ is strictly log-convex.

Let $a_1 < a_2$ be in $I$. Let any $x_0 = \lambda a_1 + (1 - \lambda) a_2$ be defined by some choice of $\lambda \in (0, 1)$. The constants $b, c, d, f$ are defined with respect to this $x_0$. Then

$$\lambda \ln(F(a_1) + G(a_1)) + (1 - \lambda) \ln(F(a_2) + G(a_2)) \geq$$

$$\lambda \ln(e^b e^{ca_1} + e^d e^{fa_1}) + (1 - \lambda) \ln(e^b e^{ca_2} + e^d e^{fa_2}) > \ln(e^b e^{ca_1} + e^d e^{fa_1})$$

$$= \ln(F(x_0) + G(x_0)),$$

where the strict inequality follows from the strict log-convexity of positive bi-exponential functions.

Example 5.1. Define $F(x) = \exp(|x|)$ and $G(x) = \exp(2|x|)$.

The support lines $b + cx$ and $d + fx$ in Theorem 5.1 may be obtained from right derivatives $F’(x)$ and $G’(x)$. Checking that $F’(x) \neq G’(x)$ is easy. Theorem 5.1 tells us that $F(x) + G(x) = \exp(|x|) + \exp(2|x|)$ is strictly log-convex. If $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$, to interpolate $F(x) + G(x)$, we would obtain $\exp(x) + \exp(2x)$ trivially. Now let $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$, we obtain $0.356134\exp(-3.318866x) + 1.643866\exp(1.814935x)$.

The proof of Theorem 5.1 is cumbersome but it could hardly be done with the help of the second derivative. On the contrary Theorem 5.1 implies the second derivative is only non-negative because strict convexity does not mean negative second derivative. The function $x^4$ is strictly convex but its second derivative is zero at zero.

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